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SPLITTING OF C.E. DEGREES AND SUPERLOWNESS

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Abstract. In this paper, we show that for any superlow c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ there exists a superlow c.e. degree $\mathbf{c}$ such that $\mathbf{c} \neq \mathbf{a}_0 \cup \mathbf{b}_0$ for all c.e. degrees $\mathbf{a}_0 \leqslant \mathbf{a}$, $\mathbf{b}_0 \leqslant \mathbf{b}$. This provides one more elementary difference between the classes of low c.e. degrees and superlow c.e. degrees. We also prove that there is a c.e. degree that is not the supremum of any two superlow not necessarily c.e. degrees degrees.

Keywords: low degree, superlow degree, jump-traceable set.

1. Introduction

In the last fifty or so years, there have emerged a large number of lowness notions associated with c.e. (as well as other classes of) sets and degrees. Some notable examples include the superlow degrees, the jump-traceable degrees, the array computable degrees, the K-trivial degrees, the contiguous degrees, the strongly jump-traceable degrees (for all of these definitions, see [1]). Some of these concepts will, where necessary, be defined later in context.

In this paper, we study and compare two lowness properties, being low and being superlow. The first problem to which the paper is devoted is the search for new elementary differences between the classes of low c.e. degrees and superlow c.e. degrees. This problem was formulated by Nies [2] and solved by Downey, Greenberg and Weber [3]. They proved that no superlow c.e. degree can bound a critical triple, but some low c.e. degree can. To find another elementary difference, we will use the

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following theorem, which is derived from the Sacks Splitting Theorem [4] [5, Ch. VII, Theorem 3.2]:

Theorem 1 (Welch [6])**.** *There exist low c.e. degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that for every c.e. degree* $\mathbf{c}$ *there are c.e. degrees* $\mathbf{a}_0 \leqslant \mathbf{a}$ *and* $\mathbf{b}_0 \leqslant \mathbf{b}$ *such that* $\mathbf{c} = \mathbf{a}_0 \cup \mathbf{b}_0$*.*

The proof of Theorem 1 is based on the fact that, by the Sacks Splitting Theorem, $\varnothing'$ is a disjunctive union of two non-intersecting low c.e. sets (hence, $\mathbf{0}' = \mathbf{a} \cup \mathbf{b}$ for some low c.e. degrees $\mathbf{a}$ and $\mathbf{b}$). Although Bickford and Mills [7] [1, Theorem 6.1.4] have shown that $\mathbf{0}' = \mathbf{a} \cup \mathbf{b}$ for some superlow c.e. degrees $\mathbf{a}$ and $\mathbf{b}$, in Theorem 1, lowness cannot be replaced by superlowness. This follows from the first result of this paper:

Theorem 3. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be superlow c.e. degrees. Then there is a superlow c.e. degree* $\mathbf{c}$ *such that* $\mathbf{c} \neq \mathbf{a}_0 \cup \mathbf{b}_0$ *for all c.e. degrees* $\mathbf{a}_0 \leqslant \mathbf{a}$, $\mathbf{b}_0 \leqslant \mathbf{b}$*.*

Thus, Theorems 1 and 3 immediately provide another elementary difference between the classes of low c.e. degrees and superlow c.e. degrees.

There are other elementary properties that distinguish the classes of low and superlow c.e. degrees. For instance, Diamondstone [8] proved that there exists a noncappable (hence, low cuppable) degree which does not cup with a superlow c.e. degree to $\mathbf{0}'$. In [9], it was proved that the semilattice generated by superlow c.e. degrees and the semilattice of all c.e. degrees (that is generated by low c.e. degrees) are not elementary equivalent. This implies that there is a low c.e degree which is not the supremum of any two superlow c.e. degrees. Downey and Ng [10] proved that such a degree can be chosen to be ultrahigh. In this paper, we prove that there is a c.e. degree that is not the supremum of any two superlow not necessarily c.e. degrees. The question of the existence of such degrees was also formulated in [2].

Recall that a set A is *low* if its jump A' is Turing-below the halting problem $\varnothing'$. The following concept is more restrictive.

Definition 1. The set A is *superlow* if $A' \leqslant_{tt} \varnothing'$. Equivalently, $A'(e) = \lim_s g(e,s)$ for a computable $0,1$-valued g such that $g(e,s)$ changes at most $b(e)$ times, for a computable function b.

