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THE QUASIVARIETY $\mathbf{SP}(L_6)$. I. AN EQUATIONAL BASIS

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Abstract. We prove that the quasivariety $\mathbf{SP}(L_6)$ is a variety and find an equational basis for this variety.

Keywords: lattice, quasivariety, variety, poset.

1. Introduction

In the present paper, we consider the finite lattice L_6, see Figure 1, which is isomorphic to the suborder lattice of a three-element chain. Suborder lattices were in the focus in a number of articles as they provide a convenient tool for proving certain embeddability results.

By a theorem by D. Bredikhin and B. Schein [1], suborder lattices are *lattice universal*; that is, each lattice is embeddable into a suitable suborder lattice. By a theorem of B. Sivák [15], a lattice L is embeddable into the suborder lattice of a finite partial order if and only if L is finite and lower bounded in the sense of R. McKenzie [8]. Suborder lattices were used for embedding lattices into the subsemigroup lattices in V. B. Repnitskiĭ [10, 11] as well as in [14].

Suborder lattices were also studied in papers [12, 13]. A general construction to embed an arbitrary lattice into a suitable suborder lattice was suggested in [12]. Based on this construction, it was shown in [13] that for arbitrary $n < \omega$, the class $\mathbf{SO}_n$ of lattices embeddable into suborder lattices of posets of length at most n

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forms a finitely based variety. An equational basis for this variety was found in [13]. There are still a number of unsolved problems which concern suborder lattices. In particular, Question 2 in [13] asks if the quasivariety generated by a finite suborder lattice is a variety.

In this paper, we give a positive answer to this question in a particular case. More specifically, we prove that the quasivariety $\mathbf{Q}(L_6)$ generated by the lattice L_6 is a finitely based variety and find a finite basis for this variety. The method we use was developed in [2]. In a subsequent article, the main result of this paper, Theorem 12, will be applied for proving a duality result for the quasivariety $\mathbf{Q}(L_6)$.

2. Definitions and auxiliary results

We assume all classes to be *abstract*; that is, closed under taking isomorphic copies of structures.

2.1. **(Quasi)varieties.** A *quasi-identity* is a universal Horn sentence of the form

$$\forall \overline{x}\ A_0(\overline{x})\ \&\ \ldots\ \&\ A_n(\overline{x}) \ \longrightarrow\ A(\overline{x}),$$

where $n < \omega$ and $A_0(\overline{x}), \ldots, A_n(\overline{x}), A(\overline{x})$ are atomic formulas of a fixed type. An *identity* is a sentence of the form

$$\forall \overline{x}\ A(\overline{x}),$$

where $A(\overline{x})$ is an atomic formula of a fixed type. A *quasivariety* is the class $\mathrm{Mod}(\Sigma)$ of models for a set Σ of quasi-identities. A *variety* is the class $\mathrm{Mod}(\Sigma)$ of models for a set Σ of identities. In this case, Σ is called a *quasi-equational basis* [an *equational basis*, respectively] of $\mathbf{K}$. It is clear that each variety is a quasi-variety.

For a type σ, let $\mathbf{K}(\sigma)$ denote the class of all structures of type σ. For an arbitrary class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ of structures, let $\mathbf{S}(\mathbf{K})$ denote the class of structures from $\mathbf{K}(\sigma)$ embeddable into structures from the class $\mathbf{K}$, and let $\mathbf{P}(\mathbf{K})$ denote the class of structures from $\mathbf{K}(\sigma)$ isomorphic to Cartesian products of structures from $\mathbf{K}$. Whenever $\mathbf{K}$ contains only one structure $\mathcal{A}$ (up to isomorphism), we write $\mathbf{O}(\mathcal{A})$ instead of $\mathbf{O}(\{\mathcal{A}\})$ for a class operator $\mathbf{O}$. Let $\mathbf{Q}(\mathbf{K})$ denote the smallest quasivariety containing $\mathbf{K}$. It is well known [7] that for a finite structure $\mathcal{A}$, the class $\mathbf{SP}(\mathcal{A})$ is a quasivariety. Thus, $\mathbf{Q}(\mathcal{A}) = \mathbf{SP}(\mathcal{A})$ for each finite structure $\mathcal{A}$.

For all the notions concerning (quasi)varieties of structures which are not defined here, we refer to A. I. Maltsev [7], V. A. Gorbunov [4], and J. Hyndman and J. B. Nation [6].

