

FINITE SIMPLE GROUPS WITH TWO MAXIMAL SUBGROUPS OF COPRIME ORDERS

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Dedicated to the memory of Vyacheslav Aleksandrovich Belonogov

ABSTRACT. In 1962, V. A. Belonogov proved that if a finite group G contains two maximal subgroups of coprime orders, then either G is one of known solvable groups or G is simple. In this short note based on results by M. Liebeck and J. Saxl on odd order maximal subgroups in finite simple groups we determine possibilities for triples (G, H, M) , where G is a finite nonabelian simple group, H and M are maximal subgroups of G with $(|H|, |M|) = 1$.

Keywords: finite group, simple group, maximal subgroup, subgroups of coprime orders.

1. INTRODUCTION

Throughout the paper we consider only finite groups, and henceforth the term group means finite group. Our terminology and notation are mostly standard and can be found in [3, 4, 6]. We denote the group $PSL_n(q)$ by $PSL_n^+(q)$ and the group $PSU_n(q)$ by $PSL_n^-(q)$. The largest integer power of a prime p dividing a positive integer k is called the p -part of k and is denoted by k_p . Given an integer a and a positive integer n coprime to a , the multiplicative order $ord_n(a)$ of a modulo n is the smallest positive integer k with

$$a^k \equiv 1 \pmod{n}.$$

In other words, the multiplicative order of a modulo n is the order of a in the multiplicative group of the units in the ring of the integers modulo n .

Following [2], we write that a group G is a *group of type A* if G is a non-special group of order pq^β , where p and q are primes, a subgroup of order q^β is a normal elementary abelian subgroup in G , $q^\beta \equiv 1 \pmod{p}$ and β is the least with this property.

In 1962, V. A. Belonogov proved the following proposition.

Proposition 1. *If a finite group G contains two maximal subgroups of coprime orders, then either G is a cyclic group of order pq , where p and q are distinct primes, or either G is a group of type A or G is simple.*

It is clear that if a finite simple group G has two maximal subgroups H and M with $(|H|, |M|) = 1$, then $|H|$ is odd or $|M|$ is odd. Maximal subgroups of

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odd orders in finite simple groups were described by M. Liebeck and J. Saxl [7]. M. Aschbacher [1] described natural geometrically defined eight families \mathfrak{C}_i for $1 \leq i \leq 8$ of subgroups of simple classical groups, which are now called Aschbacher classes, and has proved that if a maximal subgroup of a simple classical group does not belong to the union of Aschbacher classes of the group, then this maximal subgroup is almost simple. Basing on the classification of finite simple groups, results by M. Liebeck and J. Saxl [7], Aschbacher's classification and further results on maximal subgroups in finite almost simple groups [3, 6] in this short note we prove the following theorem.

Theorem 1. *Let G be a nonabelian simple group.*

(i) *If G is sporadic, then G contains two maximal subgroups H and M with $(|H|, |M|) = 1$ if and only if one of the following conditions holds:*

(1) $G \cong M_{23}$, $H \cong 23 : 11$, and M is one of the following subgroups: $PSL_3(4).2_2$, $2^4.A_7$, A_8 , $2^4 : (3 \times A_5).2$;

(2) $G \cong F_2 = B$, $H \cong 47 : 23$ and M is one of the following subgroups: $2^2.E_6(2) : 2$, $2^{9+16}.S_8(2)$, Th , $(2^2 \times F_4(2)) : 2$, $2^{2+10+20}.(M_{22} : 2 \times S_3)$, $[2^{30}].PSL_5(2)$, $S_3 \times Fi_{22} : 2$, $[2^{35}].(S_5 \times PSL_3(2))$, $HN : 2$, $O_8^+(3) : S_4$, $3^{1+8}.2^{1+6}.PSU_4(2).2$, $(3^2 : D_8 \times PSU_4(3).2^2).2$, $5 : 4 \times HS : 2$, $S_4 \times^2 F_4(2)$, $[3^{11}].(S_4 \times 2S_4)$, $S_5 \times M_{22} : 2$, $(S_6 \times PSL_3(4) : 2).2$, $5^3.PSL_3(5)$, $5^{1+4}.2^{1+4}.A_5.4$, $(S_6 \times S_6).4$, $5^2 : 4S_4 \times S_5$, $PSL_2(49).2_3$, $PSL_2(31)$, M_{11} , $PSL_3(3)$, $PSL_2(17) : 2$, $PSL_2(11) : 2$.

