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THE PSEUDO-PROJECTIVITY OF MODULES OVER SEMILOCAL RINGS

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Abstract. We discuss some problems of pseudo-projective modules over the class of classical rings. It shows that a ring is semilocal if and only if every finitely generated module with Jacobson radical zero is pseudo-projective. We also give the structure of artinian principal ideal rings. A ring is an artinian principal ideal ring if and only the class of pseudo-injective modules and the class of pseudo-projective modules coincide.

Keywords: pseudo-projective module, pseudo-injective module, QF-ring, serial ring, principal ideal ring.

1. Introduction

Throughout this article all rings are associative rings with unity and all modules are right unital modules over a ring. We denote by $|X|$ the cardinality of a set X. For a submodule N of M, we write $N \leq M$ ($N < M, N \leq^e M$) iff N is a submodule of M (respectively, a proper submodule, an essential submodule). We denote by $J(M)$ and $Soc(M)$ the radical of the module M and the socle of M, respectively. We also denote by $E(M)$ the injective envelope of M. For any term not defined here the reader is referred to [3], [5] and [15].

A module M is called *pseudo-injective* (resp., *quasi-injective*) if every monomorphism (resp., homomorphism) from each submodule of M to M is extended to an endomorphism of M ([10], [11]). It is well-known that M is pseudo-injective if M is invariant under all automorphisms of its injective envelope ([19]). These modules are called automorphism-invariant ([14]). Some properties and structure of rings via automorphism-invariant modules are studied ([1, 2, 9, 12, 13, 18, 19, 20, 22]). Dualizing the notion of a pseudo-injective module, the authors ([21]) introduced pseudo-projective modules. A module M is called *pseudo-projective* (resp., *quasi-projective*) if every epimorphism (resp., homomorphism) from M to each quotient module of M can be lifted to an endomorphism of M. On can check that every quasi-projective module is pseudo-projective. But the converse is not true in the general (see Example 1). It is well-known that over a right perfect ring, a module is pseudo-projective if and only if it is coinvariant under all automorphisms of its projective cover ([19]). This is not natural as pseudo-injective modules, because of the existence of injective envelopes and projective covers. In fact, every module have an injective envelope. But, it is not true for projective covers. It means that there are modules with no projective covers. Moreover, every module has a projective cover over perfect rings and every finitely generated module has a projective cover over semiperfect rings.

In this paper, we study pseudo-projective modules over the classical rings. Firstly, we study some properties of pseudo-projective modules over semilocal rings. In the first part, we give some characterizations of the relatively projective of direct summands of pseudo-projective modules. From these, we obtain the structure of semilocal rings via the pseudo-projectivity of modules. It shows that a ring R is semilocal if and only if every finitely generated right R-module with Jacobson radical zero is pseudo-projective (see Theorem 1). A "good"generalization of the class of semisimple artinian rings is the class of principal ideal quasi-Frobemius rings. They are called artinian principal ideal rings. These classes are studied by the famous mathematician of Ring and Module theory as Fuller, Faith, Byrd ([4, 5, 6]). Of course, they are interesting in the structure of classical rings. In classes of these rings, one-sided ideals are principal and the class of injective modules and the class of projective coincide. Moreover, every module over these rings is a direct sum of cyclic uniserial submodules which are isomorphic to one-sided ideals. In [4], Byrd proved that a ring is an artinian principal ideal ring if and only if every quasi-projective is quasi-injective. This is equivalent to every quasi-injective module is quasi-projective. In this paper, we continue study for the coincidence of pseudo-injective modules and pseudo-projective modules. It shows that a ring is an artinian principal ideal ring if and only the class of pseudo-injective modules and the class of pseudo-projective coincide (see Theorem 4).

2. RESULTS

As mentioned in the introduction, every quasi-projective module is pseudo-projective. The following example shows that the converse is not true in the general.

