S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том xx, стр. xx–xx (20xx) УДК 517.957
DOI 10.33048/semi.20xx.16.xxx MSC 37K15

INTEGRATION OF THE LOADED MKDV-SINE-GORDON EQUATION WITH A SOURCE

U.A. HOITMETOV

Abstract. The work is devoted to finding the solution of the Cauchy problem for the loaded modified Korteweg-de Vries–sine-Gordon (mKdV-sG) equation with a source in the class of rapidly decreasing functions. The problem is solved by the inverse scattering method. Several examples are given illustrating the application of the obtained results.

Keywords: loaded mKdV–sine-Gordon equation, Jost solutions, inverse scattering problem, Gelfand-Levitan-Marchenko integral equation, evolution of the scattering data.

1. Introduction

The modified Korteweg-de Vries equation (mKdV)

$$u_t + 6u^2 u_x + u_{xxx} = 0$$

is one of the most well-known models fully integrable by the inverse scattering method [33]. The mKdV equation can be applied in many areas, including Alfven waves in a collisionless plasma [22], thin elastic rods [28], hyperbolic surfaces [31], etc.

Sine-Gordon (sG) equation [1, 37]

$$u_{xt} = \sin u$$

is also a fully integrable nonlinear evolution equation. This equation has numerous applications in differential geometry [9], magnetic flux propagation in Josephson junctions [13], propagation of deformations along DNA double helix [36] and many others. Despite its long history, the sG equation and its various extensions and

---

Hoitmetov, U.A., Integration of the loaded mKdV-sine-Gordon equation with a source.

Received March, 17, 2022, published March, xx, xxxx .

generalizations still remain the object of close attention of researchers [3, 4, 6, 7, 11, 16]. The problems of the theory of solitons have been considered in many monographs and articles, of which it should be noted [2, 20, 21, 30, 32, 38].

The MKdV and the sG equations admit breather solutions which are known to describe few-cycle-pulse (FCP) optical solitons [34]. The MKdV–sG equation

$$u_{xt} + cu_x^2 u_{xx} + au_{xxxx} = b \sin u$$

with parameters a, b, and c, gives a general $(1+1)$ – dimensional model describing FCP optical soliton propagation in two-component nonlinear media [5, 23, 25, 26, 27, 34, 35]. The MKdV–sG equation was studied for supersonic motion of a crowdion, nonlinear wave propagation in an infinite one-dimensional monatomic lattice, in which the anharmonic potential competes with the dislocation potential and which can be solved by the method of inverse scattering transformation, which is ahead of the light propagation of electromagnetic solitons in a nonequilibrium medium [5, 23, 25, 26, 27, 34, 35]. The fully integrable form of the mKdV–sG equation has the following form:

$$u_{xt} + a\left(\frac{3}{2}u_x^2 u_{xx} + u_{xxxx}\right) = b \sin u.$$

The integrable model of the mKdV–sG equation describes the propagation of ultrashort optical pulses in a nonlinear Kerr medium, which demonstrates its remarkable mathematical capabilities in describing the physics of solitons with several optical cycles [5, 23, 25, 26, 27, 34, 35]. If in this integrable mKdV-sG equation $b = 0$, then we get the mKdV equation, and if $c = 0$, then we get the sG equation.

In this paper, we will consider the following

$$\text{(1)} \quad \begin{aligned} u_{xt} + P(u(x_0,t))\left(\frac{3}{2}u_x^2 u_{xx} + u_{xxxx}\right) &= Q(u(x_1,t)) \sin u + \Omega(u(x_2,t)) u_{xx} \\ &+2\sum_{k=1}^{N}\sum_{j=0}^{m_k-1} C_{m_k-1}^{j}\left(f_{k1}^{j} f_{k1}^{m_k-1-j} - f_{k2}^{j} f_{k2}^{m_k-1-j}\right), \end{aligned}$$

$$\text{(2)} \quad L(t)f_k^j = \xi_k f_k^j + j f_k^{j-1}, \quad \operatorname{Im}\xi_k > 0,\ k = \overline{1, N},\ \ j = 0, 1, \ldots, m_k - 1,$$

system of equations. Where

$$C_n^l = \frac{n!}{(n-l)!l!}, \quad L(t) = \sigma_1 \frac{d}{dx} - \frac{u_x}{2}\sigma_2 + \frac{u_x}{2}\sigma_3, \quad u_x = \frac{\partial u(x,t)}{\partial x},$$

$$u_{xt} = \frac{\partial^2 u(x,t)}{\partial x \partial t}, \quad u_{xx} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad u_{xxxx} = \frac{\partial^4 u(x,t)}{\partial x^4},$$

and $P(u(x_0,t))$, $Q(u(x_1,t))$, $\Omega(u(x_2,t))$ are some polynomials in $u(x_0,t)$, $u(x_1,t)$, $u(x_2,t)$, respectively. Here and in what follows, $\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}$ are Pauli matrices.

If in equation (1) all polynomials $P(u(x_0,t))$, $Q(u(x_1,t))$, $\Omega(u(x_2,t))$ are polynomials of degree zero, then we obtain, in the general case, the mKdV - sG equation with variable coefficients and with a source, and if at least one of the polynomials $P(u(x_0,t))$, $Q(u(x_1,t))$, $\Omega(u(x_2,t))$ is not less than the first degree, then such equations are called loaded mKdV - sG equations. Loaded differential equations in the literature [24] are called equations containing in the coefficients or on the right side some functionals of the solution, in particular, the values of the solution or its derivatives on manifolds of lower dimension. The study of such equations is of interest both from the point of view of constructing a general theory of differential equations and from the point of view of applications. Among the papers devoted to loaded equations, the papers [14, 15, 18, 19, 24, 29] and others should be especially noted.

The system of equations (1)-(2) is considered under the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{3}$$

where the initial function $u_0(x)$ $(-\infty < x < \infty)$ has the following properties:

1)

$$u_0(x) \equiv 0 (mod\, 2\pi) \quad \text{at} \quad |x| \to \infty;$$
$$\int_{-\infty}^{\infty} \big((1+|x|)|u_0'(x)| + |u_0''(x)| + |u_0'''(x)| + |u_0''''(x)|\big) dx < \infty. \tag{4}$$

2) The operator

$$L(0) = \sigma_1 \frac{d}{dx} - \frac{u_x(x,0)}{2}\sigma_2 + \frac{u_x(x,0)}{2}\sigma_3$$

has no spectral singularities and has exactly N eigenvalues $\xi_1(0), \xi_2(0), \ldots, \xi_N(0)$ with multiplicities $m_1(0), m_2(0), \ldots, m_N(0)$ in the upper half-plane of the complex plane.

It is assumed that the vector-functions $f_k^j = \left(f_{k1}^j(x,t),\ f_{k2}^j(x,t)\right)^T$ are normalized by the conditions

$$\frac{1}{(m_k-1-s)!} \int_{-\infty}^{\infty} \left(f_{k1}^{m_k-1} f_{k2}^{m_k-1-s} + f_{k2}^{m_k-1} f_{k1}^{m_k-1-s}\right) dx = A_{m_k-1-s}^k(t), \tag{5}$$

where $A_{m_k-1-s}^k(t)$ are initially given continuous functions of t $(t \ge 0)$, $k = 1, 2, \ldots, N$, $s = 0, 1, \ldots, m_k - 1$.

Let the function $u(x,t)$ have sufficient smoothness and rather quickly tend to its limits as $x \to \pm\infty$, i.e.

$$u(x,t) \equiv 0 (mod\, 2\pi) \quad \text{at} \quad |x| \to \infty;$$
$$\int_{-\infty}^{\infty} \left((1+|x|)|u_x(x,t)| + \sum_{j=2}^{4} \left|\frac{\partial^j u(x,t)}{\partial x^j}\right|\right) dx < \infty. \tag{6}$$

The main goal of this work is to obtain a representation for the solution $u(x,t)$, $f_k^j(x,t)$, $k = 1, 2, \ldots, N$, $j = 0, 1, \ldots, m_k - 1$ of problem (1)-(6) within the framework of the inverse scattering method for the operator $L(t)$. The inverse scattering problem for the operator $L(t)$ on the entire axis was studied in [10, 17].

2. Preliminaries

Consider the system of equations

$$\begin{cases} v_{1x} + i\xi v_1 = -\frac{u'(x)}{2} v_2 \\ v_{2x} - i\xi v_2 = \frac{u'(x)}{2} v_1, \end{cases} \tag{7}$$

on the entire axis ($-\infty < x < \infty$), with the potential $u(x)$ satisfying the condition

$$u(x) \equiv 0(mod\, 2\pi) \text{ for } |x| \to \infty; \quad \int_{-\infty}^{\infty} (1+|x|)|u'(x)|dx < \infty. \tag{8}$$

In this section, we present information concerning the direct and inverse scattering problems for problem (7)-(8), in the case when the operator $L(t)$ has multiple eigenvalues.