(ii) *If G is an alternating group of degree at least 5 or an exceptional group of Lie type, then G does not contain two maximal subgroups of coprime orders.*

(iii) *Let G be a simple classical group of Lie type, assume that G contains two maximal subgroups H and M with $(|H|, |M|) = 1$, and without loss of generality assume that $|H|$ is odd. Then one of the following statements holds:*

(1) $G \cong PSL_2(q)$, where $7 < q \equiv 3 \pmod{4}$, $H \cong E_q : \frac{q-1}{2}$ and $M \cong D_{2(q+1)}$;

(2) $G \cong PSL_2(q)$, where q is prime and $q \equiv 23 \pmod{24}$, $H \cong E_q : \frac{q-1}{2}$ and $M \cong S_4$;

(3) $G \cong PSL_2(q)$, where q is prime and $q \equiv 83, 107 \pmod{120}$, $H \cong E_q : \frac{q-1}{2}$ and $M \cong A_4$;

(4) $G \cong PSL_2(q)$, where q is prime and $q \equiv 59 \pmod{60}$, $H \cong E_q : \frac{q-1}{2}$ and $M \cong A_5$;

(5) $G \cong PSL_n(q)$, where n is an odd prime, q is not a power of n , and $\text{ord}_n(q) = n-1$, $H \cong \frac{q^n-1}{(q-1)(q-1, n)} : n$ and M is the stabilizer in G of a subspace of dimension m of the natural projective module of G , where $2 \leq m \leq n-2$;

(6) $G \cong PSU_n(q)$, where n is an odd prime, q is not a power of n , and either $n \equiv 1 \pmod{4}$, $\text{ord}_n(q) = n-1$ and $(n, q) \neq (5, 2)$ or $n \equiv 3 \pmod{4}$ and $\text{ord}_n(q) = \frac{n-1}{2}$, $H \cong \frac{q^n+1}{(q+1)(q+1, n)} : n$ and M is the stabilizer of a non-degenerate subspace of dimension m of the natural projective module of G , where $2 \leq m \leq \frac{n-1}{2}$;

(7) $G \cong PSU_3(q)$, where $q \neq 5$ and $(q+1)_3 = 3$, $H \cong \frac{q^3+1}{(q+1)(q+1, 3)} : 3$ and M is the stabilizer of a totally singular subspace of dimension 1 of the natural projective module of G .

(8) $G \cong PSU_n(q)$, where $n \geq 5$ is a prime, q is not a power of n , $(n, q) \neq (5, 2)$, $H \cong \frac{q^n+1}{(q+1)(q+1, n)} : n$, M is the stabilizer of a totally singular subspace of dimension m of the natural projective module of G , and one of the following statements holds:

(8i) *either $n \equiv 1 \pmod{4}$ and $\text{ord}_n(q) = n-1$ or $n \equiv 3 \pmod{4}$ and $\text{ord}_n(q) = \frac{n-1}{2}$, and $1 \leq m < n/3$;*

- (8ii) $\text{ord}_n(q) = n - 1$ and $m \geq n/3$;
 (8iii) $n \equiv 3 \pmod{4}$, $\text{ord}_n(q) = n - 1$, and $\frac{n+1}{4} < m < n/3$;
 (9) $G \cong \text{PSL}_n^\varepsilon(q)$, where $n \geq 13$ is a prime and $\varepsilon \in \{+, -\}$, $H \cong \frac{q^n - \varepsilon 1}{(q - \varepsilon 1)(q - \varepsilon 1, n)} : n$
 and M is almost simple.

Moreover, H is always a subgroup from the union of Aschbacher classes of G , and all the possibilities when M is a subgroup from the union of Aschbacher classes of G are listed in statements (iii)(1)–(iii)(8).

2. PROOF OF THEOREM 1

The following assertion is well-known and easy-proving.

Lemma 1. *Let $q > 1$ be an integer and k and m be natural numbers. Then*

- (i) $(q^k - 1, q^m - 1) = q^{(k,m)} - 1$;
 (ii) $(q^k + 1, q^m + 1) = q^{(k,m)} + 1$ if $k_2 = m_2$ and $(q^k + 1, q^m + 1) = (2, q + 1)$ otherwise;
 (iii) $(q^k - 1, q^m + 1) = q^{(k,m)} + 1$ if $k_2 > m_2$ and $(q^k - 1, q^m + 1) = (2, q + 1)$ otherwise.

The following result by M. Liebeck and J. Saxl will be useful to prove Theorem 1.