Example 1 ([9, Example 5.1])**.** *Let* $R = \begin{bmatrix} \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 \\ 0 & \mathbb{F}_2 & 0 \\ 0 & 0 & \mathbb{F}_2 \end{bmatrix}$ *where* $\mathbb{F}_2$ *is the field of two elements and* $M = e_{11}R$*. As* R *is a finite-dimensional algebra over* $\mathbb{F}_2$*, the functors*

$$\mathrm{Hom}_{\mathbb{F}_2}(-, \mathbb{F}_2) : \text{Mod-R} \to \text{R-Mod}$$

and

$$\mathrm{Hom}_{\mathbb{F}_2}(-, \mathbb{F}_2) : \text{R-Mod} \to \text{Mod-R}$$

establish a contravariant equivalence between the subcategories of left and right finitely generated modules over R*. Then,* $\mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_2)$ *is pseudo-projective left* R*-module and it is not quasi-projective.*

Next, we gve some basic properties of pseudo-projective modules and pseudo-injective modules. Firstly, Osofsky ([16]) proved that a ring is semisimple artinian if and only if every finitely generated module is injective. It is equivalent to every cyclic module is injective ([17]). We also get the similar result for pseudo-projective modules and pseudo-injective modules.

The following lemma is followed by [11, Theorem 1] and [21, Lemma 1.3].

Lemma 1. *Let R be a ring.*

(1) *If $A \oplus B$ is a pseudo-projective right R-module, every epimorphism from A to B splits.*
(2) *If $A \oplus B$ is a pseudo-injective right R-module, every monomorphism from A to B splits*

Proposition 1. *Let R be any ring. Then the following conditions are equivalent:*

(1) *R is semisimple artinian.*
(2) *Each finitely generated right R-module is pseudo-projective.*
(3) *Each finitely generated right R-module is pseudo-injective.*

Proof. (1)$\Rightarrow$ (2), (3) are obvious.

(2)$\Rightarrow$ (1) Let S be a simple right R-module. Take $N = R \oplus S$. Then, N is a finitely generated right module, and so it is pseudo-projective. Note that S is an epimorphic image of R. Then, it follows, from Lemma 1, that S is isomorphic to a direct summand of R_R, and so it is projective. We deduce that R is semisimple artinian.

(3)$\Rightarrow$ (1) is followed by the Osofsky's result. □

Recall that a module P is a *pseudo-projective cover* (resp., *projective cover*) of a right R-module M if, there exists an epimorphism $p : P \to M$ such that P is pseudo-projective (resp., *projective*) and $\mathrm{Ker}(p)$ is small in P ([21]).

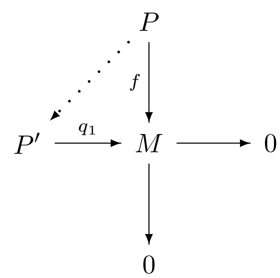
Proposition 2. *Let M be a module and let $f : P \to M$ be an epimorphism with P a projective module. Then*

(1) *M is projective if and only if $P \oplus M$ is pseudo-projective.*
(2) *M has a projective cover if and only if $P \oplus M$ has a pseudo-projective cover.*

Proof. (1) is obvious by Lemma 1.

(2) If M has a projective cover, one can check that $P \oplus M$ has a pseudo-projective cover. Assume that $P \oplus M$ has a pseudo-projective cover. We show that M is projective. Call $q : Q \to P \oplus M$ an epimorphism with small kernel and Q pseudo-projective. Take $\pi : P \oplus M \to P$ the canonical projection. Then, $\pi \circ q : Q \to P$ is an epimorphism, and so it is an splitting epimorphism (since P is projective). There is a monomorphism $\beta : P \to Q$ such that $\pi \circ q \circ \beta = 1_P$, and so $Q = \mathrm{Im}(\beta) \oplus \mathrm{Ker}(\pi \circ q)$. Let $P' = \mathrm{Ker}(\pi \circ q)$ and $q_1 = q|_{P'}$. Then, we have $q_1(P') = q(P') = \mathrm{Ker}(\pi) = M$ which implies that $q_1 : P' \to M$ is an epimorphism. One can check that $\mathrm{Ker}(q_1) = \mathrm{Ker}(q)$, and so $\mathrm{Ker}(q_1)$ is small in P'. Next, we show that P' is projective. We

consider the following diagram



Since P is projective, there is a homomorphism $g : P \to P'$ such that the above diagram is commutative. It means that $q_1 \circ g = f$. We have that $\mathrm{Ker}(q_1)$ is small in P' and obtain that g is an epimorphism. Moreover, $Q = \mathrm{Im}(\beta) \oplus P' \cong P \oplus P'$ is pseudo-projective. By Lemma 1, g splits and so P' is isomorphic to a direct summand of P. Thus, P' is projective. □