2.2. **General lattices.** Most of the following definitions correspond to R. Freese, J. Ježek, and J. B. Nation [3].

Let L be a lattice. For arbitrary two sets $A, B \subseteq L$, we say that A *refines* B and write $A \ll B$ if for each $a \in A$, there is $b \in B$ such that $a \leq b$. If $x \in L$, then A is a *join cover* of x if $\bigvee A$ exists and $x \leq \bigvee A$; we also call the inequality $x \leq \bigvee A$ a *join cover* in this case. A join cover $x \leq \bigvee A$ is *nontrivial* if $x \nleq a$ for all $a \in A$; $x \leq \bigvee A$ is *finite* if the set A is finite. A join cover $x \leq \bigvee A$ is *irredundant* if $x \nleq \bigvee B$ for all proper subsets $B \subset A$. A join cover $x \leq \bigvee A$ is *minimal* if $A \subseteq B$ for each join cover $x \leq \bigvee B$ such that $B \ll A$. The lattice L has the *complete minimal join cover refinement property* $(\mathrm{CR})_X$ for a set $X \subseteq L$ if for each nontrivial join cover A of an element $x \in X$, there is a minimal nontrivial join cover B of x such that $B \ll A$.

A non-zero element a of a lattice L is said to be *join-irreducible* if $a = b \vee c$ implies that $a \in \{b, c\}$ for all $b, c \in L$; a is said to be *completely join-irreducible* if

$a = \bigvee B$ implies that $a \in B$ for all nonempty sets $B \subseteq L$. Let $\mathrm{J}(L)$ denote the set of all join-irreducible elements in L and let $\mathrm{CJ}(L)$ denote the set of all completely join-irreducible elements in L.

Definition 1. [2] For a set $J \subseteq \mathrm{J}(L)$, we say that L is a *J-lattice* if L possesses the following properties:

(1) for each element $a \in L$, there is a subset $J_a \subseteq J$ with $a = \bigvee J_a$;
(2) for each element $a \in J$ and each nontrivial join cover $a \leq a_0 \vee \ldots \vee a_n$ with $n < \omega$ and $a_0, \ldots, a_n \in L$, there is a finite set $F \subseteq J$ such that $a \leq \bigvee F$ is a minimal join cover and $F \ll \{a_0, \ldots, a_n\}$.

We say that L is a *CJ-lattice* if L possesses the following properties:

(1) for each element $a \in L$, there is a subset $J_a \subseteq \mathrm{CJ}(L)$ with $a = \bigvee J_a$;
(2) L has the property $(\mathrm{CR})_{\mathrm{CJ}(L)}$.

In what follows, we consider the following identity of *n-distributivity*, where $1 < n < \omega$, which we denote by (D_n):

$$x \wedge (y_0 \vee y_1 \vee \ldots \vee y_n) = \bigvee_{i \leqslant n} \big[x \wedge \bigvee_{j \neq i} y_j\big].$$

This identity was introduced and considered by A. P. Huhn [5]. It is clear that (D_1) is just the identity of distributivity.

The following lemma is folklore and straightforward to prove, see for example J. B. Nation [9].

Lemma 1. *Let $n > 0$, let L be a lattice, let a set $J \subseteq \mathrm{J}(L)$ be such that for each element $a \in L$, there is a subset $J_a \subseteq J$ with $a = \bigvee J_a$. The following conditions are equivalent.*

(1) *(D_n) holds in L.*
(2) *If $a \leq b_0 \vee b_1 \vee \ldots \vee b_n$ for some $a \in J$ and some $b_0, b_1, \ldots, b_n \in L$, then there is $i \leqslant n$ such that $a \leq \bigvee_{j \neq i} b_j$.*

Corollary 2. *Let $n > 1$ and let L be a J-lattice for some set $J \subseteq \mathrm{J}(L)$. The following conditions are equivalent.*

(1) *(D_n) holds in L.*
(2) *If $a \leq b_0 \vee \ldots \vee b_m$ is a minimal nontrivial join cover for some $a, b_0, \ldots, b_m \in J$ then $0 < m < n$.*

Proposition 3. [2] *Let L be a complete dually algebraic lattice. Then the following statements hold.*

(1) *If L is n-distributive then L is a $\mathrm{J}(L)$-lattice.*
(2) *If L is in addition algebraic then L is a CJ-lattice.*

