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CAPACITIES OF GENERALIZED CONDENSERS WITH A_1-MUCKENHOUPT WEIGHT

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Abstract. We prove the relations (related to A_1-Muckenhoupt weight) between two capacities of a generalized condenser and the moduli of vector measures on the curve configuration.
Keywords: capacity of condenser, Muckenhoupt weight, modulus of vector measures.

1. Introduction

In 1999 H. Aikawa and M. Ohtsuka [2] defined the extremal length of vector measures. In particular, they established relations (related to A_p-Muckenhoupt weight w with $p \in (1, +\infty)$) between two capacities of a two plates condenser and the associated moduli of vector measures on the curve configuration.

Here we note that Aikawa-Ohtsuka constructions are essentially based on the properties of (p, w)-precise functions and absolutely continuous functions along (p, w)-a.e. curves on the open set $G \subset R^n$, $n \geq 2$. The detailed properties of absolutely continuous functions along (p, w)-a.e. curves on G for $w \in A_p$, $1 \leq p < \infty$, are given in [13, Sect. 4.1–4.3]; respectively, properties of (p, w)-precise functions for $1 < p < \infty$ are given in [13, Sect. 4.4].

In this paper we extend mentioned above Aikawa-Ohtsuka relations to the case of A_1-Muckenhoupt weights and condensers with finite number of plates.

Some relations (see Theorem 3) could be established only for Hesse condensers, for which, by definition, plates lie either inside G or one of plates is ∂G.

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2. Basic definitions, notes and auxiliary statements

For $n \geq 2$ we note by $\overline{R^n} = R^n \cup \{\infty\}$ the one-point compactification of Euclidean space R^n. All topological considerations refer to the metric space $(\overline{R^n}, q)$, where q is a chordal metric defined by stereographic projection [10].

If $F \subset \overline{R^n}$ then $\bar{F}$, ∂F notes the closure and boundary of set F in topology of $\overline{R^n}$, respectively.

The norm of point $x = (x_1, \dots, x_n) \in R^n$ is defined by $|x| = \left(\sum\limits_{k=1}^{n} x_k^2\right)^{1/2}$ and let $x \cdot y = \sum\limits_{k=1}^{n} x_k y_k$ for $y = (y_1, \dots, y_n) \in R^n$; $\mathbb{N} = \{1, 2, \dots\}$. If $a \in \overline{R^n}$ then $q(a, F)$ is a chordal distance from a to $F \subset R^n$. If $a \in R^n$ then $\operatorname{dist}(a, F)$ is the Euclidean distance from a to $F \subset R^n$. Replacing a by a set $K \subset \overline{R^n}$ ($K \subset R^n$), in the same way we define $q(K, F)$ ($\operatorname{dist}(K, F)$).

Given $r > 0$ and $x \in R^n$, let $B(x, r) = \{y \in R^n : |x - y| < r\}$. For $x = \infty$ let $B(\infty, r) = \overline{R^n \setminus B\left(0, \frac{1}{r}\right)}$. The set $O(F, \varepsilon) = \bigcup\limits_{x \in F} B(x, \varepsilon)$ we call the ε-neighborhood of set $F \subset \overline{R^n}$, where $\varepsilon > 0$.

Denote by m_k the k-dimensional Lebesgue measure, $k \in \mathbb{N}$, and set $|F| = m_n(F)$.

We will use the abbreviation "a.e." for phrase "almost every" with respect to m_n-measure. Analogously the phrases "measurable set", "measurable function", "locally integrable function" are understood in the sense of m_n-measure.

Let F be a measurable set of R^n and $u : F \to [-\infty, +\infty]$ be a measurable real-valued function. For $1 \leq p < \infty$ we let

$$\|u\|_{L_p(F)} = \left(\int\limits_F |u(x)|^p \, dx\right)^{1/p}.$$

Below in the text let G be a nonempty open set in R^n and let u be a real-valued function on G. We say that $u \in L_p(G, loc)$, if $\|u\|_{L_p(F)} < \infty$ for every compact set $F \subset G$. The class of all functions u, for which $\|u\|_{L_p(G)} < \infty$, denote by $L_p(G)$.

2.1. A_1-Muckenhoupt weight.

Following by Muckenhoupt [12], the function $w : R^n \to (0, +\infty)$ is called A_1-weight, if there exists a constant $A > 0$ such as for every ball $B = B(x, r) \subset R^n$

$$\left(\frac{1}{|B|}\int\limits_B w \, dx\right) \cdot \operatorname*{ess\,inf}_{y \in B} \frac{1}{w(y)} \leq A.$$

Denote by A_1 the class of all A_1-weights.

Next in the text let w be an A_1-weight. Let for $x \in R^n$

$$M(w(x)) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} w(y) \, dy.$$

Besides, define $L_{1,w}(G)$ as the set of all measurable functions u on G with

$$\|u\|_{L_{1,w}(G)} = \int\limits_G |u| w \, dx < \infty$$

and denote by $L_{1,w}(G, loc)$ the set of all measurable functions u on G such that $u \cdot w \in L_1(G, loc)$.

We need the following properties of A_1-weights.

Proposition 1 ([16, Remark 1.2.4])**.** *Given A_1-weight w, there exists positive constants C_1, C_2 such that the inequalities*

$$M(w(x)) \le C_1 w(x), \quad w(x) \ge \frac{C_2}{(1+|x|)^n}$$

hold a.e. on R^n. Moreover, $L_{1,w}(G) \subset L_1(G, loc)$ and if G is a bounded open set in R^n then $L_{1,w}(G) \subset L_1(G)$.

Remark 1. *It follows from Proposition 1 that* $\operatorname{ess\,inf} w$ *is locally positive in R^n.*

2.2. $(1, w)$**-modulus of family of curves, exceptional families of curves in** G**.** Let T be a numeric interval (a, b) or segment $[a, b]$, $a < b$. The curve γ in R^n is the image of T under continuous map $x = x(t)$ into R^n. Next we assume that the map $x = x(t) : T \to R^n$ is non-constant on all non-degenerate intervals from T, and define the parametrization of curve γ, in which γ is locally rectifiable curve if $T = (a, b)$. If $T = [a, b]$ then we assume that the arc $x = x(t), a_1 \le t \le b_1$, is a rectifiable curve for all segments $[a_1, b_1] \subset (a, b)$. We call such curve also a locally rectifiable. Sometimes we call it a locally rectifiable closed arc. If $x(a) = x(b)$ then γ is called a locally rectifiable closed curve.

For locally rectifiable curve (locally rectifiable closed arc) there exists the arc-length parametrization (see [13, Sec. 2.1]) $x = x(s)$, $-\infty \le a < s < b \le +\infty$ $(-\infty \le a \le s \le b \le +\infty)$ such that the length of any arc $x = x(s)$, $a < s_1 \le s \le s_2 < b$, is $s_2 - s_1$. Then the integral $\int\limits_\gamma \rho\, ds$ for $\gamma \subset G$ and Borel function $\rho : G \to [-\infty, +\infty]$, we define as integral $\int\limits_a^b \rho(x(s))\, ds$ in Lebesgue's sense. In particular, the linear measure $m_\gamma(e)$ of Borel set $e \subset \gamma$ is $\int\limits_\gamma \chi_e\, ds$, where χ_e is a characteristic function of set e. We assume that this measure m_γ is complete on γ and its element we denote as $ds = ds_\gamma$.

Let for Borel function $\rho : G \to [0, +\infty]$ and for family Γ of curves $\gamma \subset G$ the inequality $\int\limits_\gamma \rho\, ds \ge 1$ is true for all $\gamma \in \Gamma$. Then we call ρ an admissible function for Γ and write $\rho \wedge \Gamma$.

Define the $(1, w)$-modulus of family Γ of curves $\gamma \subset G$ with weight $w \in A_1$ as the quantity

$$M_{1,w}(\Gamma) = \inf \left\{ \int\limits_G \rho w\, dx : \rho \wedge \Gamma \right\}.$$

If $\Gamma = \emptyset$ then by definition $M_{1,w}(\Gamma) = 0$.

The family Γ_0 of curves in G is called an $(1, w)$-exceptional (or simply $(1, w)$-exc.) if $M_{1,w}(\Gamma_0) = 0$. If $\rho \wedge \Gamma \setminus \Gamma_0$ and Γ_0 is $(1, w)$-exc., then we write $\rho \wedge \Gamma$ $(1, w)$-a.e. It is known [13, Theorem 1.3.6] that $M_{1,w}(\Gamma) = \inf \left\{ \int\limits_G \rho w\, dx : \rho \wedge \Gamma\ (1, w) - a.e. \right\}$.

Remark 2. *Let Γ be a family of curves in G and $g : G \to [0, +\infty]$ be a measurable function in G. We call g a m_n-admissible function for Γ if there exists a Borel* *function* $\rho : G \to [0, +\infty]$, *which is equal* g *except a Borel set* $F \subset G$, $|F| = 0$, *where*

$$\int\limits_{\gamma} \rho\, ds = \int\limits_{\gamma} g\, ds \geq 1$$

for all $\gamma \in \Gamma \setminus \Gamma_0$, Γ_0 *is* $(1, w)$*-exc. Then, applying the well-known Fuglede's theorem* [8, Theorem 3]*, we deduce that*

$$M_{1,w}(\Gamma) = \inf \left\{ \int\limits_{G} gw\, dx : g \text{ is } m_n\text{-admissible function for } \Gamma \right\}.$$

Let γ is a curve in R^n with parametrization $x = x(t)$, $a < t < b$. We will say that $x_0 = s(\gamma)$ is a starting point ($y_0 = t(\gamma)$ is a terminate point) for γ, if $s(\gamma) = \lim\limits_{t \to a} x(t)$ ($t(\gamma) = \lim\limits_{t \to b} x(t)$) in $\overline{R^n}$.

Besides, if $x_0 = s(\gamma)$, $y_0 = t(\gamma)$, then we will say that γ connect x_0 and y_0. If at least one of limits $\lim\limits_{t \to a} x(t)$ and $\lim\limits_{t \to b} x(t)$ does not exist in $\overline{R^n}$ then we say that γ is oscillating.

Next, we will consider a closed arc as a locally rectifiable curve $x = x(t), a < t < b$, in G, that has starting and terminal points lying in G.

In order to provide the module description of oscillating curves and curves connecting points in $\overline{R^n}$, we use the following statement [13, Theorem 2.2.1].

Proposition 2. *Let* Γ *be a family of curves in* R^n*, for each* γ *of which there exists a compact* $K_\gamma \subset R^n$ *such that* $m_\gamma(\gamma \cap K_\gamma) = \infty$*;* $\gamma \cap K_\gamma$ *may not be connected. Then* Γ *is* $(1, w)$*-exc.*

Corollary 1. *Let* $\Gamma(R^n, \infty)$ *be the family of all curves in* R^n *each of which connect two points from* R^n *and having an infinite length. Then* $\Gamma(R^n, \infty)$ *is* $(1, w)$*-exc.*

Proof. Let Γ_k be the family of curves γ which connect points from $B(0, k)$, $k \in \mathbb{N}$. Obviously that $\gamma \cap \overline{B(0, k)}$ has an infinite length and by Proposition 2 $M_{1,w}(\Gamma_k) = 0$. Since $\Gamma(R^n, \infty) = \bigcup\limits_k \Gamma_k$, then by countable semi-additivity of modulus we deduce that $M_{1,w}(\Gamma(R^n, \infty)) = 0$. □

In the same way it is proved the next statement.

