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THE QUASIVARIETY $\mathbf{SP}(L_6)$. I. AN EQUATIONAL BASIS

A. O. BASHEYEVA, M. V. SCHWIDEFSKY, AND K. D. SULTANKULOV

ABSTRACT. We prove that the quasivariety $\mathbf{SP}(L_6)$ is a variety and find an equational basis for this variety.

Keywords: lattice, quasivariety, variety, poset.

1. INTRODUCTION

In the present paper, we consider finite lattice L_6 , see Figure 1, which is isomorphic to the suborder lattice of a three-element chain. Suborder lattices were in the focus in a number of articles as they provide a convenient tool for proving certain embeddability results.

By a theorem by D. Bredikhin and B. Schein [3], suborder lattices are *lattice universal*; that is, each lattice is embeddable into a suitable suborder lattice. By a theorem of B. Sivák [18], a lattice L is embeddable into the suborder lattice of a finite partial order if and only if L is finite and lower bounded in the sense of R. McKenzie [11]. Suborder lattices were used in proving some main results on embeddability of lattices into subsemigroup lattices of certain semigroups in V. B. Repnitskiĭ [13, 14] as well as in [17].

Suborder lattices were also studied in papers [15, 16]. In particular, it was shown in [16] that for arbitrary $n < \omega$, the class \mathbf{SO}_n of lattices embeddable into suborder lattices of posets of length at most n forms a finitely based variety. An equational basis for this variety was found in [16]. There are still a number of unsolved problems which concern suborder lattices. In particular, Question 2 in [16] asks if the quasivariety generated by a finite suborder lattice is a variety.

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In this paper, we give a positive answer to this question in a particular case. More specifically, we prove that the quasivariety $\mathbf{Q}(L_6)$ generated by the lattice L_6 is a finitely based variety and find a finite basis for this variety. The method we use was developed in [4]. In a subsequent article, the main result of this paper, Theorem 12, will be applied for proving a duality result for the quasivariety $\mathbf{Q}(L_6)$.

2. DEFINITIONS AND AUXILIARY RESULTS

We assume all classes to be *abstract*; that is, closed under taking of isomorphic copies.

2.1. (Quasi)varieties. A *quasi-identity* is a universal Horn sentence of the form

$$\forall \bar{x} A_0(\bar{x}) \ \& \ \dots \ \& \ A_n(\bar{x}) \ \longrightarrow \ A(\bar{x}),$$

where $n < \omega$ and $A_0(\bar{x}), \dots, A_n(\bar{x}), A(\bar{x})$ are atomic formulas of a fixed type. An *identity* is a sentence of the form

$$\forall \bar{x} A_0(\bar{x}) \ \& \ \dots \ \& \ A_n(\bar{x}),$$

where $n < \omega$ and $A_0(\bar{x}), \dots, A_n(\bar{x})$ are atomic formulas of a fixed type. A *quasivariety* is the class $\text{Mod}(\Sigma)$ of models for a set Σ of quasi-identities. A *variety* is the class $\text{Mod}(\Sigma)$ of models for a set Σ of identities. In this case, Σ is called a *quasi-equational basis* [an *equational basis*, respectively] of \mathbf{K} . It is clear that each variety is a quasi-variety.

For a type σ , let $\mathbf{K}(\sigma)$ denote the class of all structures of type σ . For an arbitrary class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ of algebraic structures, let $\mathbf{S}(\mathbf{K})$ denote the class of algebraic structures from $\mathbf{K}(\sigma)$ embeddable into structures from the class \mathbf{K} , and let $\mathbf{P}(\mathbf{K})$ denote the class of structures from $\mathbf{K}(\sigma)$ isomorphic to Cartesian products of structures from \mathbf{K} . Whenever \mathbf{K} contains only one structure \mathcal{A} (up to isomorphism), we write $\mathbf{O}(\mathcal{A})$ instead of $\mathbf{O}(\{\mathcal{A}\})$ for a class operator \mathbf{O} . Let $\mathbf{Q}(\mathbf{K})$ denote the smallest quasivariety containing \mathbf{K} . It is well known [10] that for a finite structure \mathcal{A} , the class $\mathbf{SP}(\mathcal{A})$ is a quasivariety. Thus, $\mathbf{Q}(\mathcal{A}) = \mathbf{SP}(\mathcal{A})$ for each finite structure \mathcal{A} .