Following [2], we denote the next identity by (C):

$$x \wedge (y_0 \vee y_1) \wedge (z_0 \vee z_1) = \bigvee_{i<2} \big[x \wedge y_i \wedge (z_0 \vee z_1)\big] \ \vee \ \bigvee_{i<2} \big[x \wedge z_i \wedge (y_0 \vee y_1)\big] \ \vee \ \bigvee_{i<2} \big[x \wedge \big((y_0 \wedge z_i) \vee (y_1 \wedge z_{1-i})\big)\big].$$

The next four statements were established in [2]. Since [2] does not contain complete proofs, we present here sketches of proofs of Lemma 4 and Lemma 6 for the sake of completeness. We emphasize that these proofs are due to the authors of [2].

Lemma 4. [2] *Let L be a lattice, let a set $J \subseteq \mathrm{J}(L)$ be such that for each element $a \in L$, there is a subset $J_a \subseteq J$ with $a = \bigvee J_a$. The following conditions are equivalent.*

1. (C) *holds in L.*
2. *If $a \leq a_0 \vee a_1$ and $a \leq b_0 \vee b_1$ are nontrivial join covers for some $a \in J$ and some $a_0, a_1, b_0, b_1 \in L$, then there are $c_0, c_1 \in L$ such that $a \leq c_0 \vee c_1$, $\{c_0, c_1\} \ll \{a_0, a_1\}$, and $\{c_0, c_1\} \ll \{b_0, b_1\}$.*

Proof. To prove that (1) implies (2), we assume that the assumptions of (2) hold. Since (C) holds in L, we have

$$a = a \wedge (a_0 \vee a_1) \wedge (b_0 \vee b_1) = \bigvee_{i<2} \big[a \wedge a_i \wedge (b_0 \vee b_1)\big] \ \vee \ \bigvee_{i<2} \big[a \wedge b_i \wedge (a_0 \vee a_1)\big] \ \vee \\ \vee \ \bigvee_{i<2} \big[a \wedge \big((a_0 \wedge b_i) \vee (a_1 \wedge b_{1-i})\big)\big].$$

As a is a join-irreducible element, it equals one of the joinands on the right-hand side of the equality above. This implies that the conclusion of (2) also holds.

To prove that (2) implies (1), we note that the inequality

$$\bigvee_{i<2} \big[x \wedge y_i \wedge (z_0 \vee z_1)\big] \ \vee \ \bigvee_{i<2} \big[x \wedge z_i \wedge (y_0 \vee y_1)\big] \ \vee \\ \vee \ \bigvee_{i<2} \big[x \wedge \big((y_0 \wedge z_i) \vee (y_1 \wedge z_{1-i})\big)\big] \leq \\ \leq x \wedge (y_0 \vee y_1) \wedge (z_0 \vee z_1)$$

holds in each lattice. Therefore, in order to prove that (C) holds in L, we have to establish that the reverse inequality holds in L. To this end, choose arbitrary elements $u, a_0, a_1, b_0, b_1 \in L$. We put

$$w = \bigvee_{i<2} \big[u \wedge a_i \wedge (b_0 \vee b_1)\big] \ \vee \ \bigvee_{i<2} \big[u \wedge b_i \wedge (a_0 \vee a_1)\big] \ \vee \ \bigvee_{i<2} \big[u \wedge \big((a_0 \wedge b_i) \vee (a_1 \wedge b_{1-i})\big)\big].$$

We have to show that

$$u \wedge (a_0 \vee a_1) \wedge (b_0 \vee b_1) \leq w.$$

According to our assumption about L, it suffices to show that each element $a \in J$ which is below $u \wedge (a_0 \vee a_1) \wedge (b_0 \vee b_1)$ is also below w. But this follows from (2). □

Corollary 5. [2] *Let L be a 2-distributive J-lattice for some set $J \subseteq \mathrm{J}(L)$. The following conditions are equivalent.*

1. (C) *holds in L.*
2. *If $a \leq a_0 \vee a_1$ and $a \leq b_0 \vee b_1$ are minimal join covers for some elements $a, a_0, a_1, b_0, b_1 \in J$, then $\{a_0, a_1\} = \{b_0, b_1\}$.*