Corollary 2. *Let* $\Gamma_{osc}(R^n)$ *be the family of oscillating curves in* R^n*. Then* $\Gamma_{osc}(R^n)$ *is* $(1, w)$*-exc.*

Below a proposition concerning the curves of a family Γ in G is said to hold for $(1, w)$-almost every $\gamma \in \Gamma$ (or $(1, w)$-a.e. $\gamma \in \Gamma$) if the subfamily of Γ for which the proposition does not hold is $(1, w)$-exc.

Let X_i be the set of all straight lines parallel to the coordinate x_i-axis, $i = 1, \ldots, n$. Index every line $l \in X_i$ as l_a where a is the point of intersection of l and hyperplane $H_i = \{x = (x_1, \ldots, x_n) : x_i = 0\}$. We will say that some property P holds for almost every lines $l_a \in X_i$ (for almost every segments parallel to x_i-axis), if the $(n-1)$-dimensional Lebesgue measure of the set $a \in H_i$, for which the property P does not hold on line $l_a \in X_i$ (resp. on segment $\tau \subset l_a \in X_i$), is zero. In this definition instead of coordinate x_i-axis we can consider any fixed straight line $l_0 \subset R^n$.

Proposition 3 ([13, Lemma 4.2.1])**.** *For every $(1,w)$-exc. family Γ of rectifiable curves in R^n almost every straight lines parallel to a fixed straight line, does not contain the curves of* Γ.

Corollary 3. *For any $(1,w)$-exc. family Γ of segments parallel to a fixed straight line l, almost all lines parallel to line l, does not contain the segments of* Γ.

2.3. **h-equivalence classes of points in G.** In this section we assume that G is a domain in R^n. Describing $(1,w)$-exc. families of curves, we apply the next assertion of Fuglede [8, Theorem 2]:

Proposition 4. *The family Γ of curves $\gamma \subset G$ is $(1,w)$-exc. if and only if there exists a nonnegative Borel function $h \in L_{1,w}(G)$ such that $\int\limits_{\gamma} h\,ds = \infty$ for every $\gamma \in \Gamma$.*

The functions similar to h allow to give the classification of points in G, through which pass the curves $\gamma \subset G$ such that $\int\limits_{\gamma} h\,ds < \infty$. Indeed, let h be a Borel nonnegative function from $L_{1,w}(G)$. We call two points x and y in G an h-equivalent if there is a curve γ in G which passes through x and y and for which $\int\limits_{\gamma} h\,ds < \infty$. The points of G are divided into h-equivalence classes. A single point for which there is no other h-equivalent point constitutes a class.

Proposition 5 ([13, Lemma 4.2.6])**.** *Let G be a domain in R^n. For any Borel function $h \in L_{1,w}(G)$, $h \geq 0$, there is an h-equivalence class which contains m_n-almost all points of G. Moreover, every curve γ in G except for a $(1,w)$-exc. family is contained in this class.*

Further the set of points in the above h-equivalence class in Proposition 5 will be called the main h-equivalence class in G.

2.4. $(1,w)$**-primitives.** Let $u = (u_1, \dots, u_n)$ be a vector field with Borel components $u_1, \dots, u_n$ in G. For locally rectifiable curve $\gamma \subset G$ with arc-length parametrization $x = x(s) = (x_1(s), \dots, x_n(s))$, $a < s < b$, we set

$$\int\limits_{\gamma} udx = \int\limits_{\gamma} (u_1 dx_1 + \cdots u_n dx_n) = \int\limits_{\gamma} \left(\sum_{i=1}^{n} u_i \frac{dx_i}{ds} \right) ds,$$

when it makes sense. Here dx is the infinitesimal vector $(dx_1, \dots, dx_n)$ along the curve γ.

Following by Fuglede [8, p. 213], function f is called an $(1,w)$-primitive for differential form udx if for $(1,w)$-a.e. curves γ integral $\int\limits_{\widetilde{cc'}} udx$ makes sense and finite; $f(c') - f(c) = \int\limits_{\widetilde{cc'}} udx$, where c and c' are any points of γ, and $\widetilde{cc'}$ is a subarc of γ connecting c and c'. Points c and c' may be coinside. In this case $\widetilde{cc'}$ is a locally rectifiable closed curve.

The following statement is true [13, Theorem 4.3.9]:

Proposition 6. *Let u be a Borel vector field in G. Then $(1,w)$-primitive exists if and only if $\int\limits_{\gamma} u\,dx = 0$ for $(1,w)$-a.e. closed curve γ in G.*

2.5. **Absolutely continuous functions on** $(1, w)$**-a.e. curves in** G**.** Let γ be a locally rectifiable curve in R^n and its arc-length parametrization is $x = x(s)$, $a < s < b$. Real-valued function $f(x)$, $x \in \gamma$, is called an absolutely continuous on γ if $f(x(s))$ is absolutely continuous on every segment $[a', b'] \subset (a, b)$. Denote by $AC^{1,w}(G)$ the class of all functions on G which are absolutely continuous on $(1, w)$-a.e. curves $\gamma \subset G$.

Due to Corollary 3 every function $f \in AC^{1,w}(G)$ belongs to the class $ACL(G)$. In other words, f is absolutely continuous on almost all segments parallel to the coordinate axes and lying in G. We use the next well-known properties of functions $f \in AC^{1,w}(G)$ below [13, Theorems 4.3.3–4.3.5].

Proposition 7. *Let $f \in AC^{1,w}(G)$. The following statements are true:*

(1) *f is finite-valued a.e. on G;*
(2) *Let Γ be the family of curves in G with end points of G. Then $f(x) \to f(x_\gamma)$, a finite value, where x tends to an end point x_γ of γ, along $(1, w)$-a.e. $\gamma \in \Gamma$.*
(3) *there exists a Borel vector field in G which is equal $\left(\dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n}\right)$ in G except a Borel set of zero m_n-measure.*

Denote by ∇f, $f \in AC^{1,w}(G)$, the vector field from Proposition 7.

Function $f \in AC^{1,w}(G)$ is called a $(1, w)$-precise in G if $|\nabla f| \in L_{1,w}(G)$.

2.6. **Sobolev space** $L^1_{1,w}(G)$**.** Denote by $L^1_{1,w}(G)$ the space of functions $f : G \to (-\infty, +\infty)$ which are locally integrable in G and having in G the distributional first-order partial derivatives that satisfy the condition

$$\int\limits_G |\nabla f| w \, dx < \infty.$$

In $L^1_{1,w}(G)$ we introduce the semi-norm

$$\|f\|_{L^1_{1,w}(G)} = \int\limits_G |\nabla f| w \, dx,$$

in which the functions that differ by a constant m_n-a.e. on each component of connectivity of set $G = \bigcup\limits_i G_i$ (constants may not coincide on different components G_i) are identified.

Below we use known statement about smooth approximation of functions $f \in L^1_{1,w}(G)$ [15, Theorem 5].

Proposition 8. *Let $f \in L^1_{1,w}(G)$ and $\{\Omega_i\}$ be a sequence of open bounded sets $\Omega_i \subset G$ such that $\overline{\Omega_i} \subset \Omega_{i+1}$ and $\bigcup\limits_i \Omega_i = G$. Then there exists a sequence of bounded functions $\psi_i \in L^1_{1,w}(G) \cap C^\infty(G)$, $i \in \mathbb{N}$, such that*

$$\int\limits_{\Omega_i} |f - \psi_i| w \, dx < \frac{1}{i}, \quad \lim_{i \to \infty} \|f - \psi_i\|_{L^1_{1,w}(G)} = 0.$$

From Proposition 8 by applying Lebesgue-Vitali theorem on the equivalence of convergence in measure and convergence in $L_{1,w}$ [4, Theorem 3.12.6], and Riesz theorem on the connection between convergence in measure and convergence almost everywhere, we will get another statement.

Corollary 4. *Let f, ψ_i, $i \in \mathbb{N}$ be from Proposition 8 then there exists a subsequence $\{i_k\}$ such that $f = \lim\limits_{k\to\infty} \psi_{i_k}$ a.e. on G.*

2.7. **Capacities of generalized condensers.** Let $E_0, \dots, E_m$, $m \in \mathbb{N}$, be a mutually disjoint non-empty compacts (in topology of $\overline{R^n}$) in $\overline{G}$; $\delta_0, \dots, \delta_m$ be different real numbers associating to these compacts. Then following by V.N. Dubinin [5] the triple $\mathcal{K} = (\mathcal{E}, \Delta, G)$, where $\mathcal{E} = \{E_i\}_{i=0}^m$, $\Delta = \{\delta_i\}_{i=0}^m$, is called a a generalized condenser (hereafter condenser) in $\overline{R^n}$. The sets E_i and $Q = G \backslash E$, where $E = \bigcup\limits_i E_i$, are called the plate and field of condenser $\mathcal{K}$, respectively. The number δ_i is called a potential of plate E_i. If $E \subset G$ or $E_0 = \partial G$, hence, $E_1, \dots, E_m \subset G$, then $\mathcal{K}$ is called a Hesse condenser [9] in $\overline{R^n}$.

Let $\mathcal{A} = \mathcal{A}(x) = (a_{ij}(x))_{i,j=1}^n$ be a symmetric matrix with measurable components $a_{ij}(x)$ in G satisfying for all $\xi = (\xi_1, \dots, \xi_n) \in R^n$ and $x \in G$ the condition

$$c_0^{-2} w(x)^2 |\xi|^2 \le \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \le c_0^2 w(x)^2 |\xi|^2, \tag{1}$$

where a constant $c_0 \ge 1$. Note that $w(x)$ has a positive values in G.

Due to (1) the matrix $\mathcal{A}$ is positively defined on G and hence for each $x \in G$ there is an orthonormal basis in R^n consists of eigenvectors of matrix $\mathcal{A}(x)$. Let $\mathcal{B}(x) = \mathcal{A}^{-1}(x) = (b_{ij}(x))_{i,j=1}^n$. Then [1, ch. XIII] we conclude that $\mathcal{B}$ is a positively defined symmetric matrix on G for which

$$c_0^{-2} w(x)^{-2} |\xi|^2 \le \sum_{i,j=1}^n b_{ij}(x)\xi_i\xi_j \le c_0^2 w(x)^{-2} |\xi|^2,$$

Besides, it is easily to see that matrices $\mathcal{A}$ and $\mathcal{B}$ may be written as $\mathcal{A} = (\sqrt{\mathcal{A}})^2$, $\mathcal{B} = (\sqrt{\mathcal{B}})^2$ with a positively defined symmetric matrices $\sqrt{\mathcal{A}}$ and $\sqrt{\mathcal{B}}$.

Let $\mathcal{A}[\xi] = \left(\sum\limits_{i,j=1}^n a_{ij}(x)\xi_i\xi_j\right)^{1/2}$ for $x \in G$ and $\xi \in R^n$; and if $\xi = \xi(x)$ is a vector-valued measurable function on $Q = G \setminus E$, then let $\mathcal{A}_1(\xi) = \int\limits_Q \mathcal{A}[\xi]\, dx$. Similarly we define $\mathcal{B}[\xi]$, $\mathcal{B}_1(\xi)$.

Then

$$\mathcal{A}[\xi] = |\sqrt{\mathcal{A}}\xi|, \quad \mathcal{B}[\xi] = |\sqrt{\mathcal{B}}\xi|, \quad \mathcal{A}_1(\xi) = \int\limits_Q |\sqrt{\mathcal{A}}\xi|\, dx, \quad \mathcal{B}_1(\xi) = \int\limits_Q |\sqrt{\mathcal{B}}\xi|\, dx$$

and for $x \in Q$

$$\begin{aligned} c_0^{-1} w(x)|\xi| \le & |\sqrt{\mathcal{A}}\xi| \le c_0 w(x)|\xi|; \\ c_0^{-1} w(x)^{-1}|\xi| \le & |\sqrt{\mathcal{B}}\xi| \le c_0 w(x)^{-1}|\xi|. \end{aligned} \tag{2}$$

Let γ be a curve with arc-length parametrization $x = x_\gamma(s)$, $a < s < b$; F is a subset of $\bar{Q}$. We will say that $x_\gamma(s) \to F$ along the curve γ if $\lim\limits_{s\to a} q(x_\gamma(s), F) = 0$ or $\lim\limits_{s\to b} q(x_\gamma(s), F) = 0$. The family of all such curves $\gamma \subset Q$ denote by $\Gamma(F, Q)$. In case $F = \emptyset$ we set $\Gamma(F, Q) = \emptyset$, $M_{1,w}(\Gamma(F, Q)) = 0$.