This notion goes back to work of Mohrherr [11], and an unpublished manuscript of Bickford and Mills [7] (where only superlow c.e. sets are studied, which are called there «abject»).

We write $J^A(e)$ for $\Phi_e(A;e)$, the jump at argument e. While lowness and superlowness restrict the *domain* $A' = \{e : J^A(e)\downarrow\}$ of J^A, *jump traceability* introduced by Nies [1,2] expresses that $J^A(e)$ has few possible values.

Definition 2. A c.e. sequence of sets $\{T_e\}_{e\in\mathbb{N}}$ is a *trace* if for some computable h,

$$\forall e\,[|T_e| \leqslant h(e)].$$

The set A is *jump-traceable* if there is a trace $\{T_e\}_{e\in\mathbb{N}}$ such that

$$\forall e\,[J^A(e)\downarrow \Rightarrow J^A(e) \in T_e].$$

In the proof of the first result of this paper, we essentially use the following theorem.

Theorem 2 (Nies [2])**.** *Superlowness and jump-traceability coincide within the c.e. sets.*

Note that none of the properties implies the other within the ω-c.e. sets.

Our notation from computability theory is mostly standard. In the following, $\varphi_e^{(n)}$ denotes the n-ary partial computable function with the Gödel number e. For unary partial computable functions we skip the upper index. We write $\varphi_e^{(n)}(x_1,\dots,x_n)\downarrow$ if this computation is defined, and $\varphi_e^{(n)}(x_1,\dots,x_n)\uparrow$ otherwise. We are assuming that

$$\varphi_{k,s}^{(n)}(x_1,\dots,x_k)\downarrow\Rightarrow\varphi_k^{(n)}(x_1,\dots,x_k)<s.$$

Let Φ_e be the Turing functional with the Gödel number e and $u(X;e,x,s)$ be the *use*-function of the computation $\Phi_{e,s}(X;x)$. We are assuming that

$$\Phi_{e,s}(X;x)\downarrow\Rightarrow\max\{\Phi_{e,s}(X;x),u(X;e,x,s)\}<s.$$

Let $W_e(X)$ be the domain of $\Phi_e(X)$. For unexplained notions we refer to Soare [5]

2. The elementary difference

In the proof of the following theorem, we let $c(x,y)$ denote the computable pairing function $2^x(2y+1)-1$. We write $c(x_0,\dots,x_{n+2})$ instead of $c(x_0,\dots,x_n,c(x_{n+1},x_{n+2}))$. Let $\mathbb{N}^{[y]}=\{c(x,y):x\in\mathbb{N}\}$ for each y.

Theorem 3. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be superlow c.e. degrees. Then there is a superlow c.e. degree* $\mathbf{c}$ *such that* $\mathbf{c}\neq\mathbf{a}_0\cup\mathbf{b}_0$ *for all c.e. degrees* $\mathbf{a}_0\leqslant\mathbf{a}$, $\mathbf{b}_0\leqslant\mathbf{b}$.

Proof. Let $\mathbf{a}=\deg_T(A)$ and $\mathbf{b}=\deg_T(B)$. We will construct a superlow c.e. set C satisfying the following requirements:

$$N_e:\exists^\infty s\,[\Phi_{e,s}(C_s;e)\downarrow]\Rightarrow\Phi_e(C;e)\downarrow,$$

$$R_{k,p,q}^{e,i,j}:W_p=\Phi_i(A)\,\&\,W_q=\Phi_j(B)\,\&\,W_p\oplus W_q=\Phi_k(C)\Rightarrow C\neq\Phi_e(W_p\oplus W_q).$$

To satisfy the requirements N_e, we will use the standard lowness strategy from the low simple set construction [5, Ch. VII, Theorem 1.1].

The strategy for satisfying the requirement $R_{k,p,q}^{e,i,j}$ is as follows. First we note that all c.e. sets A_0, B_0 with $A_0\leqslant_T A$, $B_0\leqslant_T B$ are uniformly (by Turing functionals providing these reducibilities) superlow. Therefore, one can choose integers z and m such that if the left-hand side of the implication in $R_{k,p,q}^{e,i,j}$ is true, then the sum of the number of changes in the values $J^{W_p}(z)$ and $J^{W_q}(z)$ of the jump operator does not exceed m. We then choose witnesses