As in [2], we denote the following identity by (N_5^1):

$$x \wedge \big[\big(y_0 \wedge (z_0 \vee z_1)\big) \vee y_1\big] = \big[x \wedge y_0 \wedge (z_0 \vee z_1)\big] \vee \big[x \wedge y_1\big] \vee \bigvee_{i<2} \big[x \wedge \big((y_0 \wedge z_i) \vee y_1\big)\big].$$

Lemma 6. [2] *Let L be a lattice, let a set $J \subseteq \mathrm{J}(L)$ be such that for each element $a \in L$, there is a subset $J_a \subseteq J$ with $a = \bigvee J_a$. The following conditions are equivalent.*

1. (N_5^1) *holds in L.*

(2) *If $a \leq a_0 \vee a_1$ is a nontrivial join cover and $a_0 \leq b_0 \vee b_1$ for some $a \in J$ and some $a_0, a_1, b_0, b_1 \in L$, then $a \leq (a_0 \wedge b_i) \vee a_1$ for some $i < 2$.*

Proof. To prove that (1) implies (2), we assume that $a \leq a_0 \vee a_1$ for some $a \in J$ and some $a_0, a_1 \in L$ and that $a_0 \leq b_0 \vee b_1$ for some $b_0, b_1 \in L$. Since (N_5^1) holds in L, we have

$$a = a \wedge \big[\big(a_0 \wedge (b_0 \vee b_1)\big) \vee a_1\big] = \big[a \wedge a_0 \wedge (b_0 \vee b_1)\big] \vee \big[a \wedge a_1\big] \vee \bigvee_{i<2} \big[a \wedge \big((a_0 \wedge b_i) \vee a_1\big)\big].$$

As a is a join-irreducible element, it equals one of the joinands on the right-hand side of the equality above. This implies that the conclusion of (2) also holds.

To prove that (2) implies (1), we again notice that

$$\big[x \wedge y_0 \wedge (z_0 \vee z_1)\big] \vee \big[x \wedge y_1\big] \vee \bigvee_{i<2} \big[x \wedge \big((y_0 \wedge z_i) \vee y_1\big)\big] \leq x \wedge \big[\big(y_0 \wedge (z_0 \vee z_1)\big) \vee y_1\big]$$

holds in each lattice. Therefore, in order to prove that (N_5^1) holds in L, we have to establish that the reverse inequality holds in L. In order to do this, we choose arbitrary elements $u, a_0, a_1, b_0, b_1 \in L$ and put

$$w = \big[u \wedge a_0 \wedge (b_0 \vee b_1)\big] \vee \big[u \wedge a_1\big] \vee \bigvee_{i<2} \big[u \wedge \big((a_0 \wedge b_i) \vee a_1\big)\big].$$

We have to show that

$$u \wedge \big[\big(a_0 \wedge (b_0 \vee b_1)\big) \vee a_1\big] \leq w.$$

According to our assumption about L, it suffices to show that each element $a \in J$ which is below $u \wedge \big[\big(a_0 \wedge (b_0 \vee b_1)\big) \vee a_1\big]$ is also below w. But this conclusion follows from (2). □

Corollary 7. [2] *Let L be a 2-distributive J-lattice for some set $J \subseteq \mathrm{J}(L)$. The following conditions are equivalent.*

(1) *(N_5^1) holds in L.*
(2) *If $a \leq a_0 \vee a_1$ is a minimal join cover for some $a, a_0, a_1 \in J$, then a_0 and a_1 are join-prime elements.*

We use the following notation, cf. R. Freese, J, Ježek, and J. B. Nation [3, Lemma 2.33].

Definition 2. Consider a J-lattice L, where $J \subseteq \mathrm{J}(L)$. For an element $x \in J$, we put

$$\mathfrak{M}(x) = \Big\{A \subseteq J \mid 1 < |A| < \omega,\ x \leq \bigvee A \text{ is a minimal nontrivial join cover of } x\Big\}.$$

For a set $S \subseteq J$, we put

$$S^{[0]} = S;$$
$$S^{[n+1]} = \bigcup \{A \in \mathfrak{M}(x) \mid x \in S^{[n]}\}, \quad n < \omega;$$
$$\langle S \rangle_{\mathfrak{M}} = \bigcup_{i<\omega} S^{[i]}.$$

For an element $x \in J$, we write $\langle x \rangle_{\mathfrak{M}}$ instead of $\langle \{x\} \rangle_{\mathfrak{M}}$. It is straightforward that $\langle S \rangle_{\mathfrak{M}} = \bigcup_{x \in S} \langle x \rangle_{\mathfrak{M}} \subseteq J$, whence $\langle\ \rangle_{\mathfrak{M}}$ is an algebraic closure operator on J. A set $S \subseteq J$ is $\mathfrak{M}$-*closed*, if $S = \langle S \rangle_{\mathfrak{M}}$.