Suppose first that Q is a domain in R^n. Then $\mathcal{D} = \mathcal{D}(\mathcal{K})$ define as the class of all functions u from $AC^{1,w}(Q)$ for which $u(x) \to \delta_i$ as $x \to E_i$ along $(1, w)$-a.e.

curves $\gamma \in \Gamma(E_i, Q)$, if $M_{1,w}(\Gamma(E_i, Q)) > 0$, $i = 0, 1, \ldots, m$. If $M_{1,w}(\Gamma(E_i, Q)) = 0$ for all $i = 0, 1, \ldots, m$ then by definition $\mathcal{D}(\mathcal{K}) = AC^{1,w}(Q)$ and, in particular, $u \equiv 0 \in \mathcal{D}(\mathcal{K})$.

Remark 3. *In definition $\mathcal{D}(\mathcal{K})$ due to Corollary 2 $\Gamma(E_i, Q)$ may be considered as the family of all curves connecting points from E_i and $\bar{Q}$, $0 \le i \le m$.*

Let now Q is an open set in R^n and $Q = \bigcup\limits_k Q_k$, where $Q_1, Q_2, \ldots$ are mutually disjoint components of connectivity of Q. Then define $\mathcal{D} = \mathcal{D}(\mathcal{K})$ as the class of all functions $u \in AC^{1,w}(Q)$ such that $u|_{Q_k} \in \mathcal{D}(\mathcal{K}_k)$. Here $\mathcal{K}_k = (\{E_i \cap \overline{Q_k}\}', \{\delta_i\}', Q_k)$, where $\{E_i \cap \overline{Q_k}\}'$ is a collection of sets $E_i \cap \overline{Q_k} \ne \emptyset$, $\{\delta_i\}'$ is a corresponding collection of numbers from $\{\delta_i\}_{i=0}^m$. In case $E_i \cap \overline{Q_k} = \emptyset$ for all $i = 0, 1, \ldots, m$ then we set $\mathcal{D}(\mathcal{K}_k) = AC^{1,w}(Q_k)$ and, in particular, $u|_{Q_k} \equiv 0 \in \mathcal{D}(\mathcal{K}_k)$.

Let $\mathcal{D}^* = \mathcal{D}^*(\mathcal{K})$ be another class of functions which are equal δ_i in some neighborhood E_i, $i = 0, 1, \ldots, m$, and $u \in AC^{1,w}(Q)$. Then define the capacities of condenser $\mathcal{K}$:

$$C_{\mathcal{A},1}(\mathcal{K}) = \inf\{\mathcal{A}_1(\nabla u) : u \in \mathcal{D}\},$$
$$C^*_{\mathcal{A},1}(\mathcal{K}) = \inf\{\mathcal{A}_1(\nabla u) : u \in \mathcal{D}^*\}.$$

2.8. **Moduli of configurations associated with condenser $\mathcal{K}$.** Let

$$\nu = (\nu_1, \ldots, \nu_n)$$

be a vector measure on open set Q in R^n whose components ν_i are signed Borel measures. Total variation of ν is defined as the measure

$$|\nu|(F) = \sup \sum_j \left(\sum_{i=1}^n \nu_i(F_j)^2 \right)^{1/2}$$

for Borel sets $F \subset Q$, where the supremum is taken over all finite partitions $\{F_j\}$ of set F into Borel sets. Let $\xi = (\xi_1, \ldots, \xi_n)$ be a vector function with Borel components ξ_i on Q. If $\int\limits_Q |\xi|\, d|\nu| < \infty$ then we define

$$\int\limits_Q \xi\, d\nu = \sum_{i=1}^n \int\limits_Q \xi_i\, d\nu_i.$$

It is known that

$$\left| \int\limits_Q \xi\, d\nu \right| \le \int\limits_Q |\xi|\, d|\nu|.$$

If necessary, consider that the measure ν is defined on R^n by $\nu(F) = \nu(F \cap Q)$ for all Borel sets $F \subset R^n$. Also consider that vector function $\xi \equiv 0$ on $R^n \setminus Q$.

For condenser $\mathcal{K} = (\{E_i\}, \{\delta_i\}, G)$ we define the collection of curves family $H = \{H_{01}, \ldots, H_{m-1,m}\}$ and collection of numbers $\alpha = \{\alpha_{01}, \ldots, \alpha_{m-1,m}\}$, where H_{ij} is a family of locally rectifiable curves in $Q = G \setminus E$ connecting E_i and E_j, and $\alpha_{ij} = |\delta_i - \delta_j|$, $0 \le i < j \le m$. We consider that the curve $\gamma \in H_{ij}$ is oriented from E_i to E_j in case $\delta_i < \delta_j$, and in reverse direction if $\delta_i > \delta_j$, $0 \le i < j \le m$. Then the collection $\alpha H = \{\alpha_{01} H_{01}, \ldots, \alpha_{m-1,m} H_{m-1,m}\}$ is called the main configuration for condenser $\mathcal{K}$. The numbers α_{ij}, $0 \le i < j \le m$, are used below in the definition of configuration module.

We associate to curve $\gamma \subset Q$ with arc-length parametrization $x = x_\gamma(s) = (x_{\gamma 1}(s), \dots, x_{\gamma n}(s))$, $a < s < b$, the vector measure $dx = dx_\gamma = (dx_{\gamma 1}(s), \dots, dx_{\gamma n}(s))$.

Let now $dH_{ij} = dH_{ij}(\mathcal{K}) = \{dx_\gamma : \gamma \in H_{ij}\}$, $0 \le i < j \le m$. Then the collection $\alpha dH(\mathcal{K}) = \alpha dH = \{\alpha_{01} dH_{01}, \dots, \alpha_{m-1,m} dH_{m-1,m}\}$ is called the first configuration of vector measures for $\mathcal{K}$. We define $(\mathcal{A}, 1)$-modulus of αdH as $M_{\mathcal{A},1}(\alpha dH) = \inf \mathcal{A}_1(\xi)$ where the infimum is taken over all Borel vector functions ξ on Q such that

$$\int\limits_\gamma \xi \, dx \ge \alpha_{ij} \text{ for } (1, w)\text{-a.e. } \gamma \in H_{ij}$$

$$\text{and all } H_{ij} \text{ such that } M_{1,w}(H_{ij}) > 0, 0 \le i < j \le m.$$

for this ξ we write $\xi \wedge \alpha dH$ $(1, w)$-a.e. and set $\xi \equiv (0, \dots, 0)$ on $G \backslash Q$. If $M_{1,w}(H_{ij}) = 0$ for all $0 \le i < j \le m$ then by definition any Borel vector function ξ on Q may be considered as function $\xi \wedge \alpha dH$, and, in particular, $0 \wedge \alpha dH$.

As a second configuration, we consider the collection

$$\alpha |\sqrt{\mathcal{B}} dH(\mathcal{K})| = \alpha |\sqrt{\mathcal{B}} dH| = (\alpha_{01} |\sqrt{\mathcal{B}} dH_{01}|, \dots, \alpha_{m-1,m} |\sqrt{\mathcal{B}} dH_{m-1,m}|)$$

where $|\sqrt{\mathcal{B}} dH_{ij}| = \left\{ |\sqrt{\mathcal{B}} dx| = |\sqrt{\mathcal{B}} dx_\gamma| = \sqrt{\sum\limits_{s,l=1}^{n} b_{sl} dx_{\gamma s} dx_{\gamma l}} : \gamma \in H_{ij} \right\}$,
$0 \le i < j \le m$.

A Borel function $\rho : Q \to [0, +\infty]$ is called admissible for $\alpha |\sqrt{\mathcal{B}} \, dH|$ if $\int\limits_\gamma \rho |\sqrt{\mathcal{B}} \, dx| \ge \alpha_{ij}$ for all $\gamma \in H_{ij}$ and such H_{ij} that $H_{ij} \ne \emptyset$, $0 \le i < j \le m$.

For these functions ρ we write $\rho \wedge \alpha |\sqrt{\mathcal{B}} \, dH|$. If $H_{ij} = \emptyset$ for all $0 \le i < j \le m$ then by definition any Borel function (in particular, $\rho \equiv 0$) is admissible for $\alpha |\sqrt{\mathcal{B}} \, dH|$. The module of configuration $\alpha |\sqrt{\mathcal{B}} \, dH|$ is defined as

$$M_1(\alpha |\sqrt{\mathcal{B}} \, dH|) = \inf \left\{ \int\limits_Q \rho \, dx : \rho \wedge \alpha |\sqrt{\mathcal{B}} \, dH| \right\}.$$

Similarly we define

$$M_{1,w}(\alpha |w \sqrt{\mathcal{B}} \, dH|) = \inf \left\{ \int\limits_Q \rho w \, dx : \rho \wedge \alpha |w \sqrt{\mathcal{B}} \, dH| \right\}.$$

3. On the equivalence of $L^1_{1,w}(G)$ and the class of $(1, w)$-precise functions

Since each $(1, w)$-precise function on G belongs to $L^1_{1,w}(G)$, then the equivalence of $L^1_{1,w}(G)$ and the class of $(1, w)$-precise functions in G will follow from the next theorem.

Theorem 1. *For any function $f \in L^1_{1,w}(G)$ there exists $(1, w)$-precise function f_0 in G such that $f_0 = f$, $\nabla f_0 = \nabla f$ a.e. on G.*

Proof. In view of Proposition 8 and Corollary 4 for fixed function $f \in L^1_{1,w}(G)$ there is a sequence of bounded functions $f_k \in L^1_{1,w}(G) \cap C^\infty(G)$ such that $f_k \to f$ a.e. on G and $\|\nabla f_k - \nabla f\|_{L_{1,w}(G)} \to 0$ as $k \to \infty$.

For ∇f there exists a Borel field $u = (u_1, \dots, u_n)$ which is equal to ∇f in G except a Borel set of zero m_n-measure. Obviously that $\|\nabla f_k - u\|_{L_{1,w}(G)} \to 0$ as $k \to \infty$,

$$\int_G |u| w\, dx = \int_G |\nabla f| w\, dx = \|f\|_{L^1_{1,w}(G)} < \infty.$$

Extracting, if necessary, the subsequence from ∇f_k, due to Fuglede theorem [8, Theorem 3] on properties of exceptional systems of measures, we will consider that the next condition fulfilled:

$$\int_\gamma |\nabla f_k - u|\, ds \to 0 \tag{3}$$

as $k \to \infty$ for $(1, w)$-a.e. $\gamma \subset G$.