Under condition (8), the system of equations (7) has Jost solutions with the following asymptotics

$$\left.\begin{aligned} \varphi &\sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} \\ \bar{\varphi} &\sim \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{i\xi x} \end{aligned}\right\} \text{ for } x \to -\infty; \qquad \left.\begin{aligned} \psi &\sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x} \\ \bar{\psi} &\sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} \end{aligned}\right\} \text{ for } x \to +\infty. \tag{9}$$

For real ξ, the pairs of vector functions $\{\varphi, \bar{\varphi}\}$ and $\{\psi, \bar{\psi}\}$ are pairs of linearly independent solutions for the system of equations (7), therefore

$$\varphi = a(\xi)\bar{\psi} + b(\xi)\psi, \qquad \bar{\varphi} = -\bar{a}(\xi)\psi + \bar{b}(\xi)\bar{\psi}, \tag{10}$$

where

$$\begin{gathered} a(\xi) = W\{\varphi, \psi\} \equiv \varphi_1\psi_2 - \varphi_2\psi_1, \\ b(\xi) = W\{\bar{\psi}, \varphi\}, \quad a(\xi)a(-\xi) + b(\xi)b(-\xi) = 1. \end{gathered} \tag{11}$$

The function $a(\xi)$ $(\bar{a}(\xi))$ allows an analytic continuation to the upper (lower) half-plane $\operatorname{Im}\xi > 0$, ($\operatorname{Im}\xi < 0$). For $|\xi| \to \infty$, $\operatorname{Im}\xi \geq 0$, the $a(\xi)$ function has the asymptotic behavior of $a(\xi) = 1 + O\left(\frac{1}{|\xi|}\right)$. In addition, the function $a(\xi)$ $(\bar{a}(\xi))$ can have only a finite number of zeros ξ_k $(\bar{\xi}_k)$ in the half-plane $\operatorname{Im}\xi > 0$ ($\operatorname{Im}\xi < 0$). The zeros ξ_k $(\bar{\xi}_k)$ of the function $a(\xi)$ $(\bar{a}(\xi))$ correspond to the eigenvalues of the operator L in the upper (lower) half-plane. It is assumed that the operator L has no spectral singularities, which means that the function $a(\xi)$ does not have real zeros, i.e. $a(\xi) \neq 0$, $\xi \in \mathbb{R}$. We denote by m_k the multiplicities of the eigenvalues ξ_k, ($k = 1, 2, \ldots, N$), respectively.

Functions

$$\overset{(s)}{\varphi}(x, \xi_k) \equiv \frac{\partial^s}{\partial \xi^s}\varphi(x, \xi)|_{\xi=\xi_k}, \quad s = 1, 2, \ldots, m_k - 1,$$

are called associated functions to the eigenfunction $\varphi(x, \xi_k)$. The associated functions to the eigenfunction $\psi(x, \xi_k)$ are defined similarly.

The eigenfunctions and associated functions satisfy the equations

$$L\overset{(s)}{\varphi}(x, \xi_k) = \xi_k \overset{(s)}{\varphi}(x, \xi_k) + s\overset{(s-1)}{\varphi}(x, \xi_k),$$

$$\overset{(0)}{\varphi}(x, \xi_k) \equiv \varphi(x, \xi_k), \quad k = 1, 2, \ldots, N, \;\; s = 0, 1, \ldots, m_k - 1.$$

According to the definition of eigenfunctions and associated functions, there exists a so-called chain of normalizing numbers $\{\chi_0^k,\ \chi_1^k, \ldots,\ \chi_{m_k-1}^k\}$, such that the relations hold

$$\overset{(l)}{\varphi}(x, \xi_k) = \sum_{\nu=0}^{l} \chi_{l-\nu}^k \frac{l!}{\nu!} \overset{(\nu)}{\psi}(x, \xi_k), \;\; k = 1, 2, \ldots, N, \;\; l = 0, 1, \ldots, m_k - 1. \tag{12}$$

The function ψ satisfies the following integral representation

$$\psi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x} + \int_x^\infty \mathbf{K}\,(x,s) e^{i\xi s} ds, \tag{13}$$

where $\mathbf{K}(x,s) = \begin{pmatrix} K_1(x,s) \\ K_2(x,s) \end{pmatrix}$. In representation (13), the kernel $\mathbf{K}(x,s)$ does not depend on ξ and is related to $u(x)$ by the equality

$$u'\,(x) = 4K_1\,(x,x)\,, \qquad (u'\,(x))^2 = 8\frac{dK_2\,(x,x)}{dx}\,. \tag{14}$$

The components $K_1(x,y)$, $K_2(x,y)$ of the kernel $\mathbf{K}(x,y)$, in representation (13), for $y > x$ are solutions of the Gelfand-Levitan-Marchenko integral equations

$$K_1(x,y) - F(x+y) + \int_x^\infty \int_x^\infty K_1(x,z)F(z+s)F(s+y)dsdz = 0,$$

$$K_2(x,y) + \int_x^\infty F(x+s)F(s+y)ds + \int_x^\infty \int_x^\infty K_2(x,z)F(z+s)F(s+y)dsdz = 0,$$

where

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^\infty r^+(\xi)e^{i\xi x}d\xi - i\sum_{k=1}^N \sum_{\nu=0}^{m_k-1} \chi^k_{m_k-\nu-1}\frac{1}{\nu!}\frac{d^\nu}{dz^\nu}\left[\frac{(z-\xi_k)^{m_k}}{a(z)}e^{izx}\right]|_{z=\xi_k},$$

$r^+(\xi) \equiv \frac{b(\xi)}{a(\xi)}$, $a(z)$ – analytic continuation of the function $a(\xi)$, $\operatorname{Im}\xi = 0$ to the upper half-plane of $\operatorname{Im} z > 0$. Now the potential $u(x)$ is determined from the first equality in (14).

The set

$$\left\{r^+(\xi),\, \xi \in \mathbb{R};\;\; \xi_k,\; \operatorname{Im}\xi_k > 0;\; \chi^k_j,\;\; k = 1, 2, \ldots, N,\;\; j = 0,\, 1, \ldots,\, m_k - 1\right\}$$

is called the scattering data for the system of equations (7). The direct scattering problem is to determine the scattering data from the potential $u(x)$, and the inverse problem is to reconstruct the potential $u(x)$ from the scattering data of the system of equations (7).

The following theorem is valid ([8], §6.2).

Theorem 1. *The scattering data of the operator L uniquely determine the operator L.*

In what follows, we will often use the results of the following lemma.

Lemma 1. *If the vector functions $Y(x,\zeta)$ and $Z(x,\eta)$ are solutions of the equations $LY = \zeta Y$ and $LZ = \eta Z$, then their components satisfy the equalities*

$$\frac{d}{dx}\left(y_1z_2 - y_2z_1\right) = -i\left(\zeta - \eta\right)\left(y_1z_2 + y_2z_1\right);$$

$$\frac{d}{dx}\left(y_1z_1 + y_2z_2\right) = -i\left(\zeta + \eta\right)\left(y_1z_1 - y_2z_2\right).$$

This lemma is proved by direct verification.

For convenience, in what follows we introduce the following notation

$$\overset{(n)}{\varphi}_k = \overset{(n)}{\varphi}(x,\xi_k), \quad k = 1, 2, \ldots, N.$$

Corollary 1. *Under the conditions of Lemma 1, the following equalities hold*

$$f_{k1}^{s}\varphi_1 - f_{k2}^{s}\varphi_2 = i\sum_{l=0}^{s}\frac{(-1)^l}{(\xi+\xi_k)^{l+1}}\frac{s\,!}{(s-l)\,!}\frac{d}{dx}V\left\{f_k^{s-l},\varphi\right\}, \tag{15}$$

and with $\xi \neq \xi_k$

$$f_{k1}^{s}\varphi_2 + f_{k2}^{s}\varphi_1 = i\sum_{l=0}^{s}\frac{(-1)^l}{(\xi_k-\xi)^{l+1}}\frac{s\,!}{(s-l)\,!}\frac{d}{dx}W\left\{f_k^{s-l},\varphi\right\}, \quad s=0,\,1,\ldots,\,m_k-1, \tag{16}$$

where $V\{f,g\} \equiv f_1g_1 + f_2g_2$.

Differentiating equality (15) n times by ξ and setting $\xi = \xi_k$, we obtain the following corollary.

Corollary 2. *The following equalities are valid*

$$f_{k1}^{s-1}\overset{(n)}{\varphi_2}(x,\xi_k) + f_{k2}^{s-1}\overset{(n)}{\varphi_1}(x,\xi_k) = \frac{n}{s}\left(f_{k1}^{s}\overset{(n-1)}{\varphi_2}(x,\xi_k) + f_{k2}^{s}\overset{(n-1)}{\varphi_1}(x,\xi_k)\right) + $$
$$+i\frac{d}{dx}W\left\{f_k^{s},\overset{(n)}{\varphi}(x,\xi_k)\right\}, \quad s=1,2,\ldots,m_k-1.$$

For convenience, in what follows we introduce the following notation

$$\overset{(n)}{\varphi_k} = \overset{(n)}{\varphi}(x,\xi_k), \quad k=1,2,\ldots,N.$$

Note that, according to (1), the following equalities hold:

$$\varphi_{k1}^2 + \varphi_{k2}^2 = \frac{i}{2\xi_k}\left(2u'\varphi_{k1}\varphi_{k2} + \left(\varphi_{k1}^2 - \varphi_{k2}^2\right)'\right), \tag{17}$$

$$\left(\varphi_{k1}\varphi_{k2}\right)' = \frac{u'}{2}\left(\varphi_{k1}^2 - \varphi_{k2}^2\right). \tag{18}$$

Let us prove the following lemma.