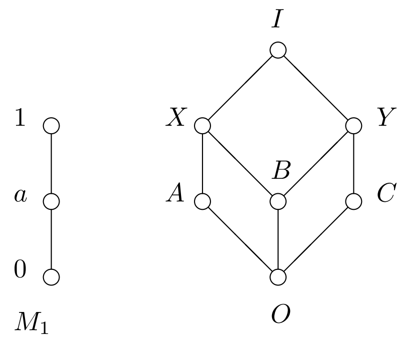

Рис. 1. Partially ordered set M_1 and lattice $L_6 \cong \mathrm{O}(M_1)$

It is clear that $\mathfrak{M}(x) = \varnothing$ whence $\langle x \rangle_{\mathfrak{M}} = \{x\}$ for each element $x \in J$ which is join-prime.

For a set $S \subseteq J$, we define a binary relation Γ_S on L as follows. If $a, b \in L$ then we put

$$(a, b) \in \Gamma_S \quad \text{if and only if } S \cap {\downarrow} a = S \cap {\downarrow} b.$$

Lemma 8. [2] *Let L be a J-lattice and let $A \subseteq J$. The following statements hold.*

(1) *If $A \subseteq B$ for some $B \subseteq J$ then $\Gamma_B \subseteq \Gamma_A$.*
(2) *If $A = \bigcup_{i \in I} A_i$ for some $A_i \subseteq J$, $i \in I$, then $\Gamma_A = \bigcap_{i \in I} \Gamma_{A_i}$.*
(3) *If A is an $\mathfrak{M}$-closed set then Γ_A is a congruence on L.*

2.3. **Suborder lattices.** Let X be a set and let $R \subseteq X^2$ be a partial order on X; that is a reflexive, antisymmetric, and transitive binary relation. In this case, we say that $(X; R)$ is a *partially ordered set* or a *poset* for short. A subset $R' \subseteq R$ is a *suborder* of R if the structure $(X; R')$ is also a poset. The set $\mathrm{O}(X, R)$ of all suborders of a partial order R on X is a partially ordered set with respect to the relation $\subseteq$ of set-theoretic inclusion. Obviously, $\Delta = \{(a, a) \mid a \in X\}$ is the least suborder of R. Thus, Δ is the smallest element in $\mathrm{O}(X, R)$. It is also obvious that R is the largest element in $\mathrm{O}(X, R)$. It is straightforward to check that for an arbitrary family $\{R_i \mid i \in I\} \subseteq \mathrm{O}(X, R)$, the relation $\bigcap_{i \in I} R_i$ is also a suborder of R; that is,

$$\bigwedge_{i \in I} R_i = \bigcap_{i \in I} R_i \in \mathrm{O}(X, R).$$

Thus, $\mathrm{O}(X, R)$ is a complete lattice, where

$$\bigvee_{i \in I} R_i = \Big(\bigcup_{i \in I} R_i\Big)^t$$

and Y^t denotes the transitive closure of a binary relation $Y \subseteq X^2$.

For more information on suborder lattices, we refer to D. Bredikhin and B. Schein [1], B. Sivák [15] as well as to [12, 13].

3. The lattice L_6

Lemma 9. *The suborder lattice $\mathrm{O}(M_1)$ is isomorphic to L_6.*

Proof. The order relation on M_1 is $R = \big\{(0,0),(0,a),(a,a),(a,1),(0,1),(1,1)\big\}$. Then the suborders of R are exactly the sets

$$\begin{aligned}
O &= \big\{(0,0),(a,a),(1,1)\big\};\\
A &= \big\{(0,a)\big\}\cup O;\\
B &= \big\{(0,1)\big\}\cup O;\\
C &= \big\{(a,1)\big\}\cup O;\\
X &= \big\{(0,a),(0,1)\big\}\cup O;\\
Y &= \big\{(a,1),(0,1)\big\}\cup O;\\
I = R &= \big\{(0,a),(a,1),(0,1)\big\}\cup O.
\end{aligned}$$

Then it follows that $\mathrm{O}(M_1)\cong L_6$, cf. Figure 1. □

4. An equational basis for $\mathbf{SP}(L_6)$

We put $\Sigma = \{(\mathrm{C}),(\mathrm{D}_2),(\mathrm{N}_5^1)\}$.