Proposition 2. (see [7, Theorem 2]) *Let G be a finite simple group and H be a maximal subgroup of odd order form G . Then one of the following statements holds:*

- (1) $G \cong A_p$, $H \cong p.((p - 1)/2)$, p is prime, $p \equiv 3 \pmod{4}$, $p \notin \{7, 11, 23\}$;
 (2) $G \cong \text{PSL}_2(q)$, $H \cong E_q.((q - 1)/2)$, $q \equiv 3 \pmod{4}$;
 (3) $G \cong \text{PSL}_n(q)$, $H \cong \left(\frac{q^n - 1}{(q - 1)(n, q - 1)}\right).n$, n is an odd prime, and $G \not\cong \text{PSL}_3(4)$;
 (4) $G \cong \text{PSU}_n(q)$, $H \cong \left(\frac{q^n + 1}{(q + 1)(n, q + 1)}\right).n$, n is an odd prime, and $G \not\cong \text{PSU}_3(3)$, $\text{PSU}_3(5)$, $\text{PSU}_5(2)$;
 (5) $G \cong M_{23}$ and $H \cong 23.11$;
 (6) $G \cong Th$ and $H \cong 31.15$;
 (7) $G \cong F_2$ and $H \cong 47.23$;
 (8) $G \cong F_1$ and $H \cong 59.29$ or $H \cong 71.35$.

G be a nonabelian simple group. Assume that M and H are maximal subgroups of G such that $(|M|, |H|) = 1$. Without loss of generality, we can assume that $|H|$ is odd. Now with respect to the classification of finite simple groups, consider simple groups case by case.

Let G be sporadic. Then one of Statements (5)–(8) of Proposition 2 holds.

Assume that $G \cong M_{23}$. Maximal subgroups of G are known, see [4]. Thus, G contains a maximal subgroup H of the form $23 : 11$, and M is a maximal subgroup of G such that $(|M|, |H|) = 1$ if and only if M is one of the following subgroups of G : $\text{PSL}_3(4).2_2$, $2^4.A_7$, A_8 , $2^4 : (3 \times A_5).2$.

Assume that $G \cong Th$. Maximal subgroups of G are known, see [8]. G contains a maximal subgroup H of the form $31 : 15$ but if M is a maximal subgroup of G which is not G -conjugate to H , then $|M|$ is divisible to 3 or 5, therefore, G does not contain a pair of maximal subgroup of coprime orders.

$G \cong F_2 = B$. Maximal subgroups of G are known, see [10], their orders are presented in [8]. By [8, 10], G contains a maximal subgroup H of the form $47 : 23$, and M is a maximal subgroup of G such that $(|M|, |H|) = 1$ if and only if M is one of the following subgroups of G : $2.^2E_6(2) : 2$, $2^{9+16}.S_8(2)$, Th , $(2^2 \times F_4(2)) : 2$,

$2^{2+10+20} \cdot (M_{22} : 2 \times S_3), [2^{30}] \cdot PSL_5(2), S_3 \times Fi_{22} : 2, [2^{35}] \cdot (S_5 \times PSL_3(2)), HN : 2, O_8^+(3) : S_4, 3^{1+8} \cdot 2^{1+6} \cdot PSU_4(2).2, (3^2 : D_8 \times PSU_4(3).2^2).2, 5 : 4 \times HS : 2, S_4 \times {}^2F_4(2), [3^{11}] \cdot (S_4 \times 2S_4), S_5 \times M_{22} : 2, (S_6 \times PSL_3(4) : 2).2, 5^3 \cdot PSL_3(5), 5^{1+4} \cdot 2^{1+4} \cdot A_5.4, (S_6 \times S_6).4, 5^2 : 4S_4 \times S_5, PSL_2(49).2_3, PSL_2(31), M_{11}, PSL_3(3), PSL_2(17) : 2, PSL_2(11) : 2.$

Let $G \cong F_1 = M$ and $H_1 \cong 59.29$ be a subgroup of G . Then H_1 is a $\{29, 59\}$ -Hall subgroup of G , and by [5, Theorem A], $\{29, 59\}$ -Hall subgroups form the unique conjugacy class in G . Note that by [8], G contains a maximal subgroup $H \cong PSL_2(59)$, and H contains a parabolic maximal subgroup which is a $\{29, 59\}$ -Hall subgroup of G . Thus, G does not contain a maximal subgroup of the form 59.29 . Let $H_1 \cong 71.35$ be a subgroup of G . By [9], H_1 is the normalizer of a Sylow 71-subgroup of G , therefore, the subgroups of the form 71.35 form the unique conjugacy class in G . Now by [8], G contains a maximal subgroup $K \cong PSL_2(71)$, and K contains a parabolic maximal subgroup of the form 71.35 . Thus, G does not contain a maximal subgroup of the form 71.35 . So, G does not contain a maximal subgroup of odd order, therefore, G does not contain a pair of maximal subgroups of coprime orders.

Thus, Statement (i) of Theorem 1 holds.

Let G be a simple alternating group. Then Statement (1) of Proposition 2 holds. Assume that G acts naturally on the set $\Omega = \{1, \dots, p\}$. Let M be a maximal subgroup of G such that p does not divide $|M|$. Then M is intransitive on Ω , and therefore, M is the stabilizer in G of a partition $\Omega = \Omega_1 \cup \Omega_2$ of Ω into disjoint subsets Ω_1 and Ω_2 . Now since p is prime, $p \equiv 3 \pmod{4}$, and $p \notin \{7, 11, 23\}$, we have that $p \geq 19$, and it is clear that $|M|$ is not coprime to $(p-1)/2$. Thus, G does not contain a pair of maximal subgroup of coprime orders and Statement (ii) of Theorem 1 holds for alternating groups.