Corollary 1 ([21, Theorem 1.16]). *A ring R is right perfect if and only if every right R-module has a pseudo-projective cover.*

Corollary 2. *Let R be a ring.*

(1) *A ring R is semiperfect if and only if every finitely generated right R-module has a pseudo-projective cover.*
(2) *A ring R is semiregular if and only if every finitely presented right R-module has a pseudo-projective cover.*

We denote by $M_n(R)$ the ring of n by n matrices over R. Now applying the same proof of [7, Theorem 3.1], we get the following,

Proposition 3. *The following conditions are equivalent for a ring R:*

(1) *R is semiperfect.*
(2) *For all $n \geq 1$, every cyclic right $M_n(R)$-module has a pseudo-projective cover.*
(3) *There exists an $n > 1$ such that every cyclic right (left) $M_n(R)$-module has a pseudo-projective cover.*

Recall that a module M is called *semiprimitive* if it's Jacobson radical is zero ([8]).

Next, we give the structure of rings via semiprimitive finitely generated modules accompanying with the pseudo-projectivity of modules.

Lemma 2. *If each semiprimitive finitely generated right R-module is pseudo-projective, then every quotient ring of R has this property.*

Proof. Let S be a quotient ring of R. Assume that M is a semiprimitive finitely generated right S-module. Then M is also a semiprimitive finitely generated right R-module. By the hypothesis, M is a pseudo-projective right R-module. It follows that M is a pseudo-projective right S-module. □

From Proposition 10.15 in [3], we have the following lemma:

Lemma 3. *Let M be a right R-module. The following conditions are equivalent:*

(1) *M is semiprimitive artinian.*

(2) *M is semiprimitive finitely cogenerated.*
(3) *M is a semisimple finitely generated module.*

Corollary 3. *A semiprimitive artinian module is pseudo-projective and pseudo-injective.*

Theorem 1. *Then the following conditions are equivalent for a ring R:*
(1) *R is a semilocal ring.*
(2) *Each semiprimitive finitely generated right R-module is artinian.*
(3) *Each semiprimitive finitely generated right R-module is pseudo-projective.*
It follows from (1) that the conditions (2)-(3) are left right symmetric.

Proof. (1) $\Rightarrow$ (2). Assume that R is semilocal. We show that every semiprimitive finitely generated right R-module is artinian. In order to conclude the proof we shall show by induction on generated elements of M. Assume that M is generated by n elements. The case $n = 1$, we have M is a cyclic module. This means that $M \cong R/K$ for some right ideal K of R. By assumption, we have $J(R/K) = 0$ or $J(R)$ is contained in K, and so $R/K \cong (R/J(R))/(K/J(R))$. We have that $R/J(R)$ is a semilocal ring and obtain that $R/J(R)$ is semisimple artinian, and so R/K is semisimple. It follows that R/K is artinian. Suppose now that each semiprimitive right R-module generated by $n = k$ elements is artinian. Call $M = m_1R + m_2R + \cdots + m_{k+1}R$ a semiprimitive finitely generated right R-module. We show that M is artinian. Indeed, we have the following short exact sequence:

$$0 \to m_1R \to M \to M/m_1R \to 0$$

The induction hypothesis can be applied to the modules m_1R and M/m_1R. It follows that m_1R and M/m_1R are artinian modules. Which implies that M is artinian. Thus, it shown that every semiprimitive finitely generated right R-module is artinian. This completes the proof of the theorem.

(2)) $\Rightarrow$ (3) by Corollary 3.

(3) $\Rightarrow$ (1) Let $\bar{R} = R/J(R)$. We show that every simple right $\bar{R}$-module is projective. Indeed, let S be an arbitrary simple right $\bar{R}$-module. Take $M = \bar{R}_{\bar{R}} \oplus S$. Then, M is a semiprimitive finitely generated $\bar{R}$-module. By (3) and Lemma 2, we have that M is pseudo-projective. Note that S is an epimorphic image of $\bar{R}_{\bar{R}}$. It follows, from Lemma 1, that S is isomorphic to a direct summand of $\bar{R}_{\bar{R}}$, and so S is projective. We deduce that $\bar{R}$ is a semilocal ring.