Lemma 2. *If* $\varphi_k = (\varphi_{k1},\,\varphi_{k2})^T$ *is an eigenvector function of the operator* L *with potential* $u(x)$ *corresponding to the eigenvalue* ξ_k*, then the equality*

$$\int_{-\infty}^{\infty}\left(\varphi_{k1}^2 + \varphi_{k2}^2\right)\sin u\,dx = 0$$

is true.

Proof. According to (17)-(18), we have

$$\int_{-\infty}^{\infty}\left(\varphi_{k1}^2 + \varphi_{k2}^2\right)\sin u\,dx$$
$$= \frac{i}{\xi_k}\left(\int_{-\infty}^{\infty}\left(\varphi_{k1}\varphi_{k2}\right)u'\sin u dx + \frac{1}{2}\int_{-\infty}^{\infty}\left(\varphi_{k1}^2 - \varphi_{k2}^2\right)'\sin u\,dx\right)$$
$$= \frac{i}{\xi_k}\left(-\int_{-\infty}^{\infty}\left(\varphi_{k1}\varphi_{k2}\right)d\cos u - \frac{1}{2}\int_{-\infty}^{\infty}\left(\varphi_{k1}^2 - \varphi_{k2}^2\right)u'\cos u\,dx\right) = 0.$$

□

3. Evolution of scattering data

Let the potential $u(x,t)$ in the system of equations (7) be the solution of the equation

$$u_{xt}+P(u(x_0,t))\left(\frac{3}{2}u_x^2u_{xx}+u_{xxxx}\right)=Q(u(x_1,t))\sin u+G, \tag{19}$$

where G tends to zero fast enough as $x\to\pm\infty$. In what follows, we will assume

$$G=\Omega(u(x_2,t))u_{xx}+2\sum_{k=1}^{N}\sum_{j=0}^{m_k-1}C_{m_k-1}^{j}\left(f_{k1}^{j}f_{k1}^{m_k-1-j}-f_{k2}^{j}f_{k2}^{m_k-1-j}\right). \tag{20}$$

Let's put

$$A=\begin{pmatrix}\frac{iQ(u(x_1,t))\cos u}{4\xi} & \frac{iQ(u(x_1,t))\sin u}{4\xi}\\ \frac{iQ(u(x_1,t))\sin u}{4\xi} & -\frac{iQ(u(x_1,t))\cos\ u}{4\xi}\end{pmatrix},$$

$$B=P(u(x_0,t))\begin{pmatrix}\left(-4i\xi^3+\frac{i\xi}{2}u_x^2\right) & \left(-2\xi^2u_x-i\xi u_{xx}+\frac{u_x^3}{4}+\frac{u_{xxx}}{2}\right)\\ \left(2\xi^2u_x-i\xi u_{xx}-\frac{u_x^3}{4}-\frac{u_{xxx}}{2}\right) & \left(4i\xi^3-\frac{i\xi}{2}u_x^2\right)\end{pmatrix}.$$

In the class of sufficiently smooth functions that are solutions of the equation $Lv=\xi v$, the following functional equalities

$$[L,A]\equiv LA-AL=-i\begin{pmatrix}0 & \frac{1}{2}Q(u(x_1,t))\sin u\\ \frac{1}{2}Q(u(x_1,t)\sin u & 0\end{pmatrix}, \tag{21}$$

$$\begin{aligned}&[L,B]\equiv LB-BL\\ &=i\begin{pmatrix}0 & \left(\frac{3}{4}u_x^2u_{xx}+\frac{1}{2}u_{xxxx}\right)P(u(x_0,t))\\ \left(\frac{3}{4}u_x^2u_{xx}+\frac{1}{2}u_{xxxx}\right)P(u(x_0,t)) & 0\end{pmatrix}.\end{aligned} \tag{22}$$

hold. From the definition of the operator $L(t)$, we get

$$\frac{\partial L}{\partial t}=i\begin{pmatrix}0 & \frac{u_{xt}}{2}\\ \frac{u_{xt}}{2} & 0\end{pmatrix}.$$

Combining the last equality with the formulas (21), (22) and with the equation (19), we get that the equation (19) is identical to the operator relation

$$\frac{\partial L}{\partial t}+[L,A]+[L,B]=iR, \tag{23}$$

where

$$R=\begin{pmatrix}0 & \frac{G}{2}\\ \frac{G}{2} & 0\end{pmatrix}.$$

Let $\varphi\left(x,\xi,t\right)$ be the Jost solution of the equation

$$L\varphi=\xi\varphi.$$

Differentiating this equality with respect to t we obtain

$$L_t\varphi+L\varphi_t=\xi\varphi_t, \tag{24}$$

Let's substitute L_t from (23) into (24), we get

$$\left(L-\xi I\right)\left(\varphi_t-A\varphi-B\varphi\right)=-iR\varphi. \tag{25}$$

as the result. We will look for the solution (25) in the form

$$\varphi_t - A\varphi - B\varphi = \alpha(x)\psi + \beta(x)\varphi. \tag{26}$$

To determine $\alpha(x)$ and $\beta(x)$ we get the equation

$$M\alpha_x\psi + M\beta_x\varphi = -R\varphi, \tag{27}$$

where

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

According to (9), we have

$$\hat{\psi}^T M\varphi = -\hat{\varphi}^T M\psi = a, \quad \hat{\psi}^T M\psi = \hat{\varphi}^T M\varphi = 0,$$

where $\hat{\varphi} = (\varphi_2 \ \ \varphi_1)^T$, $\hat{\psi} = (\psi_2 \ \ \psi_1)^T$. Multiplying (27) by $\hat{\varphi}^T$ and $\hat{\psi}^T$ we get

$$\alpha_x = \frac{\hat{\varphi}^T R\varphi}{a}, \qquad \beta_x = -\frac{\hat{\psi}^T R\varphi}{a}. \tag{28}$$

Based on (19), we have

$$\varphi_t - A\varphi - B\varphi \to \left(4i\xi^3 P(u(x_0,t)) - \frac{i}{4\xi}Q(u(x_1,t))\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x}, \ \ x \to -\infty.$$

Therefore, from (26) we deduce

$$\beta(x) \to 4i\xi^3 P(u(x_0,t)) - \frac{i}{4\xi}Q(u(x_1,t)), \quad \alpha(x) \to 0 \quad \text{at } \ x \to -\infty.$$

Solving the equations (28), we get

$$\alpha(x) = \frac{1}{a}\int_{-\infty}^{x} \hat{\varphi}^T R\varphi dx,$$

$$\beta(x) = -\frac{1}{a}\int_{-\infty}^{x} \hat{\psi}^T R\varphi dx + 4i\xi^3 P(u(x_0,t)) - \frac{i}{4\xi}Q(u(x_1,t)).$$

Thus, the equality (26) can be rewritten as

$$\begin{aligned}\varphi_t - A\varphi - B\varphi &= \frac{1}{a}\int_{-\infty}^{x} \hat{\varphi}^T R\varphi dx \cdot \psi \\ &+ \left(-\frac{1}{a}\int_{-\infty}^{x} \hat{\psi}^T R\varphi dx + 4i\xi^3 P(u(x_0,t)) - \frac{i}{4\xi}Q(u(x_1,t))\right)\varphi.\end{aligned} \tag{29}$$

Using (4) and passing in (29) to the limit at $x \to \infty$, we get

$$a_t = -\int_{-\infty}^{\infty} \hat{\psi}^T R\varphi \, dx \tag{30}$$

$$b_t = 8i\xi^3 P(u(x_0,t))b - \frac{iQ(u(x_1,t))}{2\xi}b + \frac{1}{a}\int_{-\infty}^{\infty} \hat{\varphi}^T R\varphi \, dx - \frac{b}{a}\int_{-\infty}^{\infty} \hat{\psi}^T R\varphi \, dx. \tag{31}$$

Therefore, for $\operatorname{Im}\xi = 0$ we have

$$\frac{dr^+}{dt} = \left(8i\xi^3 P(u(x_0,t)) - \frac{iQ(u(x_1,t))}{2\xi}\right) r^+ + \frac{1}{2a^2}\int_{-\infty}^{\infty} \left(G\varphi_2^2 + G\varphi_1^2\right) dx. \tag{32}$$

Lemma 3. *If vector functions* $\varphi(x,\xi)=\begin{pmatrix}\varphi_1(x,\xi)\\ \varphi_2(x,\xi)\end{pmatrix}$ *and* $\psi(x,\xi)=\begin{pmatrix}\psi_1(x,\xi)\\ \psi_2(x,\xi)\end{pmatrix}$ *are solutions of the equation (7), then their components satisfy the equalities*

$$\int_{-\infty}^{\infty} G\left(\varphi_1\psi_1+\varphi_2\psi_2\right)dx=0, \tag{33}$$

$$\int_{-\infty}^{\infty} G\left(\varphi_1^2+\varphi_2^2\right)dx=4i\xi a(\xi)b(\xi)\Omega(u(x_2,t)). \tag{34}$$

Proof. Let's introduce the following notation:

$$D=\Omega(u(x_2,t))u_{xx}\quad M=2\sum_{k=1}^{N}\sum_{j=0}^{m_k-1}C_{m_k-1}^{j}\left(f_{k1}^{j}f_{k1}^{m_k-1-j}-f_{k2}^{j}f_{k2}^{m_k-1-j}\right).$$