If G is an exceptional group of Lie type, then by Proposition 2, G does not contain a maximal subgroup of odd order, therefore, G does not contain a pair of maximal subgroup of coprime orders and Statement (ii) of Theorem 1 holds for exceptional groups of Lie type.

Let G be a finite simple classical group. Then one of Statements (2)–(4) of Proposition 2 holds.

Let Statement (2) of Proposition 2 holds, i. e. $G = PSL_2(q)$ for $3 < q \equiv 3 \pmod{4}$. Maximal subgroups of G are known (see, for example, [3, Tables 8.1, 8.2, 8.7]). Note that the group $G = PSL_2(q)$ always contains a parabolic maximal subgroup H of the form $E_q \cdot ((q-1)/2)$, and $|H|$ is odd if and only if $q \equiv 3 \pmod{4}$. We consider possibilities for M case by case.

If $M \cong D_{2(q-1)}$, then it is clear that $(|M|, |H|) \neq 1$.

Let $M \cong D_{2(q+1)}$. Then $(|H|, |M|) = 1$ if and only if $q \equiv 3 \pmod{4}$ and M is maximal in G if and only if $q \notin \{7, 9\}$. Thus, Statement (iii)(1) of Theorem 1 holds.

Let $M \cong S_4$. Then $(|M|, |H|) = 1$ if and only if $q \equiv 3 \pmod{4}$ and $q \equiv 2 \pmod{3}$, i. e. $q \equiv 11 \pmod{12}$. Note that M is maximal in G if and only if q is prime and $q \equiv \pm 1 \pmod{8}$. Therefore, M is a maximal subgroup of G such that $(|M|, |H|) = 1$ if and only if q is a prime and $q \equiv 23 \pmod{24}$. Thus, Statement (iii)(2) of Theorem 1 holds.

Let $M \cong A_4$. Then again $(|M|, |H|) = 1$ if and only if $q \equiv 11 \pmod{12}$. Note that M is maximal in G if and only if either $q = 5$ or q is prime and $q \equiv \pm 3, \pm 13$

(mod 40). Therefore, M is a maximal subgroup of G such that $(|M|, |H|) = 1$ if and only if q is a prime and $q \equiv 83, 107 \pmod{120}$. Thus, Statement (iii)(3) of Theorem 1 holds.

If $M \cong PSL_2(q_0).2$, where $q = q_0^2$, or $M \cong PSL_2(q_0)$, where $q = q_0^r$ and r is an odd prime, then it is clear that $(|M|, |H|) \neq 1$.

Let $M \cong A_5$. Note that $G = PSL_2(q)$ contains a maximal subgroup isomorphic to A_5 if and only if either q is prime and $q \equiv \pm 1 \pmod{10}$ or $q = p^2$, where p is prime and $p \equiv \pm 3 \pmod{10}$. It is easy to see that $(|M|, |H|) = 1$ if and only if $q \equiv 11 \pmod{12}$ and $q \equiv 2, 3, 4 \pmod{5}$. Note that if $q = p^2$, where $p \equiv \pm 3 \pmod{10}$, then $q \equiv 9, 49 \pmod{60}$, therefore this case does not appear. Thus, M is a maximal subgroup of G such that $(|M|, |H|) = 1$ if and only if q is prime and $q \equiv 59 \pmod{60}$. Thus, Statement (iii)(4) of Theorem 1 holds.

Let Statement (3) of Proposition 2 holds, i. e. $G \cong PSL_n(q)$, $H \cong (\frac{q^n-1}{(q-1)(n, q-1)}) \cdot n$, n is an odd prime, and $G \not\cong PSL_3(4)$. Note that by [3, Table 8.3] the group $PSL_3(4)$ does not contain a maximal subgroup of odd order, therefore, the group $PSL_3(4)$ does not contain a pair of maximal subgroup of coprime orders. Thus, we can assume that $G \not\cong PSL_3(4)$, and by [3, Tables 8.3, 8.18, 8.35, 8.70] and [6, Table 3.5.A, Proposition 4.3.6], G has a maximal subgroup $H \cong (\frac{q^n-1}{(q-1)(n, q-1)}) \cdot n$ such that $|H|$ is odd. By the Aschbacher theorem, for each maximal subgroup M of G , if M does not belong to the union of Aschbacher classes $\mathfrak{C}_i(G)$ of G for $i \in \{1, \dots, 8\}$, then M is almost simple and we write $M \in \mathfrak{S}(G)$ in this case. If $n \leq 11$, then all maximal subgroups of G are known, see [3, Tables 8.3, 8.4, 8.18, 8.19, 8.35, 8.36, 8.70, 8.71], and if $n \geq 12$, then maximal subgroups of G from the union of Aschbacher classes of G are known, see [6, Table 3.5.A]. So, we consider possibilities for M case by case.