$$x_0<x_1<\dots<x_m$$

to diagonalize the equality

$$C=\Phi_e(W_p\oplus W_q)$$

in such a way that adding each x_i, $0<i\leqslant m$, to C does not affect the restraints imposed by x_{i-1}. Thus, by sequentially adding the witnesses $x_m,x_{m-1},\dots$ to C and changing the values of $J^{W_p}(z)$ and $J^{W_q}(z)$, we get

$$C\neq\Phi_e(W_p\oplus W_q).$$

In more detail, to satisfy the requirement $R_{k,p,q}^{e,i,j}$, we construct auxiliary functionals Θ and Ξ with use-functions θ and ξ respectively. Using the Recursion theorem, we can assume that before starting the construction we know indices a and b such that $\Theta = \Phi_a$ and $\Xi = \Phi_b$. Hence, we know c.e. sequences of sets $\{T^{i,p}_x\}_{i,p,x\in\mathbb{N}}$, $\{U^{j,q}_x\}_{j,q,x\in\mathbb{N}}$ and computable sequences of functions $\{f^i_p\}_{i,p\in\mathbb{N}}$, $\{g^j_q\}_{j,q\in\mathbb{N}}$ such that

$$|T^{i,p}_x| \leqslant f^i_p(x),\ |U^{j,q}_x| \leqslant g^j_q(x), \tag{1}$$

$$W_p = \Phi_i(A)\,\&\,\Theta(W_p;x)\downarrow\Rightarrow \Theta(W_p;x) \in T^{i,p}_x, \tag{2}$$

$$W_q = \Phi_j(B)\,\&\,\Xi(W_q;x)\downarrow\Rightarrow \Xi(W_q;x) \in U^{j,q}_x \tag{3}$$

for all i, j, p, q, x. Now we choose a z from some preselected computable set and define $m = f^i_p(z)+g^j_q(z)+1$. Then we proceed according to the following instructions.

1. Choose a finite sequence $x_0, \dots, x_m$ such that

$$\Phi_{e,s}(W_{p,s} \oplus W_{q,s}; x_n)\downarrow= 0 \text{ for all } n \leqslant m,$$
$$\Phi_{k,s}(C_s; x)\downarrow \text{ for each } x \leqslant \max\{u(W_{p,s} \oplus W_{q,s}; e, x_n, s) : n \leqslant m\},$$
$$x_{n+1} > \max\{u(C_s; k, x, s) : x \leqslant u(W_{p,s} \oplus W_{q,s}; e, x_n, s)\} \text{ for each } n < m,$$

at some stage s. We define

$$\Theta(W_{p,t}; z)\downarrow= \Xi(W_{q,t}; z)\downarrow= s$$

at each stage $t \geqslant s$ and the values of their *use*-functions at z equal to s.

2. We do not allow elements less than or equal to s to be enumerated into C.

3. We start sequentially enumerating the elements $x_m, x_{m-1}, \dots$ into C whenever

$$\Theta(W_{p,t}; z) \in T^{i,p}_{z,t},\ \Xi(W_{q,t}; z) \in U^{j,q}_{z,t}$$

for some new stage t.

4. We define

$$\Theta(W_{p,u+1}; z) = \Theta(W_{p,t}; z) + 1,\ \Xi(W_{q,u+1}; z) = \Xi(W_{q,t}; z) + 1$$

for suitable t and u whenever x_n is enumerates into C.

Taking into account conditions (1)–(3) and the equality $m = f^i_p(z) + g^j_q(z) + 1$, we will have that the requirement $R^{e,i,j}_{k,p,q}$ is satisfied.

Construction

Stage 0. Let $\gamma_0 = \varnothing$, $C_0 = \varnothing$, $h_0(y) = r(y, 0) = 0$ for each y. At each next stage, we assume that the value of each parameter remains the same, unless otherwise specified.

Stage $2s + 1 = 2c(e, i, j, k, p, q, v) + 1$. Let $y = c(e, i, j, k, p, q)$, $z = c(y, h_s(y))$, and $m = f^i_p(z) + g^j_q(z) + 1$.