Using these notations, the expression (20) can be written as

$$G=D+M.$$

Let

$$M_k=2\sum_{j=0}^{m_k-1}C_{m_k-1}^{j}\left(f_{k1}^{j}f_{k1}^{m_k-1-j}-f_{k2}^{j}f_{k2}^{m_k-1-j}\right)\left(\varphi_1\psi_1+\varphi_2\psi_2\right),$$

then $M_k=M_{k1}+M_{k2}$, where

$$M_{k1}=\sum_{j=0}^{m_k-1}C_{m_k-1}^{j}\left[\left(f_{k1}^{j}\varphi_1+f_{k2}^{j}\varphi_2\right)\left(f_{k1}^{m_k-1-j}\psi_1-f_{k2}^{m_k-1-j}\psi_2\right)\right]+$$
$$+\sum_{j=0}^{m_k-1}C_{m_k-1}^{j}\left[\left(f_{k1}^{j}\varphi_1-f_{k2}^{j}\varphi_2\right)\left(f_{k1}^{m_k-1-j}\psi_1+f_{k2}^{m_k-1-j}\psi_2\right)\right],$$
$$M_{k2}=\sum_{j=0}^{m_k-1}C_{m_k-1}^{j}\left[\left(f_{k1}^{j}\varphi_2+f_{k2}^{j}\varphi_1\right)\left(f_{k1}^{m_k-1-j}\psi_2-f_{k2}^{m_k-1-j}\psi_1\right)\right]+$$
$$+\sum_{j=0}^{m_k-1}C_{m_k-1}^{j}\left[\left(f_{k1}^{j}\varphi_2-f_{k2}^{j}\varphi_1\right)\left(f_{k1}^{m_k-1-j}\psi_2+f_{k2}^{m_k-1-j}\psi_1\right)\right].$$

According to (15), we have

$$M_{k1}=\sum_{j=0}^{m_k-1}C_{m_k-1}^{j}\Big[V\{f_k^j,\varphi\}$$
$$\times\sum_{s=0}^{m_k-1-j}\frac{i(-1)^s}{(\xi+\xi_k)^{s+1}}\frac{(m_k-1-j)\,!}{(m_k-1-j-s)\,!}\frac{d}{dx}V\{f_k^{m_k-1-j-s},\psi\}\Big]$$
$$+\sum_{j=0}^{m_k-1}C_{m_k-1}^{j}\left[V\left\{f_k^{m_k-1-j},\psi\right\}\sum_{l=0}^{j}\frac{i(-1)^l}{(\xi+\xi_k)^{l+1}}\frac{j\,!}{(j-l)\,!}\frac{d}{dx}V\left\{f_k^{j-l},\varphi\right\}\right]. \tag{35}$$

It is easy to show that on the right side (35) the coefficients of

$$V\{f_k^q,\varphi\}\frac{d}{dx}V\{f_k^r,\psi\}$$

and

$$V\left\{f_k^r,\psi\right\}\frac{d}{dx}V\left\{f_k^q,\varphi\right\},\quad 0\le r+q\le m_k-1$$

are the same, so Q_{k1} is a linear combination of expressions of the form

$$\frac{d}{dx}\left(V\left\{f_k^r,\psi\right\}V\left\{f_k^q,\varphi\right\}\right),\quad 0\le r+q\le m_k-1,$$

and hence

$$\int_{-\infty}^{\infty}M_{k1}dx=0.$$

Similarly, using (16), we get

$$\int_{-\infty}^{\infty}M_{k2}dx=0.$$

Using the formulas (6), (7), (9) and (10), we have

$$\int_{-\infty}^{\infty}D\left(\varphi_1\psi_1+\varphi_2\psi_2\right)dx=\Omega(u(x_0,t))\int_{-\infty}^{\infty}\left(\varphi_1\psi_1u_{xx}+\varphi_2\psi_2u_{xx}\right)dx$$

$$=\lim_{R\to\infty}\Omega(u(x_0,t))\left[\left(\varphi_1\psi_1+\varphi_2\psi_2\right)u_x(x,t)\right]\Big|_{-R}^{R}$$

$$-\Omega(u(x_0,t))\int_{-\infty}^{\infty}\left(\varphi_1'\psi_1u_x+\varphi_1\psi_1'u_x+\varphi_2'\psi_2u_x+\varphi_2\psi_2'u\right)dx$$

$$=\lim_{R\to\infty}\Omega(u(x_0,t))\left[\left(\varphi_1\psi_1+\varphi_2\psi_2\right)u_x(x,t)\right]\Big|_{-R}^{R}$$

$$-\Omega(u(x_0,t))\int_{-\infty}^{\infty}2\big(\varphi_1'(\psi_2'-i\xi\psi_2)-\psi_1'(\varphi_2'+i\xi\varphi_2)-\varphi_2'(\psi_1'+i\xi\psi_1)-\psi_2'(\varphi_1'+i\xi\varphi_1)\big)dx$$

$$=\lim_{R\to\infty}\Omega(u(x_0,t))\left[\left(\varphi_1\psi_1+\varphi_2\psi_2\right)u_x(x,t)\right]\Big|_{-R}^{R}$$

$$-2i\xi\Omega(u(x_0,t))\int_{-\infty}^{\infty}\left(\varphi_1\psi_2+\psi_1\varphi_2\right)'dx$$

$$=\lim_{R\to\infty}\Omega(u(x_0,t))\left[\left(\varphi_1\psi_1+\varphi_2\psi_2\right)u_x(x,t)-2i\xi\left(\varphi_1\psi_2+\psi_1\varphi_2\right)\right]\Big|_{-R}^{R}=0.$$

The following integral is calculated in the same way:

$$\int_{-\infty}^{\infty}D\left(\varphi_1^2+\varphi_2^2\right)dx=\Omega(u(x_0,t))\int_{-\infty}^{\infty}\left(\varphi_1^2u_{xx}+\varphi_2^2u_{xx}\right)dx$$

$$=\lim_{R\to\infty}\Omega(u(x_0,t))\left[(\varphi_1^2+\varphi_2^2)u_x(x,t)\right]\Big|_{-R}^{R}-\Omega(u(x_0,t))\int_{-\infty}^{\infty}2(\varphi_1\varphi_1'u_x+\varphi_2\varphi_2'u_x)dx$$

$$=\lim_{R\to\infty}\Omega(u(x_0,t))\left[\left(\varphi_1^2+\varphi_2^2\right)u_x(x,t)\right]\Big|_{-R}^{R}$$

$$-\Omega(u(x_0,t))\int_{-\infty}^{\infty}\left(2\varphi_1'\left(2\varphi_2'-2i\xi\varphi_2\right)+2\varphi_2'\left(-2\varphi_1'-2i\xi\varphi_1\right)\right)dx$$

$$=\lim_{R\to\infty}\Omega(u(x_0,t))\left[\left(\varphi_1^2+\varphi_2^2\right)u_x(x,t)\right]\Big|_{-R}^{R}+4i\xi\Omega(u(x_0,t))\int_{-\infty}^{\infty}\left(\varphi_1\varphi_2\right)'dx$$

$$=4i\xi a(\xi)b(\xi)\Omega(u(x_0,t)).$$

Taking into account the form of G, we obtain the required equalities. □

Corollary 3. *According to Lemma 3 and the equality* (30), $a_t=0$, *so*

(36) $$m_k(t)=m_k(0),\quad \xi_k(t)=\xi_k(0),\quad k=1,2,\dots,N.$$

According to (32) and (34) for $\operatorname{Im}\xi = 0$ we have

$$\frac{dr^+}{dt} = \left(8i\xi^3 P(u(x_0,t)) - \frac{iQ(u(x_1,t))}{2\xi} + 2i\xi\Omega(u(x_2,t))\right) r^+. \tag{37}$$

Let us turn to finding the evolution of the normalization chain $\left\{\chi_0^n,\ \chi_1^n, \dots\ \chi_{m_n-1}^n\right\}$ corresponding to the eigenvalue ξ_n, $n = 1,2,\dots,N$. To do this, we rewrite the equality (29) in the following form

$$\begin{aligned} &\varphi_t - A\varphi - B\varphi \\ &= -\frac{1}{2a}\left(\int_{-\infty}^{x} G\left(\varphi_1\psi_1 + \varphi_2\psi_2\right) dx\ \varphi - \int_{-\infty}^{x} G\left(\varphi_1^2 + \varphi_2^2\right) dx\ \psi\right) \\ &\quad + \left(4i\xi^3 P(u(x_0,t)) - \frac{iQ(u(x_1,t))}{4\xi}\right)\varphi. \end{aligned} \tag{38}$$

Just as in the proof of Lemma 3, we represent

$$G\left(\varphi_1\psi_1 + \varphi_2\psi_2\right) = D\left(\varphi_1\psi_1 + \varphi_2\psi_2\right) + \sum_{k=1}^{N}(M_{k1} + M_{k2}),$$

where

$$D = \Omega(u(x_2,t))u_{xx},$$

$$M_{k1} = \sum_{j=0}^{m_k-1}\sum_{s=0}^{m_k-1-j} iC_{m_k-1}^{j}\frac{(-1)^s}{(\xi+\xi_k)^{s+1}}\frac{(m_k-1-j)\,!}{(m_k-1-j-s)\,!}\frac{d}{dx}\left(V\left\{f_k^{m_k-1-j-s},\psi\right\}V\left\{f_k^{j},\varphi\right\}\right),$$

$$M_{k2} = \sum_{j=0}^{m_k-1}\sum_{s=0}^{m_k-1-j} iC_{m_k-1}^{j}\frac{(-1)^s}{(\xi_k-\xi)^{s+1}}\frac{(m_k-1-j)\,!}{(m_k-1-j-s)\,!}\frac{d}{dx}\left(W\left\{f_k^{m_k-1-j-s},\psi\right\}W\left\{f_k^{j},\varphi\right\}\right).$$