Since n is odd and prime, by Tables 8.3, 8.4, 8.18, 8.19, 8.35, 8.36, 8.70, and 8.71 of [3] and [6, Table 3.5.A, Propositions 4.1.17, 4.2.9, 4.5.3, 4.6.5, 4.8.4, 4.8.5], if M is a maximal subgroup of G which is not isomorphic to H , then one of the following statements holds:

- (\mathfrak{C}_1) M is the stabilizer in G of a subspace of dimension m of the natural projective module of G , where $1 \leq m \leq n-1$;
- (\mathfrak{C}_2) $M \cong [(q-1)^{n-1}/(q-1, n)] : S_n$ is the stabilizer in G of a decomposition of the natural projective module of G into a direct sum of subspaces of dimension 1;
- (\mathfrak{C}_5) $M \cong \frac{c}{(q-1, n)} PGL_n(q_0)$ if $q = q_0^r$ for odd prime r , where $c = \frac{(q-1)}{[q_0-1, \frac{q-1}{(q-1, n)}]}$;
- (\mathfrak{C}_6) $M \cong [n^3].Sp_2(n)$ if some special conditions for n and q hold and $3^2.Q_8$ if $n = 3$ and some special conditions for q hold;
- (\mathfrak{C}_8) $M \cong PSO_n(q)$ or $M \cong PSU_n(q_0) \cdot [(q_0+1, n)c/(q-1, n)]$ if $q = q_0^2$, where $c = (q-1)/[q_0+1, (q-1)/(q-1, n)]$;
- (\mathfrak{S}) M is almost simple.

Note that by [3, Tables 8.3, 8.18, 8.35, 8.70] and [6, Proposition 4.1.17], for each $1 \leq m \leq n-1$, the stabilizer M in G of a subspace of dimension m of the natural projective module of G is a maximal subgroup of G . Moreover, by [6, Proposition 4.1.17],

$$|M| = \frac{q^{n(n-1)}(q-1)}{(q-1, n)} \prod_{i=2}^m (q^i - 1) \cdot \prod_{i=2}^{n-m} (q^i - 1).$$

By Lemma 1, is clear that $(|M|, |H|) = 1$ if and only if n does not divide $|M|$. Note that by Fermat's little theorem, if q is not a power of n , then n divides $q^{n-1} - 1$. Thus, n does not divide $|M|$ if and only if q is not a power of n , $2 \leq m \leq n - 2$, and $\text{ord}_n(q) = n - 1$. Thus, Statement (iii)(5) of Theorem 1 holds.

If $M \cong [(q-1)^{n-1}/(q-1, n)] : S_n$, then it is clear that n divides $|M|$, therefore, $|M|$ and $|H|$ are not coprime.

If $M \cong \frac{c}{(q-1, n)} PGL_n(q_0)$ for $q = q_0^r$ with r odd prime and $c = \frac{(q-1)}{[q_0-1, \frac{q-1}{(q-1, n)}}$ or $M \cong PSU_n(q_0) \cdot [(q_0+1, n)c/(q-1, n)]$ for $q = q_0^2$ and $c = (q-1)/[q_0+1, (q-1)/(q-1, n)]$, then $q_0^{n-1} - 1$ divides $|M|$, and by Fermat's little theorem, if q is not a power of n , then n divides $q_0^{n-1} - 1$. Thus, in any case n divides $|M|$, therefore, $|M|$ and $|H|$ are not coprime.

If $M \cong [n^3] \cdot Sp_2(n)$, then it is clear that n divides $|M|$, therefore, $|M|$ and $|H|$ are not coprime.

If $M \cong PSO_n(q)$, then $q^{n-1} - 1$ divides $|M|$, again using Fermat's little theorem, we conclude that n divides $|M|$. Thus, $|M|$ and $|H|$ are not coprime.

Almost simple subgroups of low-dimensional finite simple linear groups are known, see [3, Tables 8.4, 8.19, 8.36, 8.71]. Following [3] here we provide preimages of these maximal subgroups in $\tilde{G} = SL_n(q)$.