Proposition 10. *Let L be a dually algebraic lattice such that $L\models\Sigma$. Then for each element $b\in\mathrm{J}(L)$ which is not join-prime, we have*

$$\langle b\rangle_{\mathfrak{M}} = \{a,b,c\},\ \textit{where } b\le a\vee c \textit{ is a minimal join cover.}$$

Moreover, $L\in\mathbf{SP}(L_6)$.

Proof. According to Proposition 3(1), L is a J-lattice, where $J=\mathrm{J}(L)$ is the set of all join-irreducible elements of L. If $b\in J$ is not join-prime, then according to Corollary 5, $\mathfrak{M}(b)=\big\{\{a,c\}\big\}$ for some $a,c\in J$ such that $b\le a\vee c$ is a minimal nontrivial join cover. According to Corollary 7, elements a and c are join-prime, whence $\langle b\rangle_{\mathfrak{M}}=\{a,b,c\}$.

According to Lemma 8(3), $\Gamma_{\langle b\rangle_{\mathfrak{M}}}$ is a congruence on L for each $b\in J$. Since L is a J-lattice and J is an $\mathfrak{M}$-closed set, $\Gamma_J=\Delta_L$ is the least congruence on L, whence $L/\Gamma_J\cong L$. As $J=\bigcup_{x\in J}\langle x\rangle_{\mathfrak{M}}$, we conclude by Lemma 8(2) that $L\le_s \prod_{x\in J} L/\Gamma_{\langle x\rangle_{\mathfrak{M}}}$. We fix an element $x\in J$. In what follows, let $[z]$ denote the $\Gamma_{\langle x\rangle_{\mathfrak{M}}}$-equivalence class of an element $z\in L$.

If $x\in J$ is join-prime, then $\langle x\rangle_{\mathfrak{M}}=\{x\}$. Therefore, there are only two $\Gamma_{\langle x\rangle_{\mathfrak{M}}}$-equivalence classes: $[0_L]$ and $[x]$. Hence $L/\Gamma_{\langle x\rangle_{\mathfrak{M}}}\cong 2$. If $x\in J$ is not join-prime, then $\langle x\rangle_{\mathfrak{M}}=\{x,u,v\}$, where $u,v\in J$ are join-prime and $x\le u\vee v$ is a minimal nontrivial join cover. This implies in particular that u and v are incomparable and that $x\notin{\uparrow}u\cap{\uparrow}v$. The following three cases are therefore possible.

Case 1: $u\nleq x$ and $v\nleq x$. In this case, elements x, u, and v are pairwise incomparable. This implies that $x\nleq u\wedge v$, $u\nleq x\wedge v$, and $v\nleq x\wedge u$. Therefore, $[u\wedge v]=[x\wedge u]=[x\wedge v]=[0_L]$. If $v\le x\vee u$, then $v\le x$ or $v\le u$ as v is join-prime. Both cases are impossible as $\{x,u,v\}$ is an anti-chain. This implies that $[x\vee u]$ and $[x\vee v]$ are incomparable elements in the lattice $L/\Gamma_{\langle x\rangle_{\mathfrak{M}}}$. Moreover, $[u\vee v]=[1_L]$ as $x\le u\vee v$. Therefore, for an arbitrary element $z\in L$, we have in *Case 1* that one of the following cases occurs:

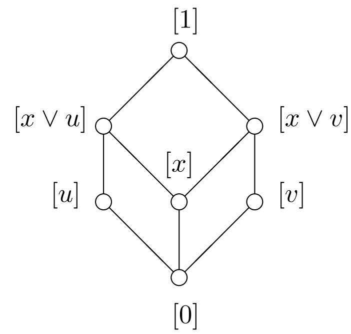

Рис. 2. Lattice $L/\Gamma_{\langle x\rangle_{\mathfrak{M}}} \cong L_6$

$$\langle x\rangle_{\mathfrak{M}} \cap {\downarrow} z = \varnothing; \quad \langle x\rangle_{\mathfrak{M}} \cap {\downarrow} z = \{x, u, v\};$$
$$\langle x\rangle_{\mathfrak{M}} \cap {\downarrow} z = \{x\}; \quad \langle x\rangle_{\mathfrak{M}} \cap {\downarrow} z = \{u\}; \quad \langle x\rangle_{\mathfrak{M}} \cap {\downarrow} z = \{v\};$$
$$\langle x\rangle_{\mathfrak{M}} \cap {\downarrow} z = \{x, u\}; \quad \langle x\rangle_{\mathfrak{M}} \cap {\downarrow} z = \{x, v\},$$