2.2. General lattices. Let L be a lattice. For arbitrary two sets $A, B \subseteq L$, we say that A *refines* B and write $A \ll B$ if for each $a \in A$, there is $b \in B$ such that $a \leq b$. If $x \in L$, then A is a *join cover* of x if $\bigvee A$ exists and $x \leq \bigvee A$; we also call the inequality $x \leq \bigvee A$ a *join cover* in this case. A join cover $x \leq \bigvee A$ is *nontrivial* if $x \not\leq a$ for all $a \in A$; $x \leq \bigvee A$ is *finite* if the set A is finite. A join cover $x \leq \bigvee A$ is *irredundant* if $x \not\leq \bigvee B$ for all proper subsets $B \subset A$. A join cover $x \leq \bigvee A$ is *minimal* if $A \subseteq B$ for each join cover $x \leq \bigvee B$ such that $B \ll A$. The lattice L has the *complete minimal join cover refinement property* $(\text{CR})_X$ for a set $X \subseteq L$ if each nontrivial join cover of each element from X can be refined to a minimal one.

For a lattice L , let $\text{J}(L)$ denote the set of all join-irreducible elements in L and let $\text{CJ}(L)$ denote the set of all completely join-irreducible elements in L .

Definition 1. [4] For a set $J \subseteq \text{J}(L)$, we say that L is a *J-lattice* if L possesses the following properties:

- (1) for each element $a \in L$, there is a subset $J_a \subseteq J$ with $a = \bigvee J_a$;
- (2) for each element $a \in J$ and each nontrivial join cover $a \leq a_0 \vee \dots \vee a_n$ with $n < \omega$ and $a_0, \dots, a_n \in L$, there is a finite set $F \subseteq J$ such that $a \leq \bigvee F$ is a minimal join cover and $F \ll \{a_0, \dots, a_n\}$.

We say that L is a *CJ-lattice* if L possesses the following properties:

- (1) for each element $a \in L$, there is a subset $J_a \subseteq \mathbf{CJ}(L)$ with $a = \bigvee J_a$;
- (2) L has the property $(\mathbf{CR})_{\mathbf{CJ}(L)}$.

It follows from the definition above that each upper continuous \mathbf{CJ} -lattice is a J -lattice for $J = \mathbf{CJ}(L)$.

In what follows, we consider the following identity of n -distributivity, where $1 < n < \omega$, which we denote by (\mathbf{D}_n) :

$$x \wedge (y_0 \vee y_1 \vee \dots \vee y_n) = \bigvee_{i \leq n} [x \wedge \bigvee_{j \neq i} y_j].$$

This identity was introduced and considered by A. P. Huhn [7]. It is clear that (\mathbf{D}_1) is just the identity of distributivity.

The following lemma is folklore and straightforward to prove, see for example J. B. Nation [12].

Lemma 1. *Let $n > 0$, let L be a lattice, let a set $J \subseteq \mathbf{J}(L)$ be such that for each element $a \in L$, there is a subset $J_a \subseteq J$ with $a = \bigvee J_a$. The following conditions are equivalent.*

- (1) (\mathbf{D}_n) holds in L .
- (2) If $a \leq b_0 \vee b_1 \vee \dots \vee b_n$ for some $a \in J$ and some $b_0, b_1, \dots, b_n \in L$, then there is $i \leq n$ such that $a \leq \bigvee_{j \neq i} b_j$.

Corollary 2. *Let $n > 1$ and let L be a J -lattice for some set $J \subseteq \mathbf{J}(L)$. The following conditions are equivalent.*

- (1) (\mathbf{D}_n) holds in L .
- (2) If $a \leq b_0 \vee \dots \vee b_m$ is a minimal nontrivial join cover for some $a, b_0, \dots, b_m \in J$ then $0 < m < n$.

Proposition 3. [4] *Let L be a complete dually algebraic lattice. Then the following statements hold.*

- (1) If L is n -distributive then L is a $\mathbf{J}(L)$ -lattice.
- (2) If L is in addition algebraic then L is a \mathbf{CJ} -lattice.