Let's introduce the next families of curves $\gamma \subset G$. Γ_1 is the family of all curves $\gamma \subset G$ for which the condition (3) does not hold; Γ_2 is the family of all curves $\gamma \subset G$ for which $\int_\gamma |u|\, ds = +\infty$; Γ_3 is a family of all oscillating curves in G; Γ_4 is a family of non-rectifiable curves that connect points in G. By virtue of Fuglede theorem mentioned above and Corollaries 1, 2 the family $\Gamma_0 = \bigcup_{i=1}^{4} \Gamma_i$ is a $(1, w)$-exc. It follows from (3) that

$$\int_\gamma \nabla f_k\, dx \to \int_\gamma u\, dx, \quad \sum_{i=1}^{n} \int_\gamma \left| \frac{\partial f_k}{\partial x_i} - u_i \right| ds \to 0, k \to \infty, \tag{4}$$

$$\int_\gamma |u|\, ds < \infty \tag{5}$$

for all $\gamma \subset G$, $\gamma \notin \Gamma_0$. Let γ be a closed curve in G. Then by definition γ connects its end points in G and if $\gamma \notin \Gamma_0$ then γ is a rectifiable curve. For these closed curves due to $f_k \in C^\infty(G)$ the equality

$$\int_\gamma \nabla f_k\, dx = 0, \quad k \in \mathbb{N},$$

is valid. Then by (3) it is true that $\int_\gamma u\, dx = 0$ for $(1, w)$-a.e. closed curves $\gamma \subset G$. Thus according to Proposition 6 there exists a primitive function g for differential form $u\, dx$ on G. By definition on $(1, w)$-a.e. curves $\gamma \subset G$, $\gamma \notin \Gamma_0$,

$$g(c') - g(c) = \int_{\widetilde{cc'}} u\, dx$$

for all points $c, c' \in \gamma$, where $\widetilde{cc'}$ is a closed subarc of γ. By the choice of γ and (5) we have

$$|g(c') - g(c)| \le \left| \int_{\widetilde{cc'}} u\, dx \right| \le \int_{\widetilde{cc'}} |u|\, ds \le \int_\gamma |u|\, ds < \infty,$$

which implies the absolutely continuous of function g on γ. Besides,

$$g(c') - g(c) = \int\limits_{\widetilde{cc'}} \frac{dg}{ds}\, ds = \int\limits_{\widetilde{cc'}} u\, dx, \tag{6}$$

$$\left|\frac{dg}{ds}\right| \leq |u| \tag{7}$$

a.e. (in the sense of linear measure m_γ) on curve γ.

Finally considering the segment $[a, b] \subset G$ parallel to x_i-axis, $i = 1, \dots, n$, as a curve $\gamma \notin \Gamma_0$, we get due to Corollary 3 that g is absolutely continuous on a.e. such segments and $\int\limits_{[a,b]} u_i\, dx_i = g(b) - g(a)$. It implies the equality $\dfrac{\partial g}{\partial x_i} = u_i$ a.e. on G.

Define ∇g as a Borel vector field which coincides with $\left(\dfrac{\partial g}{\partial x_1}, \dots, \dfrac{\partial g}{\partial x_n}\right) = (u_1, \dots, u_n)$ on G except Borel set F of zero measure. Let $\Gamma_0^* = \Gamma_0 \cup \{\gamma \subset G : m_\gamma(\gamma \cap F) > 0\}$. Obviously that Γ_0^* is a $(1, w)$-exc. Then (6) for $\gamma \notin \Gamma_0^*$ take the following form:

$$g(c') - g(c) = \int\limits_{\widetilde{cc'}} \frac{dg}{ds}\, ds = \int\limits_{\widetilde{cc'}} \nabla g\, dx. \tag{8}$$

Therefore, g is a $(1, w)$-precise function and $\|\nabla f - \nabla g\|_{L_{1,w}(G)} = 0$. This equality remains true if G is replaced by its component of connectivity G_i from representation $G = \bigcup_i G_i$ with disjoint components. Due to Proposition 1 from equality $\|\nabla f - \nabla g\|_{L_{1,w}(G_i)} = 0$ it follows that $\|\nabla f - \nabla g\|_{L_1(D_i)} = 0$ for any bounded domain $D_i \subset G_i$. It implies [11, Sect. 1.1.5] the equality $f = g + c_i$ a.e. on G_i with some constant c_i. Define $(1, w)$-precise function f_0 on G as $g + c_i$ if $x \in G_i$, $i \in \mathbb{N}$. Then $f = f_0$, $\nabla f = \nabla f_0$ a.e. on G that completes the proof of theorem. □

Since $(1, w)$-precise function f on G belongs to class $L^1_{1,w}(G)$, then from (4)–(5), (7)–(8) which hold for $(1, w)$-a.e. curves $\gamma \subset G$, it follows another statement.

Corollary 5. *Every $(1, w)$-precise function f on G has a finite limit along $(1, w)$-a.e. curves in G and $f(t(\gamma)) - f(s(\gamma)) = \int\limits_\gamma \nabla f\, dx$, where ∇f is a Borel vector field which coincides with $\left(\dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n}\right)$ a.e on G; $s(\gamma)$ and $t(\gamma)$ denote the starting and ending point of γ, respectively.*

Below, unless otherwise specified, for $f \in L^1_{1,w}(G)$ or for $(1, w)$-precise function f we will understand ∇f as a vector field from Corollary 5.

Corollary 6. *Let f_0 be a $(1, w)$-precise function on G and ξ be a direction (unit vector) in R^n. Then*

$$\frac{\partial f_0}{\partial \xi} = \nabla f_0 \cdot \xi \ \textit{a.e. on } G.$$

Proof. Let in Theorem 1 $f = f_0$. Replace in the proof of Theorem 1 ∇f_k, ∇f_0 by $\nabla f_k \cdot \xi$, $\nabla f_0 \cdot \xi$, respectively. Then due to Corollary 3, on a.e. segments $[a, b] \subset G$

parallel to ξ, we have

$$f_0(x) = \int\limits_{[a,x]} \nabla f_0 \cdot \xi \, ds + const \text{ for all } x \in [a,b],$$

that implies $\dfrac{\partial f_0}{\partial \xi} = \nabla f_0 \cdot \xi$ a.e. on G. □

Remark 4. *Below in the proofs of main statements in the above statements we will consider the field $Q = G \setminus E$ of condenser $\mathcal{K}$ instead of open set G.*

4. The equality of capacity $C_{\mathcal{A},1}(\mathcal{K})$ and moduli $M_{\mathcal{A},1}(\alpha dH)$, $M_1(\alpha|\sqrt{\mathcal{B}} dH|)$

Let $\mathcal{K} = (\{E_i\}, \{\delta_i\}, G)$ be a condenser from Section 2.6. As in the case $w \in A_p$, $p > 1$ [7, Lemma 1], it is easy to construct the function $v_0 \in C^\infty(R^n) \cap \mathcal{D}^* \cap L^1_{1,w}(R^n)$, which is equal to some constant c in some neighborhood of point $\infty \in \overline{R^n}$, where $c = 0$ if $\infty \notin E$, and $c = \delta_i$ if $\infty \in E_i$, $i = 0, \dots, m$. Hence due to $\mathcal{D}^* \subset \mathcal{D}$, (2)

$$0 \le C_{\mathcal{A},1}(\mathcal{K}) \le C^*_{\mathcal{A},1}(\mathcal{K}) \le \mathcal{A}_1(\nabla v_0) \le c_0 \int\limits_Q |\nabla v_0| w \, dx < +\infty. \tag{9}$$

Therefore $C_{\mathcal{A},1}(\mathcal{K})$, $C^*_{\mathcal{A},1}(\mathcal{K})$ are bounded quantities and we may consider that functions from $\mathcal{D}$, $\mathcal{D}^*$ are $(1,w)$-precise on Q. The boundedness of $M_1(\alpha|\sqrt{\mathcal{B}}\, dH|)$ and $M_{\mathcal{A},1}(\alpha \, dH)$ follows from the next statement.

Lemma 1.

$$0 \le M_1(\alpha|\sqrt{\mathcal{B}}\, dH|) \le M_{\mathcal{A},1}(\alpha \, dH) \le C_{\mathcal{A},1}(\mathcal{K}). \tag{10}$$

Proof. As noted above as a corollary from (9), we assume that functions from $\mathcal{D}$ are $(1,w)$-precise on Q. Then for $u \in \mathcal{D}$ due to Corollary 5

$$\int\limits_\gamma \nabla u \, dx = u(t(\gamma)) - u(s(\gamma)) = \alpha_{ij} = |\delta_i - \delta_j|$$

for $(1,w)$-a.e $\gamma \in H_{ij}$ and all H_{ij}, where $M_{1,w}(H_{ij}) > 0$, $0 \le i < j \le m$. Then $\nabla u \wedge \alpha \, dH$ $(1,w)$-a.e, and hence $\mathcal{A}_1(\nabla u) \ge M_{\mathcal{A},1}(\alpha \, dH)$. Taking the infimum over all $u \in \mathcal{D}$, we set the right inequality in (10) and boundedness of $M_{\mathcal{A},1}(\alpha \, dH)$. Next we establish the remaining inequalities of (10). For this we introduce $I_1 = \{(i,j) : 0 \le i < j \le m, H_{ij} = \emptyset\}$, $I_2 = \{(i,j) : 0 \le i < j \le m, H_{ij} \ne \emptyset, M_{1,w}(H_{ij}) = 0\}$, $I_3 = \{(i,j) : 0 \le i < j \le m, H_{ij} \ne \emptyset, M_{1,w}(H_{ij}) > 0\}$. Then there exists a sequence $\xi_k \wedge \alpha \, dH$ $(1,w)$-a.e., $|\xi_k| \in L_{1,w}(Q)$, for which $M_{\mathcal{A},1}(\alpha \, dH) \le \mathcal{A}_1(\xi_k) + \dfrac{1}{k}$, $k \in \mathbb{N}$.

By definition, for Borel vector function ξ_k there exists a $(1,w)$-exc. family $\Gamma_k \subset \bigcup\limits_{(i,j)\in I_3} H_{ij}$ such that

$$\alpha_{ij} \le \int\limits_\gamma \xi_k \, dx = \int\limits_\gamma \sqrt{\mathcal{A}}\xi \cdot \sqrt{\mathcal{B}} \, dx \le \int\limits_\gamma |\sqrt{\mathcal{A}}\xi| \cdot |\sqrt{\mathcal{B}} \, dx|$$

for all $\gamma \in H_{ij} \setminus \Gamma_k$ and all $(i,j) \in I_3$.

Associate to matrix $\mathcal{A}(x)$ the equivalent matrix $\mathcal{A}_0(x) = \{a^0_{ij}(x)\}^n_{i,j}$ with Borel components $a^0_{ij}(x)$ on G, where $\mathcal{A}_0(x)$ is $\mathcal{A}(x)$ on G except a Borel set F_0 of zero m_n-measure, and $\mathcal{A}_0(x)$ is the identity matrix J for all $x \in F_0$. If $B_0(x) = \mathcal{A}_0^{-1}(x)$ on G then

$$\sqrt{\mathcal{A}_0(x)} = \begin{cases} \sqrt{\mathcal{A}(x)}, & x \in G \setminus F_0, \\ J, & x \in F_0, \end{cases} \quad \sqrt{\mathcal{B}_0(x)} = \begin{cases} \sqrt{\mathcal{B}(x)}, & x \in G \setminus F_0, \\ J, & x \in F_0. \end{cases}$$

Let $\Gamma_0 = \left\{ \gamma \in \bigcup\limits_{(i,j)\in I_2 \cup I_3} H_{ij} : m_\gamma(\gamma \cap F_0) > 0 \right\}$ and $\Gamma_{0k} = \Gamma_k \cup \Gamma_0 \cup \left(\bigcup\limits_{(i,j)\in I_2} H_{ij} \right)$. By construction and due to (2) Γ_{0k} is $(1,w)$-exc., $\int\limits_{\gamma\cap F_0} ds = 0$ and, therefore, $\int\limits_{\gamma\cap F_0} |\sqrt{\mathcal{B}}\, dx| \le c_0 \int\limits_{\gamma \cap F_0} w^{-1}\, ds = 0$ for all $\gamma \in \bigcup\limits_{(i,j)\in I_2\cup I_3} (H_{ij} \setminus \Gamma_{0k})$, $k \in \mathbb{N}$.