□

We do not know that the hypothesis "finitely generated"in Theorem 1 is removed or not. We give the following question.

Question 1. *If each semiprimitive right R-module is pseudo-projective then R is smilocal or not?*

Next, we study properties of modules over artinian principal ideal rings.

Proposition 4 ([4, Proposition 2.1])**.** *If $A \oplus M$ is quasi-projective whenever A is injective then M is projective. Dually, if $A \oplus M$ is quasi-injective whenever A is projective then M is injective.*

A ring R is called *quasi-Frobemius* if R is one-sided artinan one-sided self-injective (see [5]). We have a characterization of quasi-Frobenius rings via the projectivity and the injectivity of modules:

Theorem 2 ([5]). *The following conditions are equivalent for a ring R:*

1. *R is a quasi-Frobenius ring.*
2. *Every injective right R-module is projective.*
3. *Every projective right R-module is injective.*

The authors Faith, Fuller and Byrd demonstrated the following theorem ([4, 5, 6]):

Theorem 3. *The following conditions are equivalent for a ring R:*

1. *R is an artinian principal ideal ring.*
2. *Every quasi-injective right R-module is quasi-projective.*
3. *Every quasi-projective right R-module is quasi-injective.*
4. *Every quotient ring of R is a quasi-Frobenius ring.*

For pseudo-projective modules and pseudo-injective modules, we also have the following result:

Theorem 4. *The following conditions are equivalent for a ring R:*

1. *R is an artinian principal ideal ring.*
2. *Every pseudo-projective right R-module is quasi-injective.*
3. *Every pseudo-projective right R-module is pseudo-injective.*
4. *Every pseudo-injective right R-module is quasi-projective.*
5. *Every pseudo-injective right R-module is pseudo-projective.*

Proof. (1) $\Rightarrow$ (2). Assume that R is an artinian principal ideal ring. Let M be a pseudo-projective right R-module. Then, by [5, Proposition 25.4.6B, 25.4.17A] we have $M = \oplus_I M_i$, where each M_i is cyclic uniserial. We show that M is quasi-projective. By [15, Corollary 4.37 and Propositions 4.32, 4.35], it suffices to show that each M_i is M_j-projective for all $i, j \in I$. Note that M is pseudo-projective, M_i is M_j-projective for all $i \neq j$ with $i, j \in I$. We show that M_i is quasi-projective for all $i \in I$. Indeed, we have that R is an artinian principal ideal ring and obtain, from [6, Theorem 5.3], that every indecomposable R-module M_i is quasi-projective. Thus, M is quasi-projective. By Theorem 3, M is quasi-injective.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are obvious.

(3) $\Rightarrow$ (1) Let T be a quotient ring of R. Since R has the property of (2), so is T. Take P a projective right T-module. We show that P is an injective right T-module. Indeed, we consider the right T-module $M := P \oplus P$. Clearly, M is a projective right T-module, and so it is pseudo-projective. By assumption, M is pseudo-injective. It follows that P is quasi-injective by [10, Theorem 2], and so M is quasi-injective. By Proposition 4, P is injective. We deduce that R is an artinian principal ideal ring by Theorem 3.

(1) $\Rightarrow$ (4). Assume that R is an artinian principal ideal ring. Let M be a pseudo-injective module. Then, $M = \oplus_I M_i$ where each M_i is cyclic uniserial. We show that M is quasi-injective. By [15, Proposition 1.18], it suffices to show that each M_j is quasi-injective and $\bigoplus_{I \backslash \{j\}} M_i$ is M_j-injective for all $j \in I$. Note that M is pseudo-injective, $\bigoplus_{I \backslash \{j\}} M_i$ is M_j-injective by [10, Theorem 2]. We show that M_j is quasi-inective for all $j \in I$. Indeed, we have that R is an artinian principal ideal ring and obtain, from [6, Theorem 5.3], that every indecomposable R-module M_i is quasi-injective. Thus, M is quasi-injective, and so it is quasi-projective.

(5) $\Rightarrow$ (1) is proved as (3) $\Rightarrow$ (1).

□

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