Following [4], we denote the next identity by (\mathbf{C}) :

$$\begin{aligned} x \wedge (y_0 \vee y_1) \wedge (z_0 \vee z_1) &= \bigvee_{i < 2} [x \wedge y_i \wedge (z_0 \vee z_1)] \vee \bigvee_{i < 2} [x \wedge z_i \wedge (y_0 \vee y_1)] \vee \\ &\vee \bigvee_{i < 2} [x \wedge ((y_0 \wedge z_i) \vee (y_1 \wedge z_{1-i}))]. \end{aligned}$$

The next four statements were established in [4]. Since [4] is not published yet, we give here proofs Lemma 4 and Lemma 6 for the sake of completeness. We emphasize that these proofs are due to the authors of [4].

Lemma 4. [4] *Let L be a lattice, let a set $J \subseteq \mathbf{J}(L)$ be such that for each element $a \in L$, there is a subset $J_a \subseteq J$ with $a = \bigvee J_a$. The following conditions are equivalent.*

- (1) (\mathbf{C}) holds in L .
- (2) If $a \leq a_0 \vee a_1$ and $a \leq b_0 \vee b_1$ are nontrivial join covers for some $a \in J$ and some $a_0, a_1, b_0, b_1 \in L$, then there are $c_0, c_1 \in L$ such that $a \leq c_0 \vee c_1$, $\{c_0, c_1\} \ll \{a_0, a_1\}$, and $\{c_0, c_1\} \ll \{b_0, b_1\}$.

Proof. We prove first that (1) implies (2). Indeed, let the assumptions of (2) hold. Since (C) holds in L , we have

$$\begin{aligned} a = a \wedge (a_0 \vee a_1) \wedge (b_0 \vee b_1) &= \bigvee_{i < 2} [a \wedge a_i \wedge (b_0 \vee b_1)] \vee \bigvee_{i < 2} [a \wedge b_i \wedge (a_0 \vee a_1)] \vee \\ &\vee \bigvee_{i < 2} [a \wedge ((a_0 \wedge b_i) \vee (a_1 \wedge b_{1-i}))]. \end{aligned}$$

As a is a join-irreducible element, it equals one of the joinands on the right-hand side of the equality above. If $a = a \wedge a_i \wedge (b_0 \vee b_1)$ or $a = a \wedge b_i \wedge (a_0 \vee a_1)$ for some $i < 2$ then $a \leq a_i$ or $a \leq b_i$ which is impossible by our assumption. Therefore, there is $i < 2$ such that $a = a \wedge ((a_0 \wedge b_i) \vee (a_1 \wedge b_{1-i})) \leq (a_0 \wedge b_i) \vee (a_1 \wedge b_{1-i})$. Putting $c_0 = a_0 \wedge b_i$, $c_1 = a_1 \wedge b_{1-i}$, we get the desired conclusion.

We prove now that (2) implies (1). As inequality

$$\begin{aligned} \bigvee_{i < 2} [x \wedge y_i \wedge (z_0 \vee z_1)] \vee \bigvee_{i < 2} [x \wedge z_i \wedge (y_0 \vee y_1)] \vee \\ \vee \bigvee_{i < 2} [x \wedge ((y_0 \wedge z_i) \vee (y_1 \wedge z_{1-i}))] &\leq \\ \leq x \wedge (y_0 \vee y_1) \wedge (z_0 \vee z_1) \end{aligned}$$

holds in each lattice, in order to prove that (C) holds in L , we have to establish that the reverse inequality holds in L . To this end, choose arbitrary elements $u, a_0, a_1, b_0, b_1 \in L$. We put

$$w = \bigvee_{i < 2} [u \wedge a_i \wedge (b_0 \vee b_1)] \vee \bigvee_{i < 2} [u \wedge b_i \wedge (a_0 \vee a_1)] \vee \bigvee_{i < 2} [u \wedge ((a_0 \wedge b_i) \vee (a_1 \wedge b_{1-i}))].$$