Since $M_{1,w}(\Gamma_{0k}) = 0$, then there exists $\rho_k \wedge \Gamma_{0k}$ such that

$$0 \le \int\limits_Q c_0 \rho_k w |\alpha| dx \le \frac{1}{k}$$

for all $k \in \mathbb{N}$, where $|\alpha| = \sqrt{\sum\limits_{0\le i<j\le m} \alpha^2_{ij}}$.

In the last integral we replace w by an equivalent Borel function $\tilde{w} : G \to [0, +\infty]$ which is equal w on G except a Borel set $F_1 \subset G$ with zero m_n-measure, where $\tilde{w} = +\infty$ on F_1. Then due to (2) $\alpha_{ij} \le \int\limits_\gamma c_0 |\alpha| \rho_k \tilde{w} |\sqrt{\mathcal{B}}\, dx|$ for all $\gamma \in \Gamma_{0k} \cap H_{ij}$ and all $(i,j) \in I_2 \cup I_3$. Then $(|\sqrt{\mathcal{A}}\xi_k| + c_0|\alpha|\rho_k\tilde{w}) \wedge \alpha|\sqrt{\mathcal{B}}\, dH|$, hence

$$0 \le M_1(|\alpha\sqrt{\mathcal{B}}\, dH|) \le \int\limits_Q |\sqrt{\mathcal{A}}\xi_k|\, dx + \int\limits_Q c_0|\alpha|\rho_k w\, dx \le M_{\mathcal{A},1}(\alpha dH) + \frac{2}{k}.$$

Letting $k \to \infty$, we establish the remaining inequalities from (10). □

Lemma 2.

$$C_{\mathcal{A},1}(\mathcal{K}) \le M_1(\alpha|\sqrt{\mathcal{B}}\, dH|).$$

Proof. Here we use the notations $\tilde{w}$, F_1, I_1, I_2, I_3 from the proof of Lemma 1. First we will establish that

$$M_{1,w}(\alpha w|\sqrt{\mathcal{B}}\, dH|) = M_1(\alpha|\sqrt{\mathcal{B}}\, dH|). \tag{11}$$

Set $w_1 = \dfrac{1}{\tilde{w}}$ on $G \setminus F_1$ and $w_1 = +\infty$ on F_1.

Given $\varepsilon > 0$ take $\rho_\varepsilon \wedge \alpha|\sqrt{\mathcal{B}}\, dH|$ such that

$$M_1(\alpha|\sqrt{\mathcal{B}}\, dH|) \le \int\limits_Q \rho_\varepsilon\, dx \le M_1(\alpha|\sqrt{\mathcal{B}}\, dH|) + \varepsilon.$$

By choice of ρ_ε and $w \cdot w_1 = +\infty$ on F_1 we have

$$\alpha_{ij} \le \int\limits_\gamma \rho_\varepsilon |\sqrt{\mathcal{B}}\, dx| \le \int\limits_\gamma \rho_\varepsilon w w_1 |\sqrt{\mathcal{B}}\, dx|$$

for all $\gamma \in H_{ij}$, $(i,j) \in I_2 \cup I_3$. Hence $\rho_\varepsilon w_1 \wedge |w\sqrt{\mathcal{B}}\, dH|$, therefore

$$M_{1,w}(\alpha|w\sqrt{\mathcal{B}}\, dH|) \le \int\limits_Q \rho_\varepsilon w_1 w\, dx = \int\limits_Q \rho_\varepsilon\, dx < M_1(\alpha|\sqrt{\mathcal{B}}\, dH|) + \varepsilon.$$

Letting $\varepsilon \to 0$, we get

$$M_{1,w}(\alpha|w\sqrt{\mathcal{B}}\, dH|) \le M_1(\alpha|\sqrt{\mathcal{B}}\, dH|).$$

Swapping the configurations $\alpha|\sqrt{\mathcal{B}}\, dH|$ and $\alpha|w\sqrt{\mathcal{B}}\, dH|$ in the above we establish the reverse inequality, that implies (11).

Let now $Q = \bigcup\limits_{k\in\mathbb{N}} Q_k$ where Q_1, Q_2,... are the components of connectivity of Q from Sect. 2.7.

Let $E_i \cap \overline{Q_k} = E_{ik}$, $H_{ij}(Q_k) = \{\gamma \in H_{ij} : \gamma \in Q_k\}$, $i = 0, 1, \dots, m$, $k \in \mathbb{N}$.

Given $\varepsilon > 0$, we choice $\rho \wedge \alpha|w\sqrt{\mathcal{B}}\, dH|$ such that

$$M_{1,w}(\alpha|w\sqrt{\mathcal{B}}\, dH|) \le \int\limits_Q \rho w\, dx < M_{1,w}(\alpha|w\sqrt{\mathcal{B}}\, dH|) + \varepsilon < \infty.$$

Below we construct the function $u_k \in \mathcal{D}(\mathcal{K}_k)$ defining it by restrictions $u|_{Q_k} = u_k \in \mathcal{D}(\mathcal{K}_k)$, where

$$\int\limits_{Q_k} \mathcal{A}[\nabla u_k]\, dx \le \int\limits_{Q_k} \rho w\, dx, \quad k \in \mathbb{N}.$$

Start this construction from component Q_1, for which the next three assumptions are possible:

(a_1) $M_{1,w}(\Gamma(E_{i1}, Q_1)) = 0$ for all $0 \le i \le m$;

(a_2) $M_{1,w}(\Gamma(E_{i1}, Q_1)) > 0$ for only one value of i. Without loss of generality we may assume that $i = 0$.

(a_3) At least for two values of i holds $M_{1,w}(\Gamma(E_{i1}, Q_1)) > 0$. For another i holds $M_{1,w}(\Gamma(E_{i1}, Q_1)) = 0$. Without loss of generality we may assume that $M_{1,w}(\Gamma(E_{i1}, Q_1)) > 0$ for only $i = 0, 1, 2$.

If (a_1) holds, set $u_1 = 0$ on Q_1. Obviously that $u_1 \in \mathcal{D}(\mathcal{K}_1)$ (see Sect. 2.7) and

$$\int\limits_{Q_1} \mathcal{A}[\nabla u_1]\, dx = 0 \le \int\limits_{Q_1} \rho w\, dx. \tag{12}$$

Under realization of (a_2), set $u_1 = \delta_0$ on Q_1, that implies $u_1 \in \mathcal{D}(\mathcal{K}_1)$ and

$$\int\limits_{Q_1} \mathcal{A}[\nabla u_1]\, dx \le \int\limits_{Q_1} \rho w\, dx. \tag{13}$$

Let now holds (a_3). Due to (2) and by choice of $\rho \in L_{1,w}(Q)$ we have

$$\int\limits_\gamma \rho|w\sqrt{\mathcal{B}}\, dx| = \int\limits_\gamma \rho w|\sqrt{\mathcal{B}}\, dx| \le c_0 \int\limits_\gamma \rho\, ds < \infty \tag{14}$$

for $(1,w)$-a.e. curves $\gamma \subset Q_1$. Let Γ_1 be family of curves $\gamma \subset Q_1$ such that either for γ does not hold (14) or γ is oscillating curve. By Corollary 3 it follows that Γ_1 is $(1,w)$-exc. Then by Proposition 4 there is a nonnegative Borel function $h_1 \in L_{1,w}(Q_1)$ such that $\int\limits_\gamma h_1\, ds = \infty$ for all $\gamma \in \Gamma_1$.

In view of Proposition 5 there exists the main h_1-equivalence class Ω_1 such that $m_n(Q_1\backslash\Omega_1)=0$ and by construction $(1,w)$-a.e. curves $\gamma\in\Gamma(E_{i1},Q_1)$ are contained in Ω_1, and any point $x\in\Omega_1$ may be connect by curve $\gamma_x\subset\Omega_1$ with E_{i1}, $i=0,1,2$. Besides, $(1,w)$-a.e. curves $\gamma\subset Q_1$ satisfying (14), contains in Ω_1.

Let Γ_i^x be a family of all curves $\gamma\in\Omega_1$ connecting E_{i1} with the point $x\in\Omega_1$, $i=0,1,2$. Then set

$$u_{i1}(x)=\delta_i+\inf_{\gamma\in\Gamma_i^x}\int\limits_{\gamma}\rho w|\sqrt{\mathcal{B}}\,dx|.$$

In points $x\in Q_1\setminus\Omega_1$ we define $u_{i1}(x)$ arbitrarily.

Show that

(i) u_{i1} is a $(1,w)$-precise function on Q_1;

(ii) for a.e. $x\in Q_1$

$$|\sqrt{\mathcal{A}(x)}\nabla u_{i1}(x)|\le\rho(x)w(x). \tag{15}$$

(iii) $\lim u_{i1}(x)=\delta_i$ as $x\to E_{i1}$ along $(1,w)$-a.e. curves $\gamma\in\Gamma(E_{i1},Q_1)$;

(iv) $\liminf u_{i1}(x)\ge\delta_j$ as $x\to E_{j1}$ along $(1,w)$-a.e. curves $\gamma\in\Gamma(E_{j1},Q_1)$, where $j=0,1,2$ and $j\ne i$.

From definition of Ω_1 and (14), for $(1,w)$-a.e. curves $\gamma\subset\Omega_1$

$$|u_{i1}(b)-u_{i1}(a)|\le\int\limits_{\widetilde{ab}}\rho|w\sqrt{\mathcal{B}}\,dx|\le c_0\int\limits_{\widetilde{ab}}\rho\,ds \tag{16}$$

for any $a,b\in\gamma$, where $\widetilde{ab}$ is an arc of γ connecting a and b. This implies that u_{i1} is absolutely continuous on γ. by Corollary 3 almost all segments $\gamma\subset Q_1$ parallel to some coordinate axis satisfy (16) and therefore by Fubini's theorem $\left|\dfrac{\partial u_{i1}}{\partial x_l}\right|\le c_0\rho$ for $l=1,\dots,n$. Thus $|\nabla u_{i1}|\in L_{1,w}(Q_1)$ and therefore u_{i1} is a $(1,w)$-precise function on Q_1, $i=0,1,2$.

In order to establish (15), take a countable dense set $\{\xi_j\}$ on the unit sphere $\{x\in R^n:|x|=1\}$. Then by Corollary 3 it is easily to see that (16) holds along almost all segments $\gamma\subset Q_1$ parallel to one of ξ_j. Hence

$$|u_{i1}(a+\lambda\xi)-u_{i1}(a)|\le\int\limits_0^{\lambda}\rho(a+t\xi_j)\left|w(a+t\xi_j)\sqrt{\mathcal{B}(a+t\xi_j)}\right|dt$$

for a.e. $a\in Q_1$. Dividing by λ and letting $\lambda\to 0$ we have due to Corollary 6

$$|\nabla u_{i1}(a)\xi_j|\le\rho(a)|w(a)\sqrt{\mathcal{B}(a)}\xi_j| \tag{17}$$

for a.e. $a\in Q_1$.