"Almost simple" maximal subgroups of $SL_n(q)$ for $n \in \{3, 5, 7, 11\}$

n	Group \tilde{G}	Maximal subgroups of \tilde{G} whose images in G are almost simple
3	$SL_3(q)$	$(q-1, 3) \times PSL_2(7)$, $3 \cdot A_6$
5	$SL_5(q)$	$(q-1, 5) \times PSL_2(11)$, M_{11} , $(q-1, 5) \times PSU_4(2)$
7	$SL_7(q)$	$(q-1, 7) \times PSU_3(3)$
11	$SL_{11}(q)$	$PSL_2(23)$, M_{24} , $(q-1, 11) \times PSL_2(23)$, $(q-1, 11) \times PSU_5(2)$

Now it is clear that if $G = PSL_n(q)$ for $n \leq 11$ and M is a maximal almost simple subgroup from $\mathfrak{S}(G)$, then n divides $|M|$, therefore, $|M|$ and $|H|$ are not coprime. Thus, Statement (iii)(9) of Theorem 1 holds.

Let Statement (4) of Proposition 2 holds, i. e. $G \cong PSU_n(q)$, $H \cong (\frac{q^n+1}{(q+1)(n, q+1)}) \cdot n$, n is an odd prime, and $G \not\cong PSU_3(3)$, $PSU_3(5)$, $PSU_5(2)$.

Note that by [3, Tables 8.5, 8.20] groups $G \not\cong PSU_3(3)$, $PSU_3(5)$, and $PSU_5(2)$ do not contain maximal subgroups of odd order, therefore, these groups do not contain pairs of maximal subgroup of coprime orders. Thus, we can assume that $G \not\cong PSU_3(3)$, $PSU_3(5)$, $PSU_5(2)$, and by [3, Tables 8.5, 8.20, 8.37, 8.72] and [6, Table 3.5.B, Proposition 4.3.6], G has a maximal subgroup $H \cong (\frac{q^n+1}{(q+1)(n, q+1)}) \cdot n$ such that $|H|$ is odd. By the Aschbacher theorem, for each maximal subgroup M of G , if M does not belong to the union of Aschbacher classes $\mathfrak{C}_i(G)$ of G for $i \in \{1, \dots, 8\}$, then M is almost simple and we write $M \in \mathfrak{S}(G)$ in this case. If $n \leq 11$, then all maximal subgroups of G are known, see [3, Tables 8.5, 8.6, 8.20, 8.21, 8.37, 8.38, 8.72, 8.73], and if $n \geq 12$, then maximal subgroups of G from the union of Aschbacher classes of G are known, see [6, Table 3.5.B]. So, we consider possibilities for M case by case.

Since n is odd and prime, by Tables 8.5, 8.6, 8.20, 8.21, 8.37, 8.38, 8.72, 8.73 of [3] and [6, Table 3.5.B, Propositions 4.1.4, 4.1.18, 4.2.9, 4.5.3, 4.5.5, 4.6.5], if M is a maximal subgroup of G which is not isomorphic to H , then one of the following statements holds:

- (C₁) M is the stabilizer in G of a subspace of dimension m , where $1 \leq m \leq (n-1)/2$, of the natural projective module of G , and either the subspace is totally singular or the subspace is non-degenerate;
- (C₂) $M \cong [(q+1)^{n-1}/(q+1, n)] : S_n$ is the stabilizer in G of a decomposition of the natural projective module of G into a direct sum of subspaces of dimension 1;
- (C₅) $M \cong \frac{c}{(q+1, n)} PGU_n(q_0)$ if $q = q_0^r$ for odd prime r , where $c = \frac{(q+1)}{[q_0+1, \frac{q+1}{(q+1, n)}]}$, or $M \cong PSO_n(q)$;
- (C₆) $M \cong [n^3].Sp_2(n)$ if some special conditions for n and q hold or $3^2.Q_8$ if $n = 3$ and some special conditions for q hold;
- (S) M is almost simple.

Note that by [3, Tables 8.5, 8.20, 8.37, 8.72] and [6, Proposition 4.1.4], for each $1 \leq m \leq \frac{n-1}{2}$, the stabilizer M in G of a non-degenerate subspace of dimension m of the natural projective module of G is a maximal subgroup of G . Moreover, by [6, Proposition 4.1.4],

$$|M| = \frac{q^{(m^2+(n-m)^2-n)/2} \cdot (q+1)}{(q+1, n)} \prod_{i=2}^m (q^i - (-1)^i) \cdot \prod_{i=2}^{n-m} (q^i - (-1)^i).$$

By Lemma 1, is clear that $(|M|, |H|) = 1$ if and only if n does not divide $|M|$. Note that by Fermat's little theorem, if q is not a power of n , then n divides $q^{n-1} - 1$ and, therefore, $ord_n(q)$ divides $n-1$ and if n does not divide $|M|$, then $m \geq 2$. Let $n-1 = k \cdot ord_n(q)$. If $k \geq 3$ is odd, then n divides $q^{\frac{n-1}{k}} - 1$ and $q^{\frac{n-1}{k}} - 1$ divides $|M|$ since $\frac{n-1}{k} < \frac{n-1}{2}$ and $\frac{n-1}{k}$ is even. If $k \geq 4$ is even, then n divides $q^{\frac{2 \cdot (n-1)}{k}} - 1$ and $q^{\frac{2 \cdot (n-1)}{k}} - 1$ divides $|M|$ since $\frac{2(n-1)}{k} \leq \frac{n-1}{2}$ and $\frac{2(n-1)}{k}$ is even. Thus, if n does not divide $|M|$, then $k \in \{1, 2\}$.