1. If $\gamma_s(z)\uparrow$ and there is a finite sequence $x_0, \dots, x_m \in \mathbb{N}^{[y]} \setminus C_s$ such that

$$\Phi_{e,s}(W_{p,s} \oplus W_{q,s}; x_n)\downarrow= 0 \text{ and } x_n > r(w, s) \text{ for all } n \leqslant m,\ w \leqslant 2y,$$
$$\Phi_{k,s}(C_s; x)\downarrow \text{ for each } x \leqslant \max\{u(W_{p,s} \oplus W_{q,s}; e, x_n, s) : n \leqslant m\},$$
$$x_{n+1} > \max\{u(C_s; k, x, s) : x \leqslant u(W_{p,s} \oplus W_{q,s}; e, x_n, s)\} \text{ for each } n < m,$$

then we define

$$\gamma_{s+1}(z) = c(m, x_0, \dots, x_m),$$
$$r(2y + 1, s + 1) = s,$$
$$\Theta(W_{p,s}; z) = \Xi(W_{q,s}; z) = s$$

with the use-values

$$\theta(W_{p,s}; z) = \xi(W_{q,s}; z) = s$$

respectively, and go to the next stage. We are assuming that

$$\theta(W_{p,t}; z) = \xi(W_{q,t}; z) = s$$

for each $t \geqslant s$.

Suppose $\gamma_s(z) \downarrow= c(m, x_0, \dots, x_m)$.

2. If there are $n \leqslant m$, $w \leqslant 2y$ such that $x_n \leqslant r(w, s)$, then we define

$$h_{s+1}(z) = h_s(z) + 1$$

and go to the next stage.

3. Suppose

$$\Theta(W_{p,s}; z) \downarrow\in T^{i,p}_{z,s},\ \Xi(W_{q,s}; z) \downarrow\in U^{j,q}_{z,s},$$
$$\forall n \leqslant m\, [\Phi_{e,s}(W_{p,s} \oplus W_{q,s}; x_n) = C_s(x_n)],$$

and there is an $l \leqslant m$ such that $x_l \notin C_s$. Let $l_0 = \max\{l \leqslant m : x_l \notin C_s\}$. If

$$\Phi_{k,s}(C_s; x) = (W_{p,s} \oplus W_{q,s})(x)$$

for each $x \leqslant \max\{u(W_{p,s} \oplus W_{q,s}; e, x_{l_0}, s) : n \leqslant l_0\}$, then we define

$$C_{s+1} = C_s \cup \{x_{l_0}\},\ r(2y+1, s+1) = s,$$

and go to the next stage.

4. Assume that

$$\Theta(W_{p,t}; z) \downarrow,\ \Xi(W_{q,t}; z) \downarrow,$$

where $t = c(e, i, j, k, p, q, v-1)$. Then we define

$$\Theta(W_{p,s}; z) = \Theta(W_{p,t}; z) + 1,\ \text{if } W_{p,s} \upharpoonright \theta(W_{p,t}; z) \neq W_{p,t} \upharpoonright \theta(W_{p,t}; z),$$
$$\Xi(W_{q,s+1}; z) = \Xi(W_{q,t}; z) + 1,\ \text{if } W_{q,s+1} \upharpoonright \xi(W_{q,t}; z) \neq W_{q,t} \upharpoonright \xi(W_{q,t}; z),$$

and go to the next stage.

Stage $2s+2$. Define

$$r(2e, s+1) = u(C_s; e, e, s)$$

for each e.

End construction

It remains to check that C is superlow. Indeed, we know the number of witnesses z belonging to a preselected computable set needed to satisfy each R-requirement. Therefore, we know the number of restraints imposed by each requirements of lower priority. Hence C is superlow. Note that similar arguments were used in the construction of a superlow simple set [1, Theorem 1.6.5]. □

Corollary 1. *The classes of low c.e. and superlow c.e. degrees are not elementary equivalent.*

The following corollary shows that in the Sacks Splitting Theorem [4] [5, Ch. VII, Theorem 3.1], in the general case, splitting sets cannot be chosen to be superlow.

Corollary 2. *There are no superlow c.e. sets A and B such that $A \cap B = \varnothing$ and $A \cup B = \varnothing'$.*

Proof. Assume that $\varnothing' = A \cup B$ for some superlow c.e. sets A and B with $A \cap B = \varnothing$. Then there exist superlow c.e. sets $\widehat{A}$ and $\widehat{B}$ such that

$$K_0 = \{c(e,x) : x \in W_e,\ e \in \mathbb{N}\} = \widehat{A} \cup \widehat{B},\ \widehat{A} \cap \widehat{B} = \varnothing.$$