see Figure 2. This implies that $L/\Gamma_{\langle x\rangle_{\mathfrak{M}}}$ has the following elements: $[0_L]$, $[x]$, $[u]$, $[v]$, $[x \vee u]$, $[x \vee v]$, $[1_L]$. Hence $L/\Gamma_{\langle x\rangle_{\mathfrak{M}}} \cong L_6$.

Case 2: $u < x$. In this case, we have $v \nleq x$, whence x and v are incomparable. Moreover, $x \nleq u \wedge v$ and $u \nleq x \wedge v$, whence $[u \wedge v] = [x \wedge v] = [0_L]$. It is also clear that $[x \vee v] = [u \vee v] = [1_L]$. Therefore, for an arbitrary element $z \in L$, we have in *Case 2* that one of the following cases occurs:

$$\langle x\rangle_{\mathfrak{M}} \cap {\downarrow} z = \varnothing; \quad \langle x\rangle_{\mathfrak{M}} \cap {\downarrow} z = \{x, u, v\};$$
$$\langle x\rangle_{\mathfrak{M}} \cap {\downarrow} z = \{x\}; \quad \langle x\rangle_{\mathfrak{M}} \cap {\downarrow} z = \{u\}; \quad \langle x\rangle_{\mathfrak{M}} \cap {\downarrow} z = \{v\},$$

see Figure 3. This implies that $L/\Gamma_{\langle x\rangle_{\mathfrak{M}}}$ has the following elements: $[0_L]$, $[x]$, $[u]$, $[v]$, $[1_L]$. Hence $L/\Gamma_{\langle x\rangle_{\mathfrak{M}}} \cong N_5 \leq L_6$.

Case 3: $v < x$. This case is symmetric to *Case 2* and therefore, $L/\Gamma_{\langle x\rangle_{\mathfrak{M}}} \cong N_5 \leq L_6$.

The above implies that L is a subdirect product of lattices isomorphic either to 2 or to N_5, or to L_6. Since both lattices, 2 and N_5, embed into L_6, we obtain that $L \in \mathbf{SP}(L_6)$. □

Corollary 11. *Let L be a bi-algebraic lattice such that $L \models \Sigma$. Then for each element $x \in \mathrm{CJ}(L)$ which is not join-prime, we have*

$$[x]_{\mathfrak{M}} = \{x, a, b\}, \text{ where } a, b \in \mathrm{CJ}_P(L),\ x \leq a \vee b.$$

In particular, $L \in \mathbf{SP}(L_6)$.

Proof. The argument is similar to the one in the proof of Proposition 10 and uses Proposition 3(2). □

Theorem 12. *The quasivariety $\mathbf{SP}(L_6)$ is a variety and Σ forms an equational basis for $\mathbf{SP}(L_6)$.*

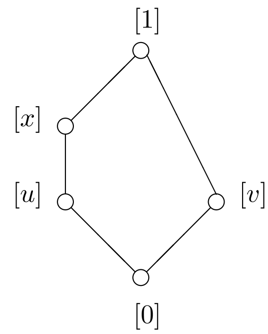

Рис. 3. Lattice $L/\Gamma_{\langle x\rangle_{\mathfrak{M}}} \cong N_5$

Proof. Let $L \models \Sigma$ and let F be the dual filter lattice of L. Then F is dually algebraic and $F \models \Sigma$. By Proposition 10, $F \in \mathbf{SP}(L_6)$ whence $L \in \mathbf{SP}(L_6)$. This proves that $\mathrm{Mod}(\Sigma) \subseteq \mathbf{SP}(L_6)$. On the other hand, the lattice L_6 has the only nontrivial join cover $b \leq a \vee c$ of a join-irreducible element. Thus, L_6 is 2-distributive by Corollary 2. Moreover, L_6 satisfies the condition (2) of Corollaries 5 and 7. This implies that $L_6 \models \Sigma$ and that $\mathbf{SP}(L_6) \subseteq \mathrm{Mod}(\Sigma)$ which proves the desired statement. □

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