We have to show that

$$u \wedge (a_0 \vee a_1) \wedge (b_0 \vee b_1) \leq w.$$

According to our assumption about L , it suffices to show that each element $a \in J$ which is below $u \wedge (a_0 \vee a_1) \wedge (b_0 \vee b_1)$ is also below w . Indeed, let $a \in J$ be such that $a \leq u \wedge (a_0 \vee a_1) \wedge (b_0 \vee b_1)$; then $a \leq u$ and $a \leq a_0 \vee a_1$, $a \leq b_0 \vee b_1$. If $a \leq a_i$ for some $i < 2$ then $a \leq u \wedge a_i \wedge (b_0 \vee b_1) \leq w$. If $a \leq b_i$ for some $i < 2$ then $a \leq u \wedge b_i \wedge (a_0 \vee a_1) \leq w$. Assume therefore that $a \leq a_0 \vee a_1$ and $a \leq b_0 \vee b_1$ are nontrivial join covers. Applying (2), we obtain that there are elements $c_0, c_1 \in L$ such that $a \leq c_0 \vee c_1$, $\{c_0, c_1\} \ll \{a_0, a_1\}$, and $\{c_0, c_1\} \ll \{b_0, b_1\}$. Let $c_0 \leq a_i$ and $c_0 \leq b_j$ for some $i, j < 2$. Then $c_1 \leq a_{1-i}$ and $c_1 \leq b_{1-j}$. Indeed, if, for example, $c_1 \leq a_i$ then we would have $a \leq c_0 \vee c_1 \leq a_i$ which would contradict our assumption that $a \leq a_0 \vee a_1$ is a nontrivial join cover. Therefore, $c_0 \leq a_i \wedge b_j$ and $c_1 \leq a_{1-i} \wedge b_{1-j}$. Hence

$$a = a \wedge (c_0 \vee c_1) \leq u \wedge ((a_i \wedge b_j) \vee (a_{1-i} \wedge b_{1-j})) \leq w,$$

which is our desired conclusion. \square

Corollary 5. [4] *Let L be a 2-distributive J -lattice for some set $J \subseteq J(L)$. The following conditions are equivalent.*

- (1) (C) holds in L .
- (2) If $a \leq a_0 \vee a_1$ and $a \leq b_0 \vee b_1$ are minimal join covers for some elements $a, a_0, a_1, b_0, b_1 \in J$, then $\{a_0, a_1\} = \{b_0, b_1\}$.

As in [4], we denote the following identity by (N_5^1) :

$$x \wedge [(y_0 \wedge (z_0 \vee z_1)) \vee y_1] = [x \wedge y_0 \wedge (z_0 \vee z_1)] \vee [x \wedge y_1] \vee \bigvee_{i < 2} [x \wedge ((y_0 \wedge z_i) \vee y_1)].$$

Lemma 6. [4] *Let L be a lattice, let a set $J \subseteq J(L)$ be such that for each element $a \in L$, there is a subset $J_a \subseteq J$ with $a = \bigvee J_a$. The following conditions are equivalent.*

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- (1) (N_5^1) holds in L .
- (2) If $a \leq a_0 \vee a_1$ is a nontrivial join cover and $a_0 \leq b_0 \vee b_1$ for some $a \in J$ and some $a_0, a_1, b_0, b_1 \in L$, then $a \leq (a_0 \wedge b_i) \vee a_1$ for some $i < 2$.

Proof. We prove first that (1) implies (2). Indeed, let $a \leq a_0 \vee a_1$ for some $a \in J$ and some $a_0, a_1 \in L$ and let $a_0 \leq b_0 \vee b_1$ for some $b_0, b_1 \in L$. Since (N_5^1) holds in L , we have

$$a = a \wedge [(a_0 \wedge (b_0 \vee b_1)) \vee a_1] = [a \wedge a_0 \wedge (b_0 \vee b_1)] \vee [a \wedge a_1] \vee \bigvee_{i < 2} [a \wedge ((a_0 \wedge b_i) \vee a_1)].$$

As a is a join-irreducible element, it equals one of the joinands on the right-hand side of the equality above. If $a = a \wedge a_0 \wedge (b_0 \vee b_1)$ then $a \leq a_0$. If $a = a \wedge a_1$ then $a \leq a_1$. In both cases, we get a contradiction with our assumption that the join cover $a \leq a_0 \vee a_1$ is nontrivial. Thus, there is $i < 2$ such that $a = a \wedge ((a_0 \wedge b_i) \vee a_1) \leq (a_0 \wedge b_i) \vee a_1$ which is our desired conclusion.