If $\nabla u_{i1}(a)=0$ then (15) is true for $x=a$. Suppose that $\nabla u_{i1}(a)\ne 0$. Then $|\sqrt{\mathcal{A}(a)}\nabla u_{i1}(a)|>0$. Take the sequence $\{\xi_{j_k}\}$ tending to $\dfrac{\mathcal{A}(a)\nabla u_{i1}(a)}{|\mathcal{A}(a)\nabla u_{i1}(a)|}$. By (17)

$$\left|\frac{\mathcal{A}(a)\nabla u_{i1}(a)\nabla u_{i1}(a)}{|\mathcal{A}(a)\nabla u_{i1}(a)|}\right|\le\rho(a)w(a)\left|\frac{\sqrt{\mathcal{B}(a)}\mathcal{A}(a)\nabla u_{i1}(a)}{|\mathcal{A}(a)\nabla u_{i1}(a)|}\right|=\frac{\rho(a)w(a)|\sqrt{\mathcal{A}(a)}\nabla u_{i1}(a)|}{|\mathcal{A}(a)\nabla u_{i1}(a)|}.$$

This and $|\mathcal{A}(a)\nabla u_{i1}(a)\nabla u_{i1}(a)| = |\sqrt{\mathcal{A}(a)}\nabla u_{i1}(a)|^2$ implies (15) for $x = a$.

In order to proof (iii), take $\gamma \in \Gamma(E_{i1}, Q_1)$, $\gamma \subset Q_1$ satisfying (14) and write (iii) in terms of arc-length parametrization of γ: $x_\gamma(t), t_0 < t < t_1$, where $x_\gamma(t) \to E_{i1}$ as $t \to t_0$. By definition of u_{i1} and (14)

$$\delta_i \le u_{i1}(x_\gamma(t)) \le \delta_i + \int\limits_{t_0}^{t} \rho w|\sqrt{\mathcal{B}}\,dx| \le \delta_i + c_0 \int\limits_{t_0}^{t} \rho\,ds \to \delta_i$$

as $t \to t_0$. Therefore $\lim u_{i1}(x) = \delta_i$ as $x \to E_{i1}$ along γ that implies (iii) for $i = 0, 1, 2$.

Let's prove (iv) by contradiction as $x \to E_{j1}$ along $(1, w)$-a.e. curves $\gamma \in \Gamma(E_{j1}, Q_1)$ for u_{i1} in case $i = 0, j = 1$. For another pairs $(i, j), i, j = 0, 1, 2, i \ne j$, the proof is similar.

Suppose that $\delta = \liminf\limits_{t\to t_1} u(x_\gamma(t)) < \delta_1$ and $x = x_\gamma(t)$ is arc-length parametrization of γ such that $x_\gamma(t) \to E_{11}$ as $t \to t_1$.

Let $\varepsilon = \delta_1 - \delta > 0$. By definition there is t, $t_0 < t < t_1$, such that

$$|u_{01}(x_\gamma(t)) - \delta| < \frac{\varepsilon}{3}, \qquad \int\limits_{t}^{t_1} \rho\,ds < \frac{\varepsilon}{3c_0}.$$

By definition of u_{01} there is $\gamma' \in \Gamma_0^x$, $\gamma' \subset \Omega_1$, with parametrization $x = x_{\gamma'}(t)$ such that

$$\delta_0 + \int\limits_{\gamma'} \rho|w\sqrt{\mathcal{B}}\,dx| < u_{01}(x) + \frac{\varepsilon}{3}.$$

Let γ'' be a subcurve of γ associated to (t, t_1). Then $\gamma' \cup \gamma'' \in H_{01}(Q_1)$ and by (14)

$$\delta_0 + \int\limits_{\gamma'\cup\gamma''} \rho|w\sqrt{\mathcal{B}}\,dx| < u_{01}(x) + \frac{\varepsilon}{3} + c_0 \int\limits_{t}^{t_1} \rho\,ds < \delta + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \delta_1 \tag{18}$$

On the other hand by choice of ρ

$$\delta_0 + \int\limits_{\gamma'\cup\gamma''} \rho|w\sqrt{\mathcal{B}}\,dx| \ge \delta_0 + \alpha_{01} = \delta_0 + |\delta_1 - \delta_0| \ge \delta_1,$$

that contradicts (18). Then (iv) is proved.

Moreover, by Corollary 5 the condition (iv) may be enhanced as follows:

$\lim u_{i1}(x) \ge \delta_j$ as $x \to E_{j1}$ along $(1, w)$-a.e. curves $\gamma \in \Gamma(E_{j1}, Q_1), i, j = 0, 1, 2, i \ne j$.

Let $u_1 = \min\{u_{01}, u_{11}, u_{21}\}$ and from (16) we immediately get that for $(1, w)$-a.e curves $\gamma \subset Q_1$

$$|u_1(b) - u_1(a)| \le \max_i |u_{i1}(b) - u_{i1}(a)| \le \int\limits_{\widetilde{ab}} \rho|w\sqrt{\mathcal{B}}\,dx| \le c_0 \int\limits_{\widetilde{ab}} \rho\,ds$$

for all $a, b \in \gamma$. As in the case u_{i1}, $i = 0, 1, 2$, it implies $u_1 \in AC^{1,w}(Q_1)$; u_1 is $(1, w)$-precise function on Q_1,

$$|\sqrt{\mathcal{A}(x)}\nabla u_1(x)| \le \rho(x)w(x) \text{ a.e. on } Q_1; \tag{19}$$

$\lim u_1(x) = \delta_i$ as $x \to E_{i1}$ along $(1, w)$-a.e. curves $\Gamma(E_{i1}, Q_1)$, $i = 0, 1, 2$. Then $u_1 \in \mathcal{D}(\mathcal{K}_1)$ and by (19)

$$\int\limits_{Q_1} |\sqrt{\mathcal{A}(x)}\nabla u_1(x)|\, dx = \int\limits_{Q_1} \mathcal{A}[\nabla u_1]\, dx \le \int\limits_{Q_1} \rho(x) w(x)\, dx.$$

Thus for each asuumptions $(a_1), (a_2), (a_3)$ there exist a function $u_1 \in \mathcal{D}(\mathcal{K}_1)$ with required properties. Replacing above the component Q_1 by Q_2, Q_3,..., similarly construct the function $u_k \in \mathcal{D}(\mathcal{K}_k)$, $k \in \mathbb{N}$, where

$$\int\limits_{Q_k} |\sqrt{\mathcal{A}(x)}\nabla u_k(x)|\, dx = \int\limits_{Q_k} \mathcal{A}[\nabla u_k]\, dx \le \int\limits_{Q_k} \rho(x) w(x)\, dx.$$

Obviously that $u : Q \to (-\infty, +\infty)$, where $u|_{Q_k} = u_k$, $k \in \mathbb{N}$, belongs to $\mathcal{D}(\mathcal{K})$ and satisfies the inequalities

$$C_{\mathcal{A},1}(\mathcal{K}) \le \int\limits_{Q} \mathcal{A}[\nabla u]\, dx \le \int\limits_{Q} \rho w\, dx \le M_{1,w}(\alpha|w\sqrt{\mathcal{B}}\, dH|) + \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain the statement of lemma. □

Connecting Lemmas 1, 2 and (11), we obtain the following theorem

Theorem 2. $C_{\mathcal{A},1}(\mathcal{K}) = M_1(\alpha|\sqrt{\mathcal{B}}\, dH|) = M_{\mathcal{A},1}(\alpha\, dH) = M_{1,w}(\alpha|w\sqrt{\mathcal{B}}\, dH|)$.

5. Equivalence $C^*_{\mathcal{A},1}(\mathcal{K})$ and $C_{\mathcal{A},1}(\mathcal{K})$ for Hesse condensers

First we will refine the requirements for $\rho \wedge \alpha|w\sqrt{\mathcal{B}}\, dH|$, $\sqrt{\mathcal{B}}$ and introduce a few notes.

As follows from definition of $\rho \wedge \alpha|w\sqrt{\mathcal{B}}\, dH|$, ρ is a Borel function on $Q = G \setminus E$. Let $\rho \equiv 0$ on $R^n \setminus Q$. Then by Vitali-Carathéodory theorem [14, Theorem 2.24] there exists a lower semicontinuous (lsc) function $g \ge \rho$ with $\|g\|_{L_{1,w}(R^n)}$ arbitrarily close to $\|\rho\|_{L_{1,w}(R^n)}$. Therefore

$$M_{1,w}(\alpha|w\sqrt{\mathcal{B}}\, dH|) = \inf\left\{\int\limits_{Q} \rho w\, dx : \rho \text{ is lsc on } Q,\ \rho \wedge \alpha|w\sqrt{\mathcal{B}}\, dH|\right\}. \tag{20}$$

Let $\mathcal{K}$ be a Hesse condenser. For $\dfrac{1}{k}$-neighborhoods $O\left(E_i, \dfrac{1}{k}\right)$ of its plates E_i, $0 \le i \le m$, $k \in \mathbb{N}$, choose $\tilde{k}_0$ such that the closures of this neighborhoods are mutually disjoint for $k = \tilde{k}_0$. Let $O_i(k) \subset O\left(E_i, \dfrac{1}{k}\right)$ be an open set with a piecewise smooth boundary and $E_i \subset \overline{O_i(k+1)} \subset O_i(k)$, $k \ge \tilde{k}_0$. Then denote $E(k) = \bigcup\limits_{i=0}^{m} \overline{O_i(k)}$, $Q(k) = G \setminus E(k)$.

By $H_{ij}(k)$ denote the family of locally rectifiable curves in $Q(k)$ connecting $\overline{O_i(k)}$ and $\overline{O_j(k)}$, $0 \le i < j \le m$. The curves from $H_{ij}(k)$ is oriented from $\overline{O_i(k)}$ to $\overline{O_j(k)}$ in case $\delta_i < \delta_j$, and in reverse direction if $\delta_i > \delta_j$, $0 \le i < j \le m$. Besides, it is easily to see that in case $H_{ij} = \emptyset$ the family $H_{ij}(k)$ also is empty for all large k. We will assume that it is true for $k \ge \tilde{k}_0$.

Next lemma goes back to Lemma 6 from [6].

Lemma 3. *Let G_k, $k \in \mathbb{N}$, be an open sets exhausting of open set $G \subset R^n$ with piecewise smooth boundaries, i.e. $\overline{G_k} \subset G_{k+1}$, $\bigcup\limits_k G_k = G$. For simplicity we set $G_0 = \emptyset$. Let β_k, $k \in \mathbb{N}$, be a sequence of positive numbers (some of which may be equal $+\infty$). There is a function $\lambda(x) \in C^\infty(G)$ such that $0 < \lambda(x) < \beta_k$, $|\nabla\lambda(x)| < \beta_k$ if $x \in G_k \setminus G_{k-1}$, $k \in \mathbb{N}$.*

Proof. For convenience we suppose that $G_{-1} = \emptyset$. There exists $\lambda_k(x) \in C^\infty(G)$ such that $\lambda_k(x) \geq 0$ on G, $\lambda_k(x) > 0$ on $G_k \setminus G_{k-1}$ and $\lambda_k(x) = 0$ on $(G \setminus G_{k+1}) \cup G_{k-2}$. Multiplying by constant, we may assume that $\lambda_k, |\nabla\lambda_k| < \dfrac{1}{3}\min\{\beta_{k-1}, \beta_k, \beta_{k+1}\}$ if $x \in G_{k+1} \setminus G_{k-2}$, where we set $\beta_0 = +\infty$. The function $\lambda(x) = \sum\limits_k \lambda_k(x)$ is required. □

Theorem 3. *If $\mathcal{K}$ is a Hesse condenser in $\overline{R^n}$ then*

$$C_{\mathcal{A},1}(\mathcal{K}) \leq C^*_{\mathcal{A},1}(\mathcal{K}) \leq C_1 c_0^2 C_{\mathcal{A},1}(\mathcal{K}), \tag{21}$$

where C_1, c_0 are constants from Proposition 1 and (2), respectively.