Likewise

$$G\left(\varphi_1^2 + \varphi_2^2\right) = D\left(\varphi_1^2 + \varphi_2^2\right) + \sum_{k=1}^{N}(F_{k1} + F_{k2}),$$

where

$$D = \Omega(u(x_2,t))u_{xx},$$

$$F_{k1} = \sum_{j=0}^{m_k-1}\sum_{s=0}^{m_k-1-j} iC_{m_k-1}^{j}\frac{(-1)^s}{(\xi+\xi_k)^{s+1}}\frac{(m_k-1-j)\,!}{(m_k-1-j-s)\,!}\frac{d}{dx}\left(V\left\{f_k^{m_k-1-j-s},\varphi\right\}V\left\{f_k^{j},\varphi\right\}\right),$$

$$F_{k2} = \sum_{j=0}^{m_k-1}\sum_{s=0}^{m_k-1-j} iC_{m_k-1}^{j}\frac{(-1)^s}{(\xi_k-\xi)^{s+1}}\frac{(m_k-1-j)\,!}{(m_k-1-j-s)\,!}\frac{d}{dx}\left(W\left\{f_k^{m_k-1-j-s},\varphi\right\}W\left\{f_k^{j},\varphi\right\}\right).$$

First, using the formulas (4), (7), we calculate the following integral:

$$\begin{aligned} &\int_{-\infty}^{x} D\left(\varphi_1^2 + \varphi_2^2\right) dx = \Omega(u(x_2,t))\int_{-\infty}^{x} u_{xx}\left(\varphi_1^2 + \varphi_2^2\right) dx \\ &= \Omega(u(x_2,t))\lim_{R\to\infty}\left(u_x\left(\varphi_1^2+\varphi_2^2\right)\right)\Big|_{-R}^{x} - \Omega(u(x_2,t))\int_{-\infty}^{x} 2u_x\left(\varphi_1\varphi_1' + \varphi_2\varphi_2'\right) dx \\ &= \Omega(u(x_2,t))u_x\left(\varphi_1^2+\varphi_2^2\right) - 2\Omega(u(x_2,t))\int_{-\infty}^{x}\left(2\varphi_1'(\varphi_2' - i\xi\varphi_2) - 2\varphi_2'(\varphi_1' + i\xi\varphi_1)\right) dx \\ &= \Omega(u(x_2,t))u_x\left(\varphi_1^2+\varphi_2^2\right) + 4i\xi\Omega(u(x_2,t))\int_{-\infty}^{x}\left(\varphi_1\varphi_2\right)' dx \end{aligned}$$

$$= \Omega(u(x_2,t))u_x\left(\varphi_1^2+\varphi_2^2\right) + 4i\xi\Omega(u(x_2,t))\varphi_1\varphi_2.$$

The validity of the following equality

$$\int_{-\infty}^{x} D\left(\varphi_1\psi_1+\varphi_2\psi_2\right)dx = \Omega(u(x_2,t))u_x\left(\varphi_1\psi_1+\varphi_2\psi_2\right) + 4i\xi\Omega(u(x_2,t))\psi_1\varphi_2$$

is shown in the same way.

Taking into account the expression for $a(\xi)$ in terms of the Wronskian of the Jost solutions and applying the equality (15) in the opposite direction, we obtain

$$\int_{-\infty}^{x} M_{k1}dx\varphi - \int_{-\infty}^{x} F_{k1}dx\psi = a(\xi)\sum_{j=0}^{m_k-1} C_{m_k-1}^{j}\int_{-\infty}^{x}\left(f_{k1}^{m_k-1-j}\varphi_1 - f_{k2}^{m_k-1-j}\varphi_2\right)dx\tilde{f}_k^j,$$

where $\tilde{f}_k^j = \left(f_{k2}^j,\ -f_{k1}^j\right)^T$. In the same way, applying the equality (16) in the opposite direction, we get

$$\int_{-\infty}^{x} M_{k2}dx\varphi - \int_{-\infty}^{x} F_{k2}dx\psi = a(\xi)\sum_{j=0}^{m_k-1} C_{m_k-1}^{j}\int_{-\infty}^{x}\left(f_{k1}^{m_k-1-j}\varphi_2 + f_{k2}^{m_k-1-j}\varphi_1\right)dx f_k^j.$$

Based on the above, the equality (38) can be rewritten as

$$\begin{aligned}\varphi_t - A\varphi - B\varphi = {} & \frac{1}{2}\Omega(u(x_2,t))u_x\begin{pmatrix}-\varphi_2\\ \varphi_1\end{pmatrix} + 2i\xi\Omega(u(x_2,t))\begin{pmatrix}0\\ \varphi_2\end{pmatrix}\\ & + \left(4i\xi^3P(u(x_0,t)) - \frac{iQ(u(x_1,t))}{4\xi}\right)\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\\ & - \frac{1}{2}\sum_{k=1}^{N}\sum_{j=0}^{m_k-1} C_{m_k-1}^{j}\int_{-\infty}^{x}\left(f_{k1}^{m_k-1-j}\varphi_1 - f_{k2}^{m_k-1-j}\varphi_2\right)dx\tilde{f}_k^j\\ & - \frac{1}{2}\sum_{k=1}^{N}\sum_{j=0}^{m_k-1} C_{m_k-1}^{j}\int_{-\infty}^{x}\left(f_{k1}^{m_k-1-j}\varphi_2 + f_{k2}^{m_k-1-j}\varphi_1\right)dx f_k^j \qquad (39)\end{aligned}$$

Differentiating the identity (39) m_n-1 times with respect to ξ and setting $\xi=\xi_n$, we get

$$\begin{aligned}&\frac{\partial \overset{(m_n-1)}{\varphi_n}}{\partial t} - \frac{iQ(u(x_1,t))}{4}V\sum_{l=0}^{m_n-1} C_{m_n-1}^{l}\frac{(-1)^{m_n-1-l}(m_n-1-l)\,!}{\xi^{m_n-l}}\overset{(l)}{\varphi_n}\\ &-B_0P(u(x_0,t))\overset{(m_n-1)}{\varphi_n} - (m_n-1)B_1P(u(x_0,t))\overset{(m_n-2)}{\varphi_n}\\ &-\frac{(m_n-1)(m_n-2)}{2}B_2P(u(x_0,t))\overset{(m_n-3)}{\varphi_n}\\ &-\frac{(m_n-1)(m_n-2)(m_n-3)}{6}B_3P(u(x_0,t))\overset{(m_n-4)}{\varphi_n}\\ &= \frac{1}{2}\Omega(u(x_2,t))u_x\begin{pmatrix}-\overset{(m_n-1)}{\varphi_{n2}}\\ \overset{(m_n-1)}{\varphi_{n1}}\end{pmatrix} + 2i(m_n-1)\Omega(u(x_2,t))\begin{pmatrix}0\\ \overset{(m_n-2)}{\varphi_{n2}}\end{pmatrix}\\ &+2i\xi_n\Omega(u(x_2,t))\begin{pmatrix}0\\ \overset{(m_n-1)}{\varphi_{n2}}\end{pmatrix} + 4i\xi_n^3P(u(x_0,t))\overset{(m_n-1)}{\varphi_n}\\ &+12i\xi_n^2P(u(x_0,t))(m_n-1)\overset{(m_n-2)}{\varphi_n} + 12i\xi_nP(u(x_0,t))(m_n-1)(m_n-2)\overset{(m_n-3)}{\varphi_n}\end{aligned}$$

$$+4i(m_n-1)(m_n-2)(m_n-3)P(u(x_0,t))\overset{(m_n-4)}{\varphi_n}$$
$$-\frac{iQ(u(x_1,t))}{4}\sum_{l=0}^{m_n-1}C_{m_n-1}^{l}\frac{(-1)^{m_n-1-l}\,(m_n-1-l)\,!}{\xi^{m_n-l}}\overset{(l)}{\varphi_n}-\frac{1}{2}\left(R_1(x)+R_2(x)\right)$$
$$-\frac{1}{2}\sum_{j=0}^{m_n-1}C_{m_n-1}^{j}\int_{-\infty}^{x}(f_{n1}^{m_n-1-j}\overset{(m_n-1)}{\varphi_{n2}}+f_{n2}^{m_n-1-j}\overset{(m_n-1)}{\varphi_{n1}})dxf_n^j \tag{40}$$

where

$$V=\begin{pmatrix}\cos u & \sin u\\ \sin u & -\cos u\end{pmatrix},\quad B_l=\frac{d^l}{d\xi^l}\left.B\right|_{\xi=\xi_n},\quad l=0,\ 1,\ 2,\ 3,$$
$$R_1(x)=\sum_{k=1}^{N}\sum_{j=0}^{m_k-1}C_{m_k-1}^{j}\int_{-\infty}^{x}\left(f_{k1}^{m_k-1-j}\overset{(m_n-1)}{\varphi_{n1}}-f_{k2}^{m_k-1-j}\overset{(m_n-1)}{\varphi_{n2}}\right)dx\tilde{f}_k^j,$$
$$R_2(x)=\sum_{\substack{k=1\\k\neq n}}^{N}\sum_{j=0}^{m_k-1}C_{m_k-1}^{j}\int_{-\infty}^{x}\left(f_{k1}^{m_k-1-j}\overset{(m_n-1)}{\varphi_{n2}}+f_{k2}^{m_k-1-j}\overset{(m_n-1)}{\varphi_{n1}}\right)dxf_k^j.$$