We prove now that (2) implies (1). As inequality

$$[x \wedge y_0 \wedge (z_0 \vee z_1)] \vee [x \wedge y_1] \vee \bigvee_{i < 2} [x \wedge ((y_0 \wedge z_i) \vee y_1)] \leq x \wedge [(y_0 \wedge (z_0 \vee z_1)) \vee y_1]$$

holds in each lattice, in order to prove that (N_5^1) holds in L , we have to establish that the reverse inequality holds in L . In order to do this, we choose arbitrary elements $u, a_0, a_1, b_0, b_1 \in L$ and put

$$w = [u \wedge a_0 \wedge (b_0 \vee b_1)] \vee [u \wedge a_1] \vee \bigvee_{i < 2} [u \wedge ((a_0 \wedge b_i) \vee a_1)].$$

We have to show that

$$u \wedge [(a_0 \wedge (b_0 \vee b_1)) \vee a_1] \leq w.$$

According to our assumption about L , it suffices to show that each element $a \in J$ which is below $u \wedge [(a_0 \wedge (b_0 \vee b_1)) \vee a_1]$ is also below w . Indeed, let $a \in J$ be such that $a \leq u \wedge [(a_0 \wedge (b_0 \vee b_1)) \vee a_1]$; then $a \leq u$ and $a \leq (a_0 \wedge (b_0 \vee b_1)) \vee a_1$. If $a \leq a_0 \wedge (b_0 \vee b_1)$ then $a \leq u \wedge a_0 \wedge (b_0 \vee b_1) \leq w$. If $a \leq a_1$ then $a \leq u \wedge a_1 \leq w$. Assume therefore that $a \leq a'_0 \vee a_1$, where $a'_0 = a_0 \wedge (b_0 \vee b_1)$, is a nontrivial join cover. Since $a'_0 \leq b_0 \vee b_1$, applying (2), we obtain that there is $i < 2$ such that

$$a \leq (a'_0 \wedge b_i) \vee a_1 = (a_0 \wedge (b_0 \vee b_1) \wedge b_i) \vee a_1 = (a_0 \wedge b_i) \vee a_1,$$

whence $a \leq u \wedge ((a_0 \wedge b_i) \vee a_1) \leq w$, which finishes the proof. \square

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Corollary 7. [4] *Let L be a 2-distributive J -lattice for some set $J \subseteq J(L)$. The following conditions are equivalent.*

- (1) (N_5^1) holds in L .
- (2) If $a \leq a_0 \vee a_1$ is a minimal join cover for some $a, a_0, a_1 \in J$, then a_0 and a_1 are join-prime elements.

We use the following notation, cf. R. Freese, J. Ježek, and J. B. Nation [5, Lemma 2.33].

Definition 2. Consider a J -lattice L , where $J \subseteq J(L)$. For an element $x \in J$, we put

$$\mathfrak{M}(x) = \left\{ A \subseteq J \mid 1 < |A| < \omega, x \leq \bigvee A \text{ is a minimal nontrivial join cover of } x \right\}.$$

For a set $S \subseteq J$, we put

$$\begin{aligned} S^{[0]} &= S; \\ S^{[n+1]} &= \bigcup \{ A \in \mathfrak{M}(x) \mid x \in S^{[n]} \}, \quad n < \omega; \\ \langle S \rangle_{\mathfrak{M}} &= \bigcup_{i < \omega} S^{[i]}. \end{aligned}$$

For an element $x \in J$, we write $\langle x \rangle_{\mathfrak{M}}$ instead of $\langle \{x\} \rangle_{\mathfrak{M}}$. It is straightforward that $\langle S \rangle_{\mathfrak{M}} = \bigcup_{x \in S} \langle x \rangle_{\mathfrak{M}} \subseteq J$, whence $\langle \rangle_{\mathfrak{M}}$ is an algebraic closure operator on J . A set $S \subseteq J$ is \mathfrak{M} -closed, if $S = \langle S \rangle_{\mathfrak{M}}$.

It is clear that $\mathfrak{M}(x) = \emptyset$ whence $\langle x \rangle_{\mathfrak{M}} = \{x\}$ for each element $x \in J$ which is join-prime.