For proof of theorem 3 we need next statements:

Lemma 4. *Given $\eta > 0$, there exists a function $\rho \wedge \alpha|w\sqrt{\mathcal{B}}\,dH|$ for which these conditions are true:*

(1) *ρ is continuous on $Q = G \setminus E$ and for every compact $K \subset R^n$, $K \cap Q \neq \emptyset$, $\inf\limits_{K\cap Q} \rho > 0$;*

(2) *$M_{1,w}(\alpha|w\sqrt{\mathcal{B}}\,dH|) \leq \int\limits_Q \rho w\,dx \leq C_1 c_0^2 M_{1,w}(\alpha|w\sqrt{\mathcal{B}}\,dH|) + \eta$.*

Proof. Given $\eta > 0$ by (20) there exists a lower semicontinuous on Q function $\rho_1 \wedge \alpha|w\sqrt{\mathcal{B}}\,dH|$ such that

$$M_{1,w}(\alpha|w\sqrt{\mathcal{B}}\,dH|) \leq \int\limits_Q \rho_1 w\,dx < M_{1,w}(\alpha|w\sqrt{\mathcal{B}}\,dH|) + \frac{\eta}{3C_1 c_0^2}.$$

By Lemma 3 for $0 < \varepsilon < 1$, taking $\beta_k = \min\{\varepsilon, \operatorname{dist}(G_k, R^n \setminus G)\}$ for $G = Q$, we find a function $\lambda(x) \in C^\infty(Q)$ with properties: $\lambda(x) < \operatorname{dist}(x, R^n \setminus Q)$, $|\nabla\lambda(x)| < \varepsilon$ on Q.

Let for $x \in Q$

$$\rho_2(x) = T(\rho_1(x)) = \frac{1}{|B(0,1)|}\int\limits_{B(0,1)} \rho_1(x + \lambda(x)y)\,dy, \tag{22}$$

where T is averaging operator introduced in [3] and studied in details in [9, Lemma 4.3]. In particular, ρ_2 is continuous on Q because $\lambda(x) > 0$ on Q.

Note that for fixed $y \in B(0,1)$ the mapping θ_y, defined by formula $z = \theta_y(x) = x + \lambda(x)y$, satisfy the inequalities

$$(1-\varepsilon)|x' - x''| \leq |\theta_y(x') - \theta_y(x'')| \leq (1+\varepsilon)|x' - x''| \tag{23}$$

for x', $x'' \in Q$. Moreover, θ_y is quasi-isometric diffeomorphism R^n onto itself, in particular, $\theta_y(Q) = Q$ [13, Lemma 2.4.2]. The Jacobian $J(x, \theta_y)$ of mapping θ_y is equal to $1 + y \cdot \nabla\lambda(x)$ [8, p. 209]. Hence

$$1 - \varepsilon \leq J(x, \theta_y) \leq 1 + \varepsilon \tag{24}$$

on Q and $\theta_y(H_{ij}) = H_{ij}$ for $0 \le i < j \le m$.

From (22) by integration we get

$$\|\rho_2\|_{L_{1,w}(Q)} = \frac{1}{|B(0,1)|} \int\limits_{B(0,1)} dy \int\limits_{Q} \rho_1(x+\lambda(x)y)w(x)\,dx.$$

Applying the change of variables $z = x + \lambda(x)y$ to the inner integral in the right side and changing the integrations, by (23), (24), get the estimate

$$\|\rho_2\|_{L_{1,w}(Q)} \le \frac{1}{1-\varepsilon} \int\limits_{Q} \left(\frac{1}{|B(0,1)|} \int\limits_{B(0,1)} w(x(z))\,dy \right) \rho_1(z)\,dz.$$

Due to Proposition 1 and $x = z - \lambda(x(z))y$ we conclude that

$$\frac{1}{|B(0,1)|} \int\limits_{B(0,1)} w(z-\lambda(x(z))y)\,dy$$

$$= \frac{1}{|B(z,\lambda(x(z)))|} \int\limits_{B(z,\lambda(x(z))} w(y)\,dy \le M(w(z)) \le C_1 w(z).$$

Therefore

$$\|\rho_2\|_{L_{1,w}(Q)} \le \frac{C_1}{1-\varepsilon} \int\limits_{Q} \rho_1(z)w(z)\,dz. \tag{25}$$

Let's show that $\rho_3 = (1+\varepsilon)c_0^2\rho_2 \wedge \alpha|w\sqrt{\mathcal{B}}\,dH|$. Consider the curve $\gamma \in H_{ij}$, $0 \le i < j \le m$. Then by (22) and Fubini theorem

$$\int\limits_{\gamma} c_0\rho_2|w(x)\sqrt{\mathcal{B}(x)}\,dx| \ge \frac{1}{|B(0,1)|} \int\limits_{B(0,1)} dy \int\limits_{\gamma} \rho_1(x+\lambda(x)y)ds_x.$$

Applying in the inner integral of right side the changing of variables $z = x+\lambda(x)y$, get the estimate

$$\int\limits_{\gamma} \rho_1(x+\lambda(x)y)ds_x$$

$$= \int\limits_{z^{-1}(\gamma)} \rho_1(z)\frac{ds_x}{ds_z}ds_z \ge c_0^{-1} \int\limits_{z^{-1}(\gamma)} \rho_1(z)|w(z)\sqrt{\mathcal{B}(z)}dz| \ge \frac{\alpha_{ij}}{c_0(1+\varepsilon)}.$$

Here we use that $z^{-1}(\gamma) \in H_{ij}$ and by (22)

$$\frac{ds_z}{ds_x} = \left|\frac{dx}{ds} + \frac{d\lambda}{ds}y\right| \le 1+\varepsilon$$

for all s from domain of arc-length parametrization except a set of zero linear measure. This implies the property $\rho_3 \wedge \alpha|w\sqrt{\mathcal{B}}\,dH|$. Besides, by choice of ρ_1, ρ_3 it follows that

$$M_{1,w}(\alpha|w\sqrt{\mathcal{B}}\,dH|) \le \int\limits_{Q} \rho_3 w\,dx \le c_0^2 C_1 M_{1,w}(\alpha|w\sqrt{\mathcal{B}}\,dH|) + \frac{\eta}{3} + o(1),$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Choose ε such that $o(1) < \frac{\eta}{3}$ and set $\rho = \rho_3 + g$ where $g \in L_{1,w}(R^n)$ is continuous and positive on R^n function with $\int\limits_Q gw\,dx < \frac{\eta}{3}$. It is obvious that ρ satisfies the conditions of lemma and the lemma is proved. □

If $w\sqrt{B}$ is continuous on Q (i.e. each component of this matrix is continuous on Q) then Lemma 4 may be refined, in the right side of 2) we may omit the constant c_0.

Lemma 5. *Let $w\sqrt{B}$ be continuous on Q. Then for $\eta > 0$, there exists a function $\rho \wedge \alpha|w\sqrt{\mathcal{B}}\,dH|$ for which these conditions are true:*

(1) *ρ is continuous on $Q = G \setminus E$ and for every compact $K \subset R^n$, $K \cap Q \neq \emptyset$, $\inf\limits_{K\cap Q} \rho > 0$;*

(2) $M_{1,w}(\alpha|w\sqrt{\mathcal{B}}\,dH|) \leq \int\limits_Q \rho w\,dx \leq C_1 M_{1,w}(\alpha|w\sqrt{\mathcal{B}}\,dH|) + \eta.$

Proof. Let $\varepsilon, \eta \in (0,1)$. There exists a lower semicontinuous on Q function $\rho_1 \wedge \alpha|w\sqrt{\mathcal{B}}\,dH|$ such that

$$M_{1,w}(\alpha|w\sqrt{\mathcal{B}}\,dH|) \leq \int\limits_Q \rho_1 w\,dx < M_{1,w}(\alpha|w\sqrt{\mathcal{B}}\,dH|) + \frac{\eta}{3C_1}.$$

Let $G = Q$ and G_k be exhaustion of G from Lemma 3. Since the function $(x,\xi) \to |w(x)\sqrt{\mathcal{B}(x)}\xi|$ is continuous on $Q \times (R^n \setminus \{0\})$ and by (2) function $|w(x)\sqrt{\mathcal{B}(x)}\xi| > c_0^{-1}|\xi| > 0$ then $\log|w(x)\sqrt{\mathcal{B}(x)}\xi|$ is continuous on $Q \times (R^n \setminus \{0\})$. Hence the last function is uniformly continuous on compact $\overline{G_k} \times \left\{\xi \in R^n : \frac{1}{2} \leq |\xi| \leq \frac{3}{2}\right\}$, $k \in \mathbb{N}$. Then there exists d_k such that for all $(x',\xi'), (x'',\xi'') \in \overline{G_k} \times \left\{\xi \in R^n : \frac{1}{2} \leq |\xi| \leq \frac{3}{2}\right\}$, $|x' - x''| < d_k$, $|\xi' - \xi''| < d_k$ we have

$$\frac{1}{1+\varepsilon} < \left|\frac{w(x')\sqrt{\mathcal{B}(x')}\xi'}{w(x'')\sqrt{\mathcal{B}(x'')}\xi''}\right| < 1 + \varepsilon. \tag{26}$$

We can assume that all $d_k < \frac{1}{2}$. Due to Lemma 3,
where $\beta_k = \min\{\varepsilon, \mathrm{dist}(G_k, R^n \setminus Q), d_{k+1}, \mathrm{dist}(G_k, \partial G_{k+1})\}$, we find a function $\lambda(x) \in C^\infty(Q)$ with properties: $\lambda(x) < \mathrm{dist}(x, R^n \setminus Q)$, $|\nabla\lambda(x)| < \varepsilon$ on Q; $\lambda(x) < \min\{d_{k+1}, \mathrm{dist}(G_k, \partial G_{k+1})\}$, $|\nabla\lambda(x)| < d_{k+1}$ if $x \in G_k \setminus G_{k-1}$, $k \in \mathbb{N}$.

Construct the mollified function ρ_2 as in the Lemma 4, we get in the same way the inequality (25). For $\gamma \in H_{ij}$, $0 \leq I < j \leq m$, consider the integral

$$\int\limits_\gamma \rho_2(x)|w(x)\sqrt{\mathcal{B}(x)}\,dx| = \frac{1}{|B(0,1)|}\int\limits_{B(0,1)} dy \int\limits_\gamma \rho_1(x + \lambda(x)y)|w(x)\sqrt{\mathcal{B}(x)}\,dx|.$$

In the inner integral on the right side let $z = z(x) = x + \lambda(x)y$. This integral is equal to

$$\int\limits_\gamma \rho_1(z(x))\frac{\left|w(x)\sqrt{\mathcal{B}(x)}\frac{dx}{ds}\right|}{\left|w(z(x))\sqrt{\mathcal{B}(z(x))}\frac{dz(x)}{ds}\right|}|w(z(x))\sqrt{\mathcal{B}(z(x))}\,dz(x)|.$$

If $x \in \gamma$ then $x \in G_k$ for some k. By properties of $\lambda(x)$, $z(x) \in G_{k+1}$. Also $|z(x) - x| = |\lambda(x)y| < d_{k+1}$, $\left|\frac{dz(x)}{ds} - \frac{dx}{ds}\right| = \left|\frac{d\lambda(x)}{ds}y\right| < d_{k+1} < \frac{1}{2}$ and $\left|\frac{dx}{ds}\right| = 1$, therefore $\frac{1}{2} < \left|\frac{dz(x)}{ds}\right| < \frac{3}{2}$. Then by (26)

$$\frac{\left|w(x)\sqrt{\mathcal{B}(x)}\frac{dx}{ds}\right|}{\left|w(z(x))\sqrt{\mathcal{B}(z(x))}\frac{dz(x)}{ds}\right|} > \frac{1}{1+\varepsilon}.$$

Thus

$$\int\limits_{\gamma} \rho_2(x)|w(x)\sqrt{\mathcal{B}(x)}\,dx| \geq \frac{1}{1+\varepsilon}\int\limits_{z^{-1}(\gamma)} \rho_1(z)|w(z)\sqrt{\mathcal{B}(z)}\,dz| \geq \frac{\alpha_{ij}}{1+\varepsilon},$$

since $z^{-1}(\gamma) \in H_{ij}$ for $\gamma \in H_{ij}$, $0 \leq i < j \leq m$. Therefore $\rho_3 = (1+\varepsilon)\rho_2 \wedge |w\sqrt{\mathcal{B}}\,dH|$. Further as in Lemma 4 we put $\rho = \rho_3 + g$, where g is continuous and positive in R^n, $\|g\|_{L_{1,w}(R^n)} < \dfrac{\eta}{3}$. Taking ε sufficiently small, we obtain the statements of this lemma. □

Lemma 6. *Let $\mathcal{K}$ be a Hesse condenser in $\overline{R^n}$ and function $\rho \wedge \alpha|w\mathcal{B}\,dH|$ satisfies the conditions of Lemma 4 with given $\eta > 0$. Then for $\varepsilon \in (0,1)$ and all $H_{ij} \neq \emptyset$, $0 \leq i < j \leq m$, there exists $k_0 \geq \tilde{k}_0$ such that*

$$\int\limits_{\gamma} \rho|w\sqrt{\mathcal{B}}\,dx| \geq \alpha_{ij}(1-\varepsilon) \tag{27}$$

for all $\gamma \in H_{ij}(k)$ and all $k \geq k_0$. Here $\tilde{k}_0$, $H_{ij}(k)$ defined in the beginning of Section 5.