Note that according to (15) and (16)

$$\lim_{x\to\infty}R_1(x)=\lim_{x\to\infty}R_2(x)=0. \tag{41}$$

Using Corollary 2 of Lemma 1, we can show that

$$\sum_{j=0}^{m_n-1}C_{m_n-1}^{j}\int_{-\infty}^{x}\left(f_{n1}^{m_k-1-j}\overset{(m_n-1)}{\varphi_{n2}}+f_{n2}^{m_k-1-j}\overset{(m_n-1)}{\varphi_{n1}}\right)dxf_n^j=$$

(42)
$$=\sum_{j=0}^{m_n-1}C_{m_n-1}^{j}\int_{-\infty}^{x}\left(f_{n1}^{m_n-1}\overset{(m_n-1-j)}{\varphi_{n2}}+f_{n2}^{m_n-1}\overset{(m_n-1-j)}{\varphi_{n1}}\right)dxf_n^j+\sum_{j=1}^{m_n-1}Q_j(x)f_n^j,$$

where $Q_j(x)$ is a linear combination of expressions of the form $W\{f_n^r,\overset{(q)}{\varphi_n}\},\ (rq=j)$, and therefore

$$\lim_{x\to\infty}Q_j(x)=0. \tag{43}$$

According to the definition of the functions f_n^s and $\overset{(s)}{\varphi_n}$ $s=0,1,2,\ldots,m_n-1$, there are numbers $d_0,d_1,\ldots,d_{m_n-1}$ such that

$$f_n^j=\sum_{s=0}^{j}C_j^s d_{j-s}\overset{(s)}{\varphi_n},\quad j=0,1,2,\ldots,m_n-1.$$

Therefore, the following is true

$$\sum_{j=0}^{m_n-1}C_{m_n-1}^{j}\int_{-\infty}^{x}\left(f_{n1}^{m_n-1}\overset{(m_n-1-j)}{\varphi_{n2}}+f_{n2}^{m_n-1}\overset{(m_n-1-j)}{\varphi_{n1}}\right)dxf_n^j$$
$$=\sum_{j=0}^{m_n-1}C_{m_n-1}^{j}\int_{-\infty}^{x}\left(f_{n1}^{m_n-1}\overset{(m_n-1-j)}{\varphi_{n2}}+f_{n2}^{m_n-1}\overset{(m_n-1-j)}{\varphi_{n1}}\right)dx\sum_{s=0}^{j}C_j^s d_{j-s}\overset{(s)}{\varphi_n}$$

$$= \sum_{s=0}^{m_n-1} \sum_{j=s}^{m_n-1} C_{m_n-1}^{j} C_j^s d_{j-s} \int_{-\infty}^{x} \left(f_{n1}^{m_n-1} \overset{(m_n-1-j)}{\varphi_{n2}} + f_{n2}^{m_n-1} \overset{(m_n-1-j)}{\varphi_{n1}} \right) dx \, \overset{(s)}{\varphi_n}$$

$$= \sum_{s=0}^{m_n-1} C_{m_n-1}^{s} \left(\sum_{k=0}^{m_n-1-s} C_{m_n-1-s}^{k} d_{m_n-1-k-s} \int_{-\infty}^{x} \left(f_{n1}^{m_n-1} \overset{(k)}{\varphi_{n2}} + f_{n2}^{m_n-1} \overset{(k)}{\varphi_{n1}} \right) dx \right) \overset{(s)}{\varphi_n}$$

$$= \sum_{s=0}^{m_n-1} C_{m_n-1}^{s} \cdot \int_{-\infty}^{x} \left(f_{n1}^{m_n-1} f_{n2}^{m_n-1-s} + f_{n2}^{m_n-1} f_{n1}^{m_n-1-s} \right) dx \, \overset{(s)}{\varphi_n} .$$

Thus, according to (42), the equality (40) can be rewritten as

$$\frac{\partial \overset{(m_n-1)}{\varphi_n}}{\partial t} - \frac{iq(t)}{4} V \sum_{l=0}^{m_n-1} C_{m_n-1}^{l} \frac{(-1)^{m_n-1-l}\,(m_n-1-l)\,!}{\xi^{m_n-l}} \overset{(l)}{\varphi_n} - B_0 P(u(x_0,t)) \overset{(m_n-1)}{\varphi_n}$$

$$-(m_n-1) B_1 P(u(x_0,t)) \overset{(m_n-2)}{\varphi_n} - \frac{(m_n-1)(m_n-2)}{2} B_2 P(u(x_0,t)) \overset{(m_n-3)}{\varphi_n}$$

$$-\frac{(m_n-1)(m_n-2)(m_n-3)}{6} B_3 P(u(x_0,t)) \overset{(m_n-4)}{\varphi_n} = \frac{1}{2}\Omega(u(x_2,t)) u_x \begin{pmatrix} -\overset{(m_n-1)}{\varphi_{n2}} \\ \overset{(m_n-1)}{\varphi_{n1}} \end{pmatrix}$$

$$+2i(m_n-1)\Omega(u(x_2,t)) \begin{pmatrix} 0 \\ \overset{(m_n-2)}{\varphi_{n2}} \end{pmatrix} + 2i\xi_n \Omega(u(x_2,t)) \begin{pmatrix} 0 \\ \overset{(m_n-1)}{\varphi_{n2}} \end{pmatrix} + 4i\xi_n^3 P(u(x_0,t)) \overset{(m_n-1)}{\varphi_n}$$

$$+12i\xi_n^2 P(u(x_0,t))(m_n-1) \overset{(m_n-2)}{\varphi_n} + 12i\xi_n P(u(x_0,t))(m_n-1)(m_n-2) \overset{(m_n-3)}{\varphi_n}$$

$$+4i(m_n-1)(m_n-2)(m_n-3) P(u(x_0,t)) \overset{(m_n-4)}{\varphi_n}$$

$$-\frac{iq(t)}{4} \sum_{l=0}^{m_n-1} C_{m_n-1}^{l} \frac{(-1)^{m_n-1-l}\,(m_n-1-l)\,!}{\xi^{m_n-l}} \overset{(l)}{\varphi_n} - \frac{1}{2}\left(R_1(x) + R_2(x)\right)$$

(44)

$$-\frac{1}{2} \sum_{l=0}^{m_n-1} C_{m_n-1}^{l} \int_{-\infty}^{x} (f_{n1}^{m_n-1} f_{n2}^{m_n-1-l} + f_{n2}^{m_n-1} f_{n1}^{m_n-1-l}) dx \, \overset{(l)}{\varphi_n} - \frac{1}{2} \sum_{j=1}^{m_n-1} Q_j(x) f_n^j$$

Using (4), (12), (41), (43) we pass in the equality (45) to the redistribution at $x \to \infty$. Equating the coefficients of $\begin{pmatrix} 0 \\ 1 \end{pmatrix} (ix)^l \cdot e^{i\xi_n x}$, with $l = m_n - 1,\, m_n - 2, \ldots, 0$ we get

$$\frac{d\chi_0^n}{dt} = \left(8i\xi_n^3 P(u(x_0,t)) - \frac{iQ(u(x_1,t))}{2\xi_n} + 2i\xi_n \Omega(u(x_2,t)) - \frac{1}{2} A_0^n(t) \right) \chi_0^n,$$

$$\frac{d\chi_1^n}{dt} = \left(8i\xi_n^3 P(u(x_0,t)) - \frac{iQ(u(x_1,t))}{2\xi_n} + 2i\xi_n \Omega(u(x_2,t)) - \frac{1}{2} A_0^n(t) \right) \chi_1^n$$
$$+ \left(24i\xi_n^2 P(u(x_0,t)) + \frac{iQ(u(x_1,t))}{2\xi_n^2} + 2i\Omega(u(x_2,t)) - \frac{1}{2} A_1^n(t) \right) \chi_0^n,$$

$$\frac{d\chi_2^n}{dt} = \left(8i\xi_n^3 P(u(x_0,t)) - \frac{iQ(u(x_1,t))}{2\xi_n} + 2i\xi_n \Omega(u(x_2,t)) - \frac{1}{2} A_0^n(t) \right) \chi_2^n$$
$$+ \left(24i\xi_n^2 P(u(x_0,t)) + \frac{iQ(u(x_1,t))}{2\xi_n^2} + 2i\Omega(u(x_2,t)) - \frac{1}{2} A_1^n(t) \right) \chi_1^n$$
$$+ \left(24i\xi_n P(u(x_0,t)) - \frac{iQ(u(x_1,t))}{2\xi_n^3} - \frac{1}{2} A_2^n(t) \right) \chi_0^n,$$