Let n be an integer such that $W_n \not\equiv_T A_0 \oplus B_0$ for all c.e. sets $A_0 \leqslant_T \widehat{A}$ and $B_0 \leqslant_T \widehat{B}$. We have

$$W_n \equiv_T \{c(n,x) : c(n,x) \in \widehat{A}\} \oplus \{c(n,x) : c(n,x) \in \widehat{B}\}.$$

That is a contradiction. □

3. A c.e. degree that cannot be split into two superlow degrees

For a binary function $\varphi_e^{(2)}$, we let

$$\widehat{\varphi}_e(x) = \begin{cases} \varphi_e^{(2)}(x, \min\{n : \varphi_e^{(2)}(x,n)\downarrow\}), & \text{if } \exists n\, [\varphi_e^{(2)}(x,n)\downarrow], \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

It follows from [12] that X is superlow iff there exists an e such that $X' = \widehat{\varphi}_e$ iff for every i there exists an e such that $W_i(X) = \widehat{\varphi}_e$.

Theorem 4. *There exists a c.e. degree* $\mathbf{c}$ *such that* $\mathbf{c} \neq \mathbf{a} \cup \mathbf{b}$ *for all superlow degrees* $\mathbf{a}$ *and* $\mathbf{b}$.

Proof. We will construct a c.e. set C and Turing functionals Θ, Ξ satisfying the following requirements:

$$R^{e,i,j}_{k,p,q} : C = \Phi_e(\Phi_i(C) \oplus \Phi_j(C))\,\&\, \Phi_i(C) \oplus \Phi_j(C) = \Phi_k(C) \Rightarrow$$
$$\Rightarrow H(\Phi_i(C)) \neq \widehat{\varphi}_p \vee S(\Phi_j(C)) \neq \widehat{\varphi}_q,$$

where $H(X) = \operatorname{dom}\Theta(X)$ and $S(X) = \operatorname{dom}\Xi(X)$.

The strategy for satisfying the requirement $R^{e,i,j}_{k,p,q}$ is similar to the strategy from Theorem 3. The main difference between these strategies is as follows. We need to diagonalize the condition

$$H(\Phi_i(C)) = \widehat{\varphi}_p \,\&\, S(\Phi_j(C)) = \widehat{\varphi}_q,$$

but we cannot immediately choose z and m as in Theorem 3 (that is why C cannot be made superlow). Instead, we are looking for z, m_p, m_q such that $\varphi_{p,s}(z, m_p)\downarrow$ and $\varphi_{q,s}(z, m_q)\downarrow$. After that we choose witnesses

$$x_0 < x_1 < \cdots < x_m,$$

where $m = m_p + m_q + 1$, for the diagonalization in such a way that adding each x_i, $0 < i \leqslant m$, to C does not affect the restraints imposed by x_{i-1}. Thus, by sequentially adding the witnesses $x_m, x_{m-1}, \ldots$ to C and changing the values of $H(\Phi_i(C))$ and $H(\Phi_j(C))$, we get

$$H(\Phi_i(C)) \neq \widehat{\varphi}_p \vee S(\Phi_j(C)) \neq \widehat{\varphi}_q.$$

In more detail, to satisfy the requirement $R^{e,i,j}_{k,p,q}$, we proceed according to the following instructions.

1. Choose integers z and m_p, m_q such that $\varphi_{p,s}(z, m_p)\downarrow$ and $\varphi_{q,s}(z, m_q)\downarrow$. Let $m = m_p + m_q + 1$.

2. Choose a finite sequence $x_0, \ldots, x_m$ such that

$$\Phi_{e,s}(\Phi_{i,s}(C_s) \oplus \Phi_{j,s}(C_s); x_n) = 0,$$

$$\Phi_{k,s}(C_s;x)\downarrow \text{ for each } x \leqslant \max\{u(\Phi_{i,s}(C_s)\oplus\Phi_{j,s}(C_s);e,x_n,s) : n\leqslant m\},$$
$$x_{n+1} > \max\{u(C_s;k,x,s) : x \leqslant u(\Phi_{i,s}(C_s)\oplus\Phi_{j,s}(C_s);e,x_n,s)\} \text{ for each } n<m.$$

3. We do not allow elements less than or equal to s to be enumerated into C.

4. We start sequentially enumerating the elements $x_m, x_{m-1}, \dots$ into C whenever

$$\widehat{\varphi}_{p,s}(z) = H_s(\Phi_{i,s}(C_s))(z) = \widehat{\varphi}^{(2)}_{q,s}(z) = S_s(\Phi_{j,s}(C_s))(z) = 1.$$

Taking into account the equality $m = m_p + m_q + 1$, we will have that the requirement $R^{e,i,j}_{k,p,q}$ is satisfied.