For a set $S \subseteq J$, we define a binary relation Γ_S on L as follows. If $a, b \in L$ then we put

$$(a, b) \in \Gamma_S \quad \text{if and only if } S \cap \downarrow a = S \cap \downarrow b.$$

Lemma 8. [4] *Let L be a J -lattice and let $A \subseteq J$. The following statements hold.*

- (1) *If $A \subseteq B$ for some $B \subseteq J$ then $\Gamma_B \subseteq \Gamma_A$.*
- (2) *If $A = \bigcup_{i \in I} A_i$ for some $A_i \subseteq J$, $i \in I$, then $\Gamma_A = \bigcap_{i \in I} \Gamma_{A_i}$.*
- (3) *If A is an \mathfrak{M} -closed set then Γ_A is a congruence on L .*

2.3. Suborder lattices. Let X be a set and let $R \subseteq X^2$ be a partial order on X ; that is a reflexive, antisymmetric, and transitive binary relation. In this case, we say that $(X; R)$ is a *partially ordered set* or a *poset* for short. A subset $R' \subseteq R$ is a *suborder* of R if the structure $(X; R')$ is also a poset. The set $O(X, R)$ of all suborders of a partial order R on X is a partially ordered set with respect to the relation \subseteq of set-theoretic inclusion. Obviously, $\Delta = \{(a, a) \mid a \in X\}$ is a least suborder of R . Thus, Δ is the smallest element in $O(X, R)$. It is also obvious that R is the largest element in $O(X, R)$. It is straightforward to check that for an arbitrary family $\{R_i \mid i \in I\} \subseteq O(X, R)$, the relation $\bigcap_{i \in I} R_i$ is also a suborder of R ; that is,

$$\bigwedge_{i \in I} R_i = \bigcap_{i \in I} R_i \in O(X, R).$$

Thus, $O(X, R)$ is a complete lattice, where

$$\bigvee_{i \in I} R_i = \left(\bigcup_{i \in I} R_i \right)^t$$

and Y^t denotes the transitive closure of a binary relation $Y \subseteq X^2$.

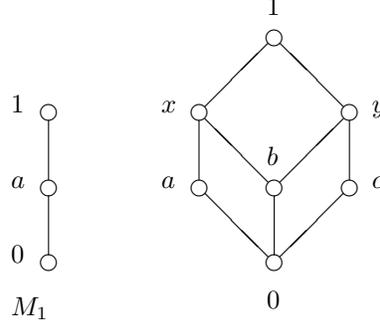


Рис. 1. Partially ordered set M_1 and lattice $L_6 \cong O(M_1)$

3. THE LATTICE L_6

Lemma 9. *The suborder lattice $O(M_1)$ is isomorphic to L_6 .*

Proof. The order relation on M_1 is $R = \{(0, 0), (0, a), (a, a), (a, 1), (0, 1), (1, 1)\}$. Then the suborders of R are exactly the sets

$$\begin{aligned} O &= \{(0, 0), (a, a), (1, 1)\}; \\ A &= \{(0, a)\} \cup O; \\ B &= \{(0, 1)\} \cup O; \\ C &= \{(a, 1)\} \cup O; \\ X &= \{(0, a), (0, 1)\} \cup O; \\ Y &= \{(a, 1), (0, 1)\} \cup O; \\ I = R &= \{(0, a), (a, 1), (0, 1)\} \cup O. \end{aligned}$$

Then it follows that $O(M_1) \cong L_6$, cf. Figure 1. □

4. AN EQUATIONAL BASIS FOR $\mathbf{SP}(L_6)$

We put $\Sigma_6 = \{(C), (D_2), (N_5^1)\}$.

Proposition 10. *Let L be a dually algebraic lattice such that $L \models \Sigma_6$. Then for each element $b \in J(L)$ which is not join-prime, we have*

$$\langle b \rangle_{\mathfrak{M}} = \{a, b, c\}, \text{ where } b \leq a \vee c \text{ is a minimal join cover.}$$

Moreover, $L \in \mathbf{SP}(L_6)$.

Proof. According to Proposition 3(1), L is a J -lattice, where $J = J(L)$ is the set of all join-irreducible elements of L . If $b \in J$ is not join-prime, then according to Corollary 5, $\mathfrak{M}(b) = \{\{a, c\}\}$ for some $a, c \in J$ such that $b \leq a \vee c$ is a minimal nontrivial join cover. According to Corollary 7, elements a and c are join-prime, whence $\langle b \rangle_{\mathfrak{M}} = \{a, b, c\}$.