Assume that $n \equiv 1 \pmod{4}$ and, therefore, $\frac{n-1}{2}$ is even and $q^{\frac{n-1}{2}} - 1$ divides $|M|$. Thus, in this case, if $k = 2$, then n divides $|M|$. Suppose that $k = 1$ and n divides $|M|$. Then n divides $q^i + 1$ for some odd $i \leq n-2$. Therefore, $n-1 = ord_n(q)$ divides $2i \leq 2n-4$. Thus, $2i = n-1$ and i is even, a contradiction.

Assume that $n \equiv 3 \pmod{4}$ and, therefore, $\frac{n-1}{2}$ is odd and $q^{\frac{n-1}{2}} + 1$ divides $|M|$. Thus, in this case, if $k = 1$, then n divides $|M|$. Suppose that $k = 2$ and n divides $|M|$. Then since $\frac{n-1}{2}$ is odd, we have $q^{\frac{n-1}{2}} - 1$ does not divide $|M|$, therefore, n divides $q^i + 1$ for some odd $i \leq n-2$. Thus, $\frac{n-1}{2} = ord_n(q)$ divides $2i \leq 2n-4$. Thus, $2i = n-1$ and $i = \frac{n-1}{2}$, a contradiction. Thus, Statement (iii)(6) of Theorem 1 holds.

Note that by [3, Tables 8.5, 8.20, 8.37, 8.72] and [6, Proposition 4.1.18], for each $1 \leq m \leq \frac{n-1}{2}$, the stabilizer M in G of a totally singular subspace of dimension m of the natural projective module of G is a maximal subgroup of G . Moreover, by [6, Proposition 4.1.18],

$$|M| = \frac{q^{n(n-1)/2} \cdot (q^2 - 1)}{(q+1, n)} \prod_{i=2}^m (q^{2i} - 1) \cdot \prod_{i=2}^{n-2m} (q^i - (-1)^i).$$

By Lemma 1, is clear that $(|M|, |H|) = 1$ if and only if n does not divide $|M|$. Note that by Fermat's little theorem, if q is not a power of n , then n divides $q^{n-1} - 1$ and, therefore, $ord_n(q)$ divides $n-1$.

Let $n = 3$. Then $m = 1$, $|M| = \frac{q^3 \cdot (q^2 - 1)}{(q+1, 3)}$ and 3 does not divide $|M|$ if and only if $(q+1)_3 = 3$. Thus, Statement (iii)(7) of Theorem 1 holds.

Let $n \geq 5$. Assume that $m \geq n/3$. Then the number $\prod_{i=2}^{n-2m} (q^i - (-1)^i)$ divides the number $\prod_{i=2}^m (q^{2i} - 1)$ and, therefore, n does not divide $|M|$ if and only if $\text{ord}_n(q^2) > m > n/3$. It is clear that $\text{ord}_n(q^2)$ divides $\frac{n-1}{2}$, therefore, n does not divide $|M|$ if and only if $\text{ord}_n(q^2) = \frac{n-1}{2}$ which is equivalent to $\text{ord}_n(q) = n-1$.

Assume that $m < n/4$. Then the number $\prod_{i=2}^m (q^{2i} - 1)$ divides the number $\prod_{i=2}^{n-2m} (q^i - (-1)^i)$. Thus, as in the proof of Statement (iii)(6), we obtain that n does not divide $|M|$ if and only if either $n \equiv 1 \pmod{4}$ and $\text{ord}_n(q) = n-1$ or $n \equiv 3 \pmod{4}$ and $\text{ord}_n(q) = \frac{n-1}{2}$.

Finally, assume that $n/4 \leq m < n/3$. Let $n-1 = k \cdot \text{ord}_n(q)$. If $k \geq 3$ and $\frac{n-1}{k}$ is even, then n divides $q^{\frac{n-1}{k}} - 1$ and $q^{\frac{n-1}{k}} - 1$ divides $\prod_{i=2}^m (q^{2i} - 1)$ since $\frac{n-1}{k} < \frac{n-1}{2} < 2 \cdot n/4$. If $k \geq 4$ and $\frac{n-1}{k}$ is odd, then n divides $q^{\frac{n-1}{k}} - 1$ and $q^{\frac{n-1}{k}} - 1$ divides $\prod_{i=2}^m (q^{2i} - 1)$ since $2 \cdot \frac{n-1}{k} \leq 2 \cdot \frac{n-1}{4} < 2 \cdot n/4$. Thus, $k \in \{1, 2\}$.