Construction

Stage 0. Let $\gamma_0 = \varnothing$, $C_0 = \varnothing$, $h_0(y) = r(y,0) = 0$ for each y. At each next stage, we assume that the value of each parameter remains the same, unless otherwise specified.

Stage $s+1 = c(e,i,j,k,p,q,v)+1$. Let $y = c(e,i,j,k,p,q)$, $z = c(y,h_s(y))$. Let

$$m = \begin{cases} m_p+m_q+1, & \text{if } m_p = \min\{n : \varphi_{p,s}(z,n)\downarrow\},\ m_q = \min\{n : \varphi_{q,s}(z,n)\downarrow\},\\ -1, & \text{if } \forall n\,[\varphi_{p,s}(z,n)\uparrow] \vee \forall n\,[\varphi_{q,s}(z,n)\uparrow]. \end{cases}$$

.

1. If $\gamma_s(z)\uparrow$ and there is a finite sequence $x_0,\dots,x_m$ such that

$$\Phi_{e,s}(\Phi_{i,s}(C_s)\oplus\Phi_{j,s}(C_s);x_n) = 0 \text{ and } x_n > \max\{r(w,s) : w<y\} \text{ for each } n\leqslant m,$$

where $\max\varnothing = -1$,

$$\Phi_{k,s}(C_s;x)\downarrow \text{ for each } x \leqslant \max\{u(\Phi_{i,s}(C_s)\oplus\Phi_{j,s}(C_s);e,x_n,s) : n\leqslant m\},$$
$$x_{n+1} > \max\{u(C_s;k,x,s) : x \leqslant u(\Phi_{i,s}(C_s)\oplus\Phi_{j,s}(C_s);e,x_n,s)\} \text{ for each } n<m,$$

then we define

$$\gamma_{s+1}(z) = c(m,x_0,\dots,x_m),$$
$$r(y,s+1) = s,$$
$$\Theta(\Phi_{i,s}(C_s);z) = \Xi(\Phi_{j,s}(C_s);z) = 0$$

with the use-values

$$\theta(\Phi_{i,s}(C_s);z) = \xi(\Phi_{j,s}(C_s);z) = s$$

respectively, and go to the next stage. We are assuming that

$$\theta(\Phi_{i,t}(C_t);z) = \xi(\Phi_{j,t}(C_t);z) = s$$

for each $t \geqslant s$.

Suppose $\gamma_s(z)\downarrow = c(m,x_0,\dots,x_m)$.

2. If there are $n\leqslant m$, $w<y$ such that $x_n \leqslant r(w,s)$, then we define

$$h_{s+1}(z) = h_s(z)+1$$

and go to the next stage.

3. Assume that

$$\varphi^{(2)}_{p,s}(z,m_p) = H_s(\Phi_{i,s}(C_s))(z),\ \varphi^{(2)}_{q,s}(z,m_q) = S_s(\Phi_{j,s}(C_s))(z).$$

If

$$z \notin H_s(\Phi_{i,s}(C_s)) \text{ or } z \notin S_s(\Phi_{j,s}(C_s)),$$

then we define

$$\Theta(\Phi_{i,s}(C_s); z) = 0 \text{ or } \Xi(\Phi_{j,s}(C_s); z) = 0$$

respectively. If

$$z \in H_s(\Phi_{i,s}(C_s)) \cap S_s(\Phi_{j,s}(C_s))$$

and there exists an $l \leqslant m$ such that $x_l \notin C_s$, then we define

$$C_{s+1} = C_s \cup \{x_{l_0}\}$$

for the greatest l_0 with $x_{l_0} \notin C$, and

$$r(y, s+1) = s.$$

End construction

It is not hard to see that the construction combines strategies of satisfying each R-requirement by the finite injury priority method. This completes the proof of the theorem. □

Note that we can easily combine the satisfying of the R-requirements from last theorem with the satisfying of the N-requirements form Theorem 3. Hence, there exists a low c.e. degree $\mathbf{c}$ such that $\mathbf{c} \neq \mathbf{a} \cup \mathbf{b}$ for all superlow degrees $\mathbf{a}$ and $\mathbf{b}$.

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