$$\frac{d\chi_3^n}{dt} = \left(8i\xi_n^3 P(u(x_0,t)) - \frac{iQ(u(x_1,t))}{2\xi_n} + 2i\xi_n\Omega(u(x_2,t)) - \frac{1}{2}A_0^n(t)\right)\chi_3^n$$
$$+\left(24i\xi_n^2 P(u(x_0,t)) + \frac{iQ(u(x_1,t))}{2\xi_n^2} + 2i\Omega(u(x_2,t)) - \frac{1}{2}A_1^n(t)\right)\chi_2^n$$
$$+\left(24i\xi_n P(u(x_0,t)) - \frac{iQ(u(x_1,t))}{2\xi_n^3} - \frac{1}{2}A_2^n(t)\right)\chi_1^n$$
$$+\left(8iP(u(x_0,t)) + \frac{iQ(u(x_1,t))}{2\xi_n^4} - \frac{1}{2}A_3^n(t)\right)\chi_0^n,$$
$$\frac{d\chi_{m_n-1-\nu}^n}{dt} = \left(8i\xi_n^3 P(u(x_0,t)) + 2i\xi_n\Omega(u(x_2,t))\right)\chi_{m_n-1-\nu}^n$$
$$+\left(24i\xi_n^2 P(u(x_0,t)) + 2i\Omega(u(x_2,t))\right)\chi_{m_n-2-\nu}^n + 24i\xi_n P(u(x_0,t))\chi_{m_n-3-\nu}^n$$
$$+8iP(u(x_0,t))\chi_{m_n-4-\nu}^n - \sum_{s=\nu}^{m_n-1}\left(\frac{(-1)^{m_n-1-s} iQ(u(x_1,t))}{2\xi_n^{m_n-s}} + \frac{1}{2}A_{m_n-1-s}^n(t)\right)\chi_{s-\nu}^n,$$
$$n = 1, 2, \ldots, N, \quad \nu = m_n - 1, m_n - 2, \ldots, 0.$$

Thus, the following theorem is proved.

Theorem 2. *If the functions $u(x,t)$, $f_k^j(x,t)$, $k = 1, 2, \ldots, N$, $j = 0, 1, \ldots, m_k - 1$ are a solution to the (1)-(6) problem, then the scattering data of the operator $L(t)$ change with respect to t as follows*

$$\frac{dr^+}{dt} = \left(8i\xi^3 P(u(x_0,t)) - \frac{iQ(u(x_1,t))}{2\xi} + 2i\xi\Omega(u(x_2,t))\right)r^+ \quad (\operatorname{Im}\xi = 0)$$
$$m_n(t) = m_n(0), \qquad \frac{d\xi_n}{dt} = 0,$$
$$\frac{d\chi_0^n}{dt} = \left(8i\xi_n^3 P(u(x_0,t)) - \frac{iQ(u(x_1,t))}{2\xi_n} + 2i\xi_n\Omega(u(x_2,t)) - \frac{1}{2}A_0^n(t)\right)\chi_0^n,$$
$$\frac{d\chi_1^n}{dt} = \left(8i\xi_n^3 P(u(x_0,t)) - \frac{iQ(u(x_1,t))}{2\xi_n} + 2i\xi_n\Omega(u(x_2,t)) - \frac{1}{2}A_0^n(t)\right)\chi_1^n$$
$$+\left(24i\xi_n^2 P(u(x_0,t)) + \frac{iQ(u(x_1,t))}{2\xi_n^2} + 2i\Omega(u(x_2,t)) - \frac{1}{2}A_1^n(t)\right)\chi_0^n,$$
$$\frac{d\chi_2^n}{dt} = \left(8i\xi_n^3 P(u(x_0,t)) - \frac{iQ(u(x_1,t))}{2\xi_n} + 2i\xi_n\Omega(u(x_2,t)) - \frac{1}{2}A_0^n(t)\right)\chi_2^n$$
$$+\left(24i\xi_n^2 P(u(x_0,t)) + \frac{iQ(u(x_1,t))}{2\xi_n^2} + 2i\Omega(u(x_2,t)) - \frac{1}{2}A_1^n(t)\right)\chi_1^n$$
$$+\left(24i\xi_n P(u(x_0,t)) - \frac{iQ(u(x_1,t))}{2\xi_n^3} - \frac{1}{2}A_2^n(t)\right)\chi_0^n,$$
$$\frac{d\chi_3^n}{dt} = \left(8i\xi_n^3 P(u(x_0,t)) - \frac{iQ(u(x_1,t))}{2\xi_n} + 2i\xi_n\Omega(u(x_2,t)) - \frac{1}{2}A_0^n(t)\right)\chi_3^n$$
$$+\left(24i\xi_n^2 P(u(x_0,t)) + \frac{iQ(u(x_1,t))}{2\xi_n^2} + 2i\Omega(u(x_2,t)) - \frac{1}{2}A_1^n(t)\right)\chi_2^n$$
$$+\left(24i\xi_n P(u(x_0,t)) - \frac{iQ(u(x_1,t))}{2\xi_n^3} - \frac{1}{2}A_2^n(t)\right)\chi_1^n$$
$$+\left(8iP(u(x_0,t)) + \frac{iQ(u(x_1,t))}{2\xi_n^4} - \frac{1}{2}A_3^n(t)\right)\chi_0^n,$$

$$\frac{d\chi^n_{m_n-1-\nu}}{dt} = \left(8i\xi_n^3 P(u(x_0,t)) + 2i\xi_n \Omega(u(x_2,t))\right)\chi^n_{m_n-1-\nu}$$

$$+\left(24i\xi_n^2 P(u(x_0,t)) + 2i\Omega(u(x_2,t))\right)\chi^n_{m_n-2-\nu} + 24i\xi_n P(u(x_0,t))\chi^n_{m_n-3-\nu}$$

$$+8iP(u(x_0,t))\chi^n_{m_n-4-\nu} - \sum_{s=\nu}^{m_n-1}\left(\frac{(-1)^{m_n-1-s} iQ(u(x_1,t))}{2\xi_n^{m_n-s}} + \frac{1}{2}A^n_{m_n-1-s}(t)\right)\chi^n_{s-\nu},$$

$$n = 1,\, 2,\, \ldots,\, N, \quad \nu = m_n - 1,\, m_n - 2,\, \ldots\, 0.$$

Remark 1. The obtained equalities completely determine the evolution of the scattering data, which makes it possible to apply the inverse scattering method to solve the problem (1)-(6).

Let a function $u_0(x)$ satisfying the condition (4) be given. Then the solution of the problem (1)-(6) is found by the following algorithm.

- We solve the direct scattering problem with the initial function $u_0(x)$ and get the scattering data
$$\left\{r^+(\xi),\ \xi_k,\ \chi_j^k,\ k = 1, 2, \ldots, N,\ \ j = 0,\, 1, \ldots,\, m_k - 1\right\}$$
for the operator $L(0)$.
- Using the results of Theorem 2, we find the scattering data for $t > 0$:
$$\left\{r^+(\xi,t),\ \xi_k(t),\ \chi_j^k(t),\ k = 1, 2, \ldots, N,\ \ j = 0,\, 1, \ldots,\, m_k - 1\right\}$$
- Using the method based on the Gelfand-Levitan-Marchenko integral equation, we solve the inverse scattering problem, i.e. we find the unique (according to Theorem 1) $u(x,t)$ from the scattering data for $t > 0$ obtained at the previous step.
- After that, we solve the direct problem for the operator $L(t)$ with the potential $u(x,t)$ and find the functions $f_k^j(x,t),\ j = 0, 1, \ldots, m_{k-1},\ k = \overline{1, N}$.

Example 1. Consider the following problem

$$u_{xt} + \frac{1}{2t+1}\left(\frac{3}{2}u_x^2 u_{xx} + u_{xxxx}\right) = \frac{1}{4t+2}\sin u + \frac{2}{2t+1}u_{xx} + 2\left(f_{11}^2 - f_{12}^2\right), \tag{45}$$

$$Lf_1 = \xi_1 f_1 \tag{46}$$

$$u(x,0) = 4\arctan\left(e^{-2x}\right). \tag{47}$$

Here

$$2\int_{-\infty}^{\infty} f_{11}(x,\,t) f_{12}(x,t)dx = A_0(t) = -\frac{1}{4t+2}.$$

It can be seen that in equation (45) all polynomials $P(u(x_0,t)),\ Q(u(x_1,t)),$ $\Omega(u(x_2,t))$ of order zero, i.e. equation (45) is an mKdV-sG equation with variable coefficients and with source.