According to Lemma 8(3), $\Gamma_{\langle b \rangle_{\mathfrak{M}}}$ is a congruence on L for each $b \in J$. Since L is a J -lattice and J is an \mathfrak{M} -closed set, $\Gamma_J = \Delta_L$ is a least congruence on L , whence $L/\Gamma_J \cong L$. As $J = \bigcup_{x \in J} \langle x \rangle_{\mathfrak{M}}$, we conclude by Lemma 8(2) that $L \leq_s$

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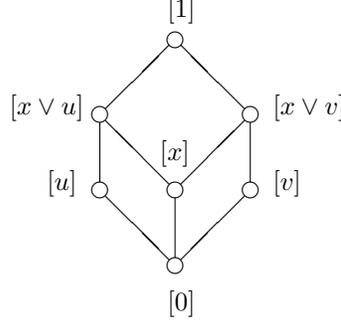


Рис. 2. Lattice $L/\Gamma_{\langle x \rangle_{\mathfrak{M}}} \cong L_6$

$\prod_{x \in J} L/\Gamma_{\langle x \rangle_{\mathfrak{M}}}$. We fix an element $x \in J$. In what follows, let $[z]$ denote the $\Gamma_{\langle x \rangle_{\mathfrak{M}}}$ -equivalence class of an element $z \in L$.

If $x \in J$ is join-prime, then $\langle x \rangle_{\mathfrak{M}} = \{x\}$. Therefore, there are only two $\Gamma_{\langle x \rangle_{\mathfrak{M}}}$ -equivalence classes: $[0_L]$ and $[x]$. Hence $L/\Gamma_{\langle x \rangle_{\mathfrak{M}}} \cong 2$. If $x \in J$ is not join-prime, then $\langle x \rangle_{\mathfrak{M}} = \{x, u, v\}$, where $u, v \in J$ are join-prime and $x \leq u \vee v$ is a minimal nontrivial join cover. This implies in particular that u and v are incomparable and that $x \notin \uparrow u \cap \uparrow v$. The following three cases are therefore possible.

Case 1: $u \not\leq x$ and $v \not\leq x$. In this case, elements x , u , and v are pairwise incomparable. This implies that $x \not\leq u \wedge v$, $u \not\leq x \wedge v$, and $v \not\leq x \wedge u$. Therefore, $[u \wedge v] = [x \wedge u] = [x \wedge v] = [0_L]$. If $v \leq x \vee u$, then $v \leq x$ or $v \leq u$ as v is join-prime. Both cases are impossible as $\{x, u, v\}$ is an anti-chain. This implies that $[x \vee u]$ and $[x \vee v]$ are incomparable elements in the lattice $L/\Gamma_{\langle x \rangle_{\mathfrak{M}}}$. Moreover, $[u \vee v] = [1_L]$ as $x \leq u \vee v$. Therefore, for an arbitrary element $z \in L$, we have in *Case 1* that one of the following cases occurs:

$$\begin{aligned} \langle x \rangle_{\mathfrak{M}} \cap \downarrow z &= \emptyset; & \langle x \rangle_{\mathfrak{M}} \cap \downarrow z &= \{x, u, v\}; \\ \langle x \rangle_{\mathfrak{M}} \cap \downarrow z &= \{x\}; & \langle x \rangle_{\mathfrak{M}} \cap \downarrow z &= \{u\}; & \langle x \rangle_{\mathfrak{M}} \cap \downarrow z &= \{v\}; \\ \langle x \rangle_{\mathfrak{M}} \cap \downarrow z &= \{x, u\}; & \langle x \rangle_{\mathfrak{M}} \cap \downarrow z &= \{x, v\}, \end{aligned}$$

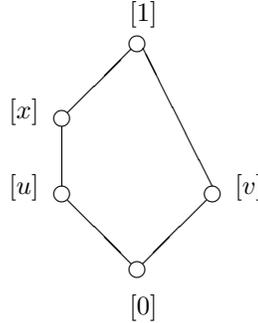
see Figure 2. This implies that $L/\Gamma_{\langle x \rangle_{\mathfrak{M}}}$ has the following elements: $[0_L]$, $[x]$, $[u]$, $[v]$, $[x \vee u]$, $[x \vee v]$, $[1_L]$. Hence $L/\Gamma_{\langle x \rangle_{\mathfrak{M}}} \cong L_6$.

