

Referee's report on the manuscript

“The quasivariety $\mathbf{SP}(L_6)$. I. An equational basis”

The authors consider the lattice L_6 of suborders of a three-element chain and prove that the quasivariety generated by L_6 is a variety; moreover, they explicitly find a finite basis for identities of this variety.

This is an interesting result supporting a conjecture on quasivarieties generated by finite lattices of suborders from an article of the second author.

The approach is interesting too. Namely, the authors follow a general scheme for studying lattices with the use of their join-irreducible elements. Notice that the scheme was suggested by the second author and W. Dziobiak; it is new and unpublished.

The text of the manuscript is well written: there are words on motivation for the research, necessary definitions, preliminary results, a formulation and correct proof of the main result.

I strongly recommend to publish the manuscript.

Since several places in the text were difficult for the referee to read and understand (from the mathematical point of view, not because of authors' English), such places are listed below with brief comments.

(1) Articles

The first sentence of the introduction: “...we consider **the** finite lattice...” is better. In the first paragraph of 2.3, Δ is “**the** least” suborder. In the second line from the bottom of page 7, $\Gamma_J = \Delta_L$ is “**the** least” congruence.

(2) Citations

Please, check in the third paragraph of introduction: In the first sentence, [15, 16] is mentioned but only [16] (three times) is cited later.

There is shortage of references together with extra bibliographic items. Say, sources 1, 2, 6, 8, and 9 are never mentioned; on the other hand, it would be more convenient for reader to find references to undefined terms and notation at the beginning of each subsection of Sec. 2. Say, in 2.2, it is not difficult to either recall the definitions of irreducible, completely irreducible, and prime elements or provide a reference to a source on lattices.

(3) Terminology

The definition of an identity in 2.1 generalises the usual one. In fact, for $n > 0$, we obtain the conjunction of “usual” identities. I doubt if this generalisation is needed because the basis for $\mathbf{SP}(L_6)$ consists of “usual” identities; moreover, author's identity is no longer a partial case of a quasi-identity.

In the second paragraph of 2.1, “structures” are used together with “algebraic structures.” I believe, it would be better to use the first (shorter) term everywhere.

(4) Notation

I suggest to use capital letters for suborders at Fig. 1, as in Lemma 9 below (notice that these elements are also mentioned in the fourth line of the proof of Theorem 12).

(5) Three points that need clarification

(a) The second paragraph of the introduction, the last sentence: too many indefinites. “Suborder lattices were used for embedding lattices into the lattices of subsemigroups of semigroups in [13, 14] and [17].” or something like that is shorter and easier to understand.

(b) In the first paragraph of 2.2, the definition of $(CR)_X$ in the last sentence sounds strange. What does that “refined to a minimal” mean? The definition of “minimal” from the previous sentence says that a minimal cover A is a *subset* of each cover B that *refines* A . I think, a more rigorous definition is needed.

(c) On page 3, “upper continuous” lattices are mentioned in lines 3–4. Since this type of lattices is never mentioned later, I suggest to remove the sentence.

(6) Problem with an unpublished source

There are six assertions in 2.2 that are taken from [4]. Proposition 3 and Lemma 8 come without proofs, the two corollaries are immediate from the corresponding lemmas and definitions, and proofs of two lemmas are presented with indication that they are due to the authors of another yet unpublished manuscript. That looks a bit inconsistent. Basically, there are two questions: Are the proofs of Proposition 3 and Lemma 8 too long to be presented? Are the proofs (not assertions) of Lemmas 4 and 6 essential for the proof of the main result?

But there may be the third one: Assume that a part of a manuscript (say, the above mentioned [4]) is published as a part of another article; will it be legal to include the same text and publish elsewhere? I think the editors may clarify this situation.

My suggestion is to provide short outlines of all proofs and mention [4] as the source of further details.