It is easy to find scattering data for the operator $L(0)$:

$$r^+(\xi,0) = 0, \quad \xi_1(0) = i, \quad \chi_1(0) = -2i.$$

By virtue of Theorem 2, we have

$$r^+(\xi,t) = 0, \quad \xi_1(t) = i, \quad \chi_1(t) = -2i(2t+1)$$

Hence $F(x) = (4t+2)e^{-x}$. Solving the Gelfand-Levitan-Marchenko integral equation, we have

$$K_1(x, y) = \frac{2(2t+1)e^{-x-y}}{1+(2t+1)^2 e^{-4x}}.$$

Where do we find the solution of the Cauchy problem (45)-(47)

$$u(x, t) = 4\arctan\left((2t+1)e^{-2x}\right),$$

$$f_{11}(x, t) = -\frac{(2t+1)e^{-3x}}{1+(2t+1)^2 e^{-4x}}, \quad f_{12}(x, t) = \frac{e^{-x}}{1+(2t+1)^2 e^{-4x}}.$$

Example 2. Consider the following specific problem

$$\begin{aligned} &u_{xt} + \frac{u(0,t)}{4\arctan\left(\frac{1}{t+1}\right)}\left(\frac{3}{2}u_x^2 u_{xx} + u_{xxxx}\right) \\ &= \frac{(4-(t+1)^2)u\left(-\ln 2, t\right)}{8\arctan\left(\frac{4}{t+1}\right)}\sin u + \frac{u\left(\ln 2, t\right)}{\arctan\left(\frac{1}{4t+4}\right)}u_{xx} + f_{11}^2 - f_{12}^2, \end{aligned} \tag{48}$$

$$Lf_1 = \xi_1 f_1 \tag{49}$$

$$u(x, 0) = 4\arctan\left(e^{-2x}\right), \quad x \in \mathbb{R}. \tag{50}$$

Here

$$A_0(t) = -\frac{t+1}{2}.$$

It can be seen that in the equation (48) all polynomials $P(u(x_0, t))$, $Q(u(x_1, t))$, $\Omega(u(x_2, t))$ of the first order, i.e. the equation (48) is a loaded mKdV-sG equation with a source.

In this case, the solution to the problem (48)-(50) has the following form:

$$u(x,\, t) = 4\arctan\left(\frac{e^{-2x}}{t+1}\right); \quad f_{11} = -\frac{(t+1)e^{-3x}}{(t+1)^2 + e^{-4x}}, \quad f_{12} = \frac{(t+1)^2 e^{-x}}{(t+1)^2 + e^{-4x}}.$$

References

[1] M.J. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur, *Method for solving the sine-Gordon equation*, Phys. Rev. Lett., **30** (1973), 1262–1264.

[2] M.J. Ablowitz, H. Segur, *Solitons and the inverse scattering transform*, SIAM, Philadelphia, 1981.

[3] T. Aktosun, F. Demontis, C. van der Mee, *Exact solutions to the sine-Gordon equation*, J. Math. Phys., **51**:12 (2010), 123521.

[4] L. Alzaleq, V. Manoranjan, *Analytical solutions for the generalized sine-Gordon equation with variable coefficients.* Phys. Scr., **96**:5 (2021), 055218.

[5] D.-Y. Chen, D.-J. Zhang, S.-F. Deng, *The Novel Multi-Soliton Solutions of the MKdV–Sine Gordon Equations*, J. Phys. Soc. Jpn., **71** (2002), 658–659.

[6] J.L. Cieśliński, T. Nikiciuk, K. Waśkiewicz, *The sine-Gordon equation on time scales*, J. Math. Anal. Appl., **423** (2015), 1219–1230.

[7] A.T. Deresse, Y.O. Mussa, A.K. Gizaw, *Analytical Solution of Two-Dimensional Sine-Gordon Equation*, Adv. Math. Phys. **2021** (2021), 6610021.

[8] R.K. Dodd, J.C. Eilbeck, J.D. Gibbon, H.C Morris, *Solitons and Nonlinear Wave Equations*, MIR, Moscow, 1988.

[9] L.P. Eisenhart, *A treatise on the difierential geometry of curves and surfaces*, Dover Publ., New York, 1960.

[10] I.S. Frolov, *Inverse scattering problem for a Dirac system on the whole axis*, Dokl. Akad. Nauk SSSR, **13** (1972), 1468–1472.

[11] V.A. Gani, A.M. Marjaneh, A. Askari, E. Belendryasova, D. Saadatmand, *Scattering of the double sine-Gordon kinks*, Eur. Phys. Jour. C., **78** (2018), 345.
[12] C.S. Gardner, I.M. Greene, M.D. Kruskal, R.M. Miura, *Method for Solving the Korteweg-de Vries Equation*, Phys. Rev. Lett., **19** (1967), 1095–1097.
[13] D.R. Gulevich, M.B. Gaifullin, F.V. Kusmartsev, *Controlled dynamics of sine-Gordon breather in long Josephson junctions*, Eur. Phys. Jour. B., **85**:1 (2012), 24.
[14] A.B. Hasanov, U.A. Hoitmetov, *On Integration of the Loaded Korteweg-de Vries Equation in the Class of Rapidly Decreasing Functions*, Proc. Inst. Math. Mech. Natl. ASc. Azerb., **47**:2 (2021), 250–261.
[15] U.A. Hoitmetov, *Integration of the loaded KdV equation with a self-consistent source of integral type in the class of rapidly decreasing complex-valued functions*, Matem. Trudy, **24**:2 (2021), 181–198.
[16] G. Hovhannisyan, *3 soliton solution to Sine-Gordon equation on a space scale*, J. Math. Phys. **60** (2019), 103502.
[17] A.B. Khasanov, *An inverse problem in scattering theory for a system of two first-order nonselfadjoint differential equations*, Dokl. Akad. Nauk SSSR. **277**:3 (1984), 559–562.
[18] A.B. Khasanov, U.A. Hoitmetov, *On Integration of the Loaded mKdV Equation in the Class of Rapidly Decreasing Functions*, Bull. Irkutsk State Univ. Ser. Math., **38** (2021), 19–35.
[19] A.B. Khasanov, U.A. Hoitmetov, *Integration of the General Loaded Korteweg–de Vries Equation with an Integral Type Source in the Class of Rapidly Decreasing Complex-Valued Functions*, Russ. Math., **65**:7 (2021), 43–57.
[20] A.B. Khasanov, G.U. Urazboev, *On the sine-Gordon equation with a self-consistent source*, Sib. Adv. Math. **19**:(1) (2009), 13–23.
[21] A.B. Khasanov, G.U. Urazboev, *Integration of the sine-Gordon equation with a self-consistent source of the integral type in the case of multiple eigenvalues*, Rus. Math. **53**:3, (2009), 45–55.
[22] A.H. Khater, O.H. El-Kalaawy, D.K. Callebaut, *Backlund Transformations and Exact Solutions for Alfven Solitons in a Relativistic Electron–Positron Plasma*, Phys. Scr., **58**:6 (1998), 545–548.
[23] K. Konno, W. Kameyama, H. Sanuki, *Effect of weak dislocation potential on nonlinear wavepropagation in anharmonic crystal*, J. Phys. Soc. Jap. **37** (1974), 171–176.
[24] A.I. Kozhanov, *Nonlinear loaded equations and inverse problems*, Comput. Math. Math. Phys., **44**:4 (2004), 657–675.
[25] H. Leblond, D. Mihalache, *Models of few optical cycle solitons beyond the slowly varying envelope approximation*, Phys. Rep., **523** (2013), 61–126.
[26] H. Leblond, D. Mihalache, *Few-optical-cycle solitons: modified Korteweg–de Vries sine-Gordon equation versus other non-slowly-varying-envelope-approximation models*, Phys. Rev. A, **79** (2009), 063835.
[27] H. Leblond, S.V. Sazonov, I.V. Melnikov, D. Mihalache, F. Sanchez, *Few-cycle nonlinear optics of multicomponent media*, Phys. Rev. A, **74** (2006), 063815.
[28] S. Matsutani, H. Tsuru, *Reflectionless Quantum Wire*, J. Phys. Soc. Jap., **60**:11 (1991), 3640–3644.
[29] A.M. Nakhushev, *Equations of Mathematical Biology*, Vysshaya Shkola, Moscow, 1995.
[30] S.P. Novikov, S.V. Manakov, L.B. Pitaevskii, V.E. Zakharov, *Theory of solitons. The inverse scattering method*, Plenum Press, New York, 1984.
[31] W. Schief, *An infinite hierarchy of symmetries associated with hyperbolic surfaces*, Nonlinearity, **8**:1, 1–9.
[32] G.U. Urazboev, U.A. Xoitmetov, A.K. Babadjanova, *Integration of the matrix modified Korteweg–de Vries equation with an integral type source*, Theor. Math. Phys., **203**:3 (2020), 734–746.
[33] M. Wadati, *The exact solution of the modified Korteweg–de Vries equation*, J. Phys. Soc. Jpn., **32** (1972), 1681.
[34] A.M. Wazwaz, *N-soliton solutions for the integrable modified KdV–sine-Gordon equation*, Phys. Scr., **89** (2014), 065805.
[35] A.M. Wazwaz, *Travelling wave solutions for the MKdV–sine-Gordon and the MKdV–sinh-Gordon equations by using a variable separated ODE method*, Appl. Math. Comput. **181** (2006), 1713–1719.
[36] L.V. Yakushevich, *Nonlinear physics of DNA. 2nd ed.*, Wiley, Chichester, 2004.

[37] V.E. Zakharov, L.A. Takhtadzhyan, L.D. Faddeev, *Complete description of solutions of the sine-Gordon equation*, Sov. Phys. Dokl., **19** (1975), 824–826.
[38] S.-L. Zhao, H.-H. Hao, D.-J. Zhang, *The multi-soliton solutions for the non-isospectral mKdV–sine Gordon equation*, Modern Physics Letters B, **23**:4 (2009), 607–621.

Umid Azadovich Hoitmetov
Urgench Stete University,
Kh. Alimdjan str., 14,
220100, Urgench, Uzbekistan
Email address: x_umid@mail.ru