Case 2: $u < x$. In this case, we have $v \not\leq x$, whence x and v are incomparable. Moreover, $x \not\leq u \wedge v$ and $u \not\leq x \wedge v$, whence $[u \wedge v] = [x \wedge v] = [0_L]$. It is also clear that $[x \vee v] = [u \vee v] = [1_L]$. Therefore, for an arbitrary element $z \in L$, we have in *Case 2* that one of the following cases occurs:

$$\begin{aligned} \langle x \rangle_{\mathfrak{M}} \cap \downarrow z &= \emptyset; & \langle x \rangle_{\mathfrak{M}} \cap \downarrow z &= \{x, u, v\}; \\ \langle x \rangle_{\mathfrak{M}} \cap \downarrow z &= \{x\}; & \langle x \rangle_{\mathfrak{M}} \cap \downarrow z &= \{u\}; & \langle x \rangle_{\mathfrak{M}} \cap \downarrow z &= \{v\}, \end{aligned}$$

see Figure 3. This implies that $L/\Gamma_{\langle x \rangle_{\mathfrak{M}}}$ has the following elements: $[0_L]$, $[x]$, $[u]$, $[v]$, $[1_L]$. Hence $L/\Gamma_{\langle x \rangle_{\mathfrak{M}}} \cong N_5 \leq L_6$.

Case 3: $v < x$. This case is symmetric to *Case 2* and therefore, $L/\Gamma_{\langle x \rangle_{\mathfrak{M}}} \cong N_5 \leq L_6$.

Рис. 3. Lattice $L/\Gamma_{\langle x \rangle_{\mathfrak{M}}} \cong N_5$

The above implies that L is a subdirect product of lattices isomorphic either to 2 or to N_5 , or to L_6 . Since both lattices, 2 and N_5 , embed into L_6 , we obtain that $L \in \mathbf{SP}(L_6)$. \square

5-6-0

Corollary 11. *Let L be a bi-algebraic lattice such that $L \models \Sigma_6$. Then for each element $x \in \text{CJ}(L)$ which is not join-prime, we have*

$$[x]_{\mathfrak{M}} = \{x, a, b\}, \text{ where } a, b \in \text{CJ}_P(L), x \leq a \vee b.$$

In particular, $L \in \mathbf{SP}(L_6)$.

Proof. The argument is similar to the one in the proof of Proposition 10 and uses Proposition 3(2). \square

5-7

Theorem 12. *The quasivariety $\mathbf{SP}(L_6)$ is a variety and Σ_6 forms an equational basis for $\mathbf{SP}(L_6)$.*

Proof. Let $L \models \Sigma_6$ and let F be the dual filter lattice of L . Then F is dually algebraic and $F \models \Sigma_6$. By Proposition 10, $F \in \mathbf{SP}(L_6)$ whence $L \in \mathbf{SP}(L_6)$. This proves that $\text{Mod}(\Sigma_6) \subseteq \mathbf{SP}(L_6)$. On the other hand, the lattice L_6 has the only nontrivial join cover $b \leq a \vee c$ of a join-irreducible element. Thus, L_6 is 2-distributive by Corollary 2. Moreover, L_6 satisfies the condition (2) of Corollaries 5 and 7. This implies that $L_6 \models \Sigma_6$ and that $\mathbf{SP}(L_6) \subseteq \text{Mod}(\Sigma_6)$ which proves the desired statement. \square

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AYNUR ORYNBASAROVNA BASHEYEVA
 L. N. GUMILEV EURASIAN NATIONAL UNIVERSITY,
 KAZHYMUKAN STR., 13,
 010008, NUR-SULTAN, KAZAKHSTAN
E-mail address: basheeva@mail.ru

MARINA VLADIMIROVNA SCHWIDEFSKY
 SOBOLEV INSTITUTE OF MATHEMATICS SB RAS,
 ACAD. KOPTYUG AVE., 4,
 630090, NOVOSIBIRSK, RUSSIA
E-mail address: semenova@math.nsc.ru

KUANYSH DAULETBEKOVICH SULTANKULOV
 L. N. GUMILEV EURASIAN NATIONAL UNIVERSITY,
 KAZHYMUKAN STR., 13,
 010008, NUR-SULTAN, KAZAKHSTAN
E-mail address: kuanysh.sultankulov@edu.kz