Lemma 6 is proved by scheme proposed in [7, Lemma 4] for arbitrary condenser in $\overline{R^n}$ and weighted capacity with A_p-weight, $1 < p < \infty$. Moreover, due to topological condition $(\partial G \setminus E) \cap E = \emptyset$ for Hesse condenser this scheme is simplified and allows replace the exponent $p > 1$ by $p = 1$. Therefore we omit the proof of this lemma.

Proof of Theorem 3. By (9) and Theorem 2, it is sufficiently to prove that $C^*_{\mathcal{A},1}(\mathcal{K}) \leq C_1 c_0^2 M_{1,w}(\alpha|w\sqrt{\mathcal{B}}\,dH|)$. Let $\eta \in (0,1)$ and function $\rho \wedge \alpha|w\sqrt{\mathcal{B}}\,dH|$ satisfies the conditions of Lemma 4 with given η. Applying the Lemma 6 to this function ρ with given $\varepsilon \in \left(0, \dfrac{1}{2}\right)$, and find for ρ, ε the number $k_0 = k_0(\varepsilon, \rho)$ for which the inequalities (27) holds with $k \geq k_0$.

Let

$$\rho_0 = \begin{cases} \frac{\rho}{1-2\varepsilon}, & x \in Q(k_0); \\ 0, & x \in \overline{R^n} \setminus Q(k_0). \end{cases}$$

Then $\rho_0 \wedge \alpha|w\sqrt{\mathcal{B}}\,dH(k)|$, $k \geq k_0$, by construction ρ_0 is continuous on $Q(k_0) = G \setminus E(k_0)$ and therefore ρ_0 is locally bounded on $Q_{k_0+1} = G \cup \left(\bigcup\limits_{i=0}^{m} O_i(k_0+1)\right) \setminus \{\infty\}$ and continuous on Q_{k_0+1} a.e.

Here we let $\sqrt{\mathcal{A}(x)} = J \cdot w(x)$, $\sqrt{\mathcal{B}(x)} = J \cdot w(x)^{-1}$ on $G_{k_0+1} \setminus G$, where J is an identity matrix of order n.

Let $u_i = \delta_i$ on $\overline{O_i(k_0+2)}$ and

$$u_i(x) = \delta_i + \inf_{\gamma_x} \int\limits_{\gamma_x} \rho_0 |w\sqrt{\mathcal{B}}\, dx|,$$

for all $x \in Q_{k_0+1} \setminus \overline{O_i(k_0+2)}$, where the infimum is taken over all rectifiable curves $\gamma_x \subset Q_{k_0+1} \setminus \overline{O_i(k_0+2)}$ connecting the point x and $\overline{O_i(k_0+2)}$, $i = 0, \dots, m$. If there is no such curves, then let $u_i = \max\limits_{0 \le j \le m} \delta_j$. Due to local boundedness of ρ_0 on Q_{k_0+1} the function $u_i(x)$ satisfies locally the Lipschitz condition and therefore by Rademacher theorem $u_i(x)$ is differentiable a.e. on G_{k_0+1}. Besides, applying arguments from proofs of conditions (i), (ii), (iii) of Lemma 2, we establish that

(1) $|\nabla u_i(x)| \le c_0 \rho_0(x)$ a.e. on Q_{k_0+1};
(2) $u_i(x)$ is a $(1, w)$-precise function on Q_{k_0+1};
(3) $|\sqrt{\mathcal{A}(x)} \nabla u_i(x)| \le \rho_0(x) w(x)$ a.e. on Q_{k_0+1};
(4) $u_i(x) = \delta_i$ in some neighborhood of E_i and for $x \in O_j(k_0+2) \setminus \{\infty\}$, where $0 \le j \le m$ and $j \ne i$, either $u_i(x) \ge \delta_i + \alpha_{ij} \ge \delta_j$ or $u_i(x) = \max\limits_{0 \le l \le m} \delta_l \ge \delta_j$. If $\infty \in O_j(k_0+2)$ then let $u_i(\infty) = \max\limits_{0 \le l \le m} \delta_l \ge \delta_j$.

Let $u(x) = \min\limits_{1 \le i \le m} u_i(x)$ on Q_{k_0+1} and $u(\infty) = \min\limits_{1 \le i \le m} u_i(\infty)$ if $\infty \in \bigcup\limits_{i=0}^{m} O_i(k_0+2)$.

By construction $u(x) = \delta_i$ in some neighborhood of E_i, $i = 0, \dots, m$. Besides, for any rectifiable curve $\gamma \subset Q_{k_0+1}$

$$|u(b) - u(a)| \le \max_{0 \le i \le m} |u_i(b) - u_i(a)| \le \int\limits_{\widetilde{ab}} \rho_0 |w\sqrt{\mathcal{B}}\, dx| \le \int\limits_{\widetilde{ab}} c_0 \rho_0\, ds \tag{28}$$

for all $a, b \in \gamma$, where $\widetilde{ab}$ if a subcurve of γ connecting points a and b.

By (28), applying the arguments from proof of the conditions (i)–(iii) in Lemma 2, we obtain that $|\nabla u| \le c_0 \rho_0$ a.e. on Q_{k_0+1}; $u(x)$ is a $(1, w)$-precise function on Q_{k_0+1}; $|\sqrt{\mathcal{A}} \nabla u| \le \rho_0 w$ a.e. on Q_{k_0+1}. Hence $u \in \mathcal{D}^*(\mathcal{K})$ and

$$C^*_{\mathcal{A},1}(\mathcal{K}) \le \int\limits_Q \rho_0 w\, dx \le \int\limits_Q \frac{\rho}{1-2\varepsilon} w\, dx = \int\limits_Q \rho w\, dx + o(1) \le$$
$$C_1 c_0^2 M_{1,w}(\alpha |w\sqrt{\mathcal{B}}\, dH|) + o(1) + \eta,$$

where $o(1) \to 0$ as $\varepsilon \to 0$.

Letting $\varepsilon \to 0$, $\eta \to 0$, by Theorem 2 we conclude that

$$C_{\mathcal{A},1}(\mathcal{K}) \le C^*_{\mathcal{A},1}(\mathcal{K}) \le c_0^2 C_1 C_{\mathcal{A},1}(\mathcal{K}).$$

Therefore inequalities (21) in Theorem 3 are established. □

Assuming the continuity of $w\sqrt{\mathcal{B}}$ on Q, replacing Lemma 4 by Lemma 5, similarly we set the following result.

Theorem 4. *Let $w\sqrt{\mathcal{B}}$ be continuous on Q. Then $C_{\mathcal{A},1}(\mathcal{K}) \le C^*_{\mathcal{A},1}(\mathcal{K}) \le C_1 C_{\mathcal{A},1}(\mathcal{K})$. In particular, if $w \equiv 1$ then $C_1 = 1$ and $C_{\mathcal{A},1}(\mathcal{K}) = C^*_{\mathcal{A},1}(\mathcal{K})$.*

Remark 5. *This theorem extends* [2, Theorem 4] *to the case $p = 1$, in which uniform continuity of $w\sqrt{\mathcal{B}}$ is required. Moreover, above arguments allows to prove Theorem 4 from* [2] *for $1 < p < \infty$ with a weaker condition of continuity for $w^{1/p}\sqrt{\mathcal{B}}$.*

References

[1] P.S. Aleksandrov, *Lektsii po analiticheskoy geometrii*, Nauka, Moscow, 1968.

[2] H. Aikawa, M. Ohtsuka, *Extremal lenth of vector measures*, Ann. Acad. Sci. Fenn., Math., **24**:1 (1999), 61–88. Zbl 0940.31006

[3] V.V. Aseev, *On a modulus property*, Sov. Math., Dokl., **12** (1971), 1409–1411. Zbl 0233.46059

[4] V.I. Bogachev, O.G. Smolyanov, *Real and functional analysis*, Springer, Cham, 2020. Zbl 1466.26002

[5] V.N. Dubinin, *Asymptotics for the capacity of a condenser with variable potential levels*, Sib. Math. J., **61**:4 (2020), 626–631. Zbl 1455.30016

[6] Y.V. Dymchenko, *Equality of the capacity and module of a condenser on a surface*, J. Math. Sci., New York, **118**:1 (2003), 4795–4807. Zbl 1066.31005

[7] Y.V. Dymchenko, V.A. Shlyk, *On a problem of Dubinin for the capacity of a condenser with a finite number of plates*, Math. Notes, **103**:6 (2018), 901–910. Zbl 1397.30025

[8] B. Fuglede, *Extremal length and functional completion*, Acta Math., **98** (1957), 171–219. Zbl 0079.27703

[9] J. Hesse, *A p-extremal length and p-capacity equality*, Ark. Mat., **13** (1975), 131–144. Zbl 0302.31009

[10] O. Martio, V. Ryazanov, U. Srebro, E. Yakubov, *Moduli in modern mapping theory*, Springer, New York, 2009. Zbl 1175.30020

[11] V.G. Maz'ya, *Sobolev spaces*, Springer, Berlin etc., 1985. Zbl 0692.46023

[12] B. Muckenhoupt, *Weighted norm unequalities for the Hardy maximal function*, Trans. Am. Math. Soc., **165** (1972), 207–226. Zbl 0236.26016

[13] M. Ohtsuka, *Extremal length and precise functions. With a preface by Fumi-Yuki Maeda*, GAKUTO International Series. Mathematical Sciences and Applications, **19**, Gakkōtosho, Tokyo, 2003. Zbl 1075.31001

[14] W. Rudin, *Real and complex analysis. 3rd Ed.*, McGraw-Hill, New York, 1987. Zbl 0925.00005

[15] I.M. Tarasova, V.A. Shlyk, *Weighted Sobolev spaces, capacities and exceptional sets*, Sib. Èlectron. Mat. Izv., **17** (2020), 1552–1570. Zbl 1466.46031

[16] B.O. Turesson, *Nonlinear potential theory and weighted Sobolev spaces*, Springer, Berlin, 2000. Zbl 0949.31006

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