Assume that $n \equiv 1 \pmod{4}$ and, therefore, $\frac{n-1}{2}$ is even. Since $n/4 \leq m$, we have $q^{\frac{n-1}{2}} - 1$ divides $|M|$. Thus, in this case, if $k = 2$, then n divides $|M|$. Suppose that $k = 1$ and n divides $|M|$. Then n divides $q^i - 1$ for some $i \leq 2n/3$ or n divides $q^i + 1$ for some odd $i \leq n-2$. Note that in the last case $n-1 = \text{ord}_n(q)$ divides $2i \leq 2n-4$ and, therefore, $2i = n-1$ and i is even. In both cases we have a contradiction.

Assume that $n \equiv 3 \pmod{4}$ and, therefore, $\frac{n-1}{2}$ is odd. Suppose that $k = 2$ and n divides $|M|$. Then since $\frac{n-1}{2}$ is odd, we have $q^{\frac{n-1}{2}} - 1$ does not divide $|M|$, therefore, n divides $q^i - 1$ for some even $i \leq 2n/3$ or n divides $q^i + 1$ for some odd $i \leq n-2$. In the last case, $\frac{n-1}{2} = \text{ord}_n(q)$ divides $2i \leq 2n-4$. Thus, $2i = n-1$ and $i = \frac{n-1}{2}$, a contradiction. Assume that $k = 1$. If $m = \frac{n+1}{4}$, then n divides $q^{\frac{n-1}{2}} + 1$ and $q^{\frac{n-1}{2}} + 1$ divides $|M|$. Let $m > \frac{n+1}{4}$. Suppose that n divides $|M|$. Then n divides $q^{2i} - 1$ for some $i < n/3$ or n divides $q^i - 1$ for some even $i < \frac{n-1}{2}$ or n divides $q^i + 1$ for some odd $i < \frac{n-1}{2}$. In the last case, n divides $q^{2i} - 1$ and $2i < n-1$. In all cases we have contradictions. Thus, Statement (iii)(8) of Theorem 1 holds.

If $M \cong [(q+1)^{n-1}/(q+1, n)] : S_n$, then it is clear that n divides $|M|$, therefore, $|M|$ and $|H|$ are not coprime.

If $M \cong \frac{c}{(q+1, n)} PGU_n(q_0)$ for $q = q_0^r$ with r odd prime and $c = \frac{(q+1)}{[q_0+1, \frac{q+1}{(q+1, n)}]}$, then $q_0^{n-1} - 1$ divides $|M|$, and by Fermat's little theorem, if q is not a power of n , then n divides $q_0^{n-1} - 1$. Thus, in any case n divides $|M|$, therefore, $|M|$ and $|H|$ are not coprime.

Similar, $M \cong PSO_n(q)$, then $q^{n-1} - 1$ divides $|M|$, again using Fermat's little theorem, we conclude that n divides $|M|$. Thus, $|M|$ and $|H|$ are not coprime.

If $M \cong [n^3].Sp_2(n)$, then it is clear that n divides $|M|$, therefore, $|M|$ and $|H|$ are not coprime.

Almost simple subgroups of low-dimensional finite simple unitary groups are known, see [3, Tables 8.6, 8.21, 8.38, 8.73]. Following [3] here we provide preimages of these maximal subgroups in $\tilde{G} = SU_n(q)$.

"Almost simple" maximal subgroups of $SU_n(q)$ for $n \in \{3, 5, 7, 11\}$

n	Group \tilde{G}	Maximal subgroups of \tilde{G} whose images in G are almost simple
3	$SU_3(q)$	$(q+1, 3) \times PSL_2(7)$, $3 \cdot A_6$, $3 \cdot A_6 2_3$, $3 \cdot A_7$
5	$SU_5(q)$	$(q+1, 5) \times PSL_2(11)$, $(q+1, 5) \times PSU_4(2)$
7	$SU_7(q)$	$(q+1, 7) \times PSU_3(3)$
11	$SU_{11}(q)$	$(q+1, 11) \times PSL_2(23)$, $(q+1, 11) \times PSU_5(2)$

Now it is clear that if $G = PSU_n(q)$ for $n \leq 11$ and M is a maximal almost simple subgroup from $\mathfrak{S}(G)$, then n divides $|M|$, therefore, $|M|$ and $|H|$ are not coprime. Thus, Statement (iii)(9) of Theorem 1 holds.

Note that H is always a subgroup from the union of Aschbacher classes of G , and all the possibilities when M is a subgroup from the union of Aschbacher classes of G are listed in statements (iii)(1)–(iii)(9).

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