S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 16, стр. 144–150 (2022) УДК 512.5
DOI 10.33048/semi.2022.16.xxx MSC 13A99

A QUADRATIC PART OF A BENT FUNCTION CAN BE ANY

N.N. TOKAREVA

Abstract. Boolean functions in n variables that are on the maximal possible Hamming distance from all affine Boolean functions in n variables are called bent functions (n is even). They are intensively studied since sixties of XX century in relation to applications in cryptography and discrete mathematics. Often, bent functions are represented in their algebraic normal form (ANF). It is well known that the linear part of ANF of a bent function can be arbitrary. In this note we prove that a quadratic part of a bent function can be arbitrary too.

Keywords: Boolean function, bent function, linear function, quadratic function, homogeneous function.

1. Introduction

Recall that Boolean functions in even number of variables that are on the maximal possible Hamming distance from the set of all affine Boolean functions are called bent functions [8]. Bent functions play an important role in constructions of symmetric ciphers since they help to defend ciphers against linear cryptanalysis[4] and have many applications in discrete mathematics and communications, see [9]. It is well known that every Boolean function can be in the unique way represented in its Algebraic Normal Form (ANF). This representation is used very often for property description and realization of a Boolean function. It is known that bent functions are too far from classification. No conditions on ANF of a Boolean function are known in order to say that the function is bent.

Tokareva, N.N., A quadratic part of a bent function can be any.

The work is supported by Mathematical Center in Akademgorodok under agreement No. 075-15-2019-1613 with the Ministry of Science and Higher Education of the Russian Federation and Laboratory of Cryptography JetBrains Research.

Received March, 1, 2022, published March, 1, 2022.

In this paper a new problem in bent functions is stated and studied: is it true that an arbitrary homogeneous Boolean function of degree k in n variables (n is even) is a k-degree part in ANF of some bent function in n variables? For small k it can be formulated like this. Is it true that linear (quadratic, cubic, etc.) part of ANF of a bent function can be arbitrary? For sure, this question is interesting nor only for bent functions.

It is well known that a linear part in ANF of a bent function can be arbitrary. Moreover, any linear function can be added to a bent function without changing its property to be bent. In this paper we prove that a quadratic part of a bent function can also be arbitrary. Namely, we prove that an arbitrary quadratic homogeneous Boolean function in n variables is a quadratic part of some bent function in n variables, where n is even, $n \geqslant 6$. For cubic parts the question remains open.

2. Preliminaries

We use the following standard notation:

$\mathbb{F}_2^n$ — the vector space over $\mathbb{F}_2$;
$x = (x_1, \ldots, x_n)$ — a binary vector;
$f, g : \mathbb{F}_2^n \to \mathbb{F}_2$ — Boolean functions;
$dist(f, g)$ — *Hamming distance* between f and g, i. e. the number of coordinates in which their vectors of values differ;
$a_1x_1 \oplus \ldots \oplus a_nx_n \oplus b$ — an *affine function* in variables $x_1, \ldots, x_n$, where $a \in \mathbb{F}_2^n$ and $b \in \mathbb{F}_2$, sign $\oplus$ stands for addition modulo 2 (XOR);
bent function — a Boolean function in n variables (n is even) that is on the maximal possible Hamming distance from the set of all affine functions. It is known [8] that this distance is equal to $2^{n-1} - 2^{(n/2)-1}$;
$\mathcal{A}_n$ — the set of all affine functions in n variables;
$\mathcal{B}_n$ — the set of all bent functions in n variables.
Recall that any Boolean function can be uniquely represented in its *algebraic normal form* (ANF):

$$f(x_1, \ldots, x_n) = \left(\bigoplus_{k=1}^{n} \bigoplus_{i_1, \ldots, i_k} a_{i_1, \ldots, i_k} \, x_{i_1} \cdot \ldots \cdot x_{i_k} \right) \oplus a_0,$$

where for each k indices $i_1, \ldots, i_k$ are pairwise distinct and sets $\{i_1, \ldots, i_k\}$ are exactly all different nonempty subsets of the set $\{1, \ldots, n\}$; coefficients $a_{i_1, \ldots, i_k}$, a_0 take values from $\mathbb{F}_2$. For a Boolean function f the number of variables in the longest item of its ANF is called the *algebraic degree* of a function (or briefly *degree*) and is denoted by $deg(f)$. A Boolean function is *affine*, *quadratic*, *cubic* and so on if its degree is not more than 1, or equal to 2, 3, etc.

In what follows let n be an even number.

According to O.Rothaus (1966, 1976) [8] and V. A. Eliseev, O. P. Stepchenkov (1962) [9], degree $deg(f)$ of a bent function f in $n \geqslant 4$ variables is not more than $n/2$. If $n = 2$ a bent function is quadratic. For any possible degree from 2 to $n/2$ it is not difficult to construct a bent function of such degree.

Several restrictions on ANF of bent functions can be naturally considered. A bent function is called *homogeneous* if all monomials of its ANF are of the same degree. C. Qu, J. Seberri and J. Pieprzyk proved [15] that there are 30 homogeneous bent functions of degree 3 in 6 variables. Partial results on classification of cubic

homogeneous bent functions in 8 variables were obtained by C.Charnes et al. in [1]. C. Charnes, M. Rotteler and T. Beth [2] have proved the following fact that we will use further.

Proposition 1. *There exist cubic homogeneous bent functions in each even number of variables n for $n \geqslant 6$.*

For the homogeneous bent functions of higher degrees it is known only a little.

3. On the quadratic part of ANF of a bent function

It is well known that the class of bent functions is closed under addition of affine functions and under affine transformations of variables, see [3]. In other words it holds

Proposition 2. *For any bent function g in n variables (n is even, $n \geqslant 2$) the function $g'(x) = g(Ax \oplus b) \oplus c_1x_1 \oplus \ldots \oplus c_nx_n \oplus d$ is also bent, where A is a nonsingular matrix, b, c are arbitrary binary vectors of length n, d is a constant from $\mathbb{F}_2$.*

Functions g and g' are called *EA-equivalent*.

Note that we can add an arbitrary affine function to a bent function without changing its property to be bent. Recall that it is not possible to find a non affine Boolean function that does the same, since for any non affine Boolean function f there exists a bent function g such that $f \oplus g$ is not bent, see [13], [10]. For instance, it is not possible even to add a quadratic function to all bent functions in order to save their property to be bent. But we want to prove that it is possible to find a bent function with an arbitrary quadratic part of ANF!

In this section we show that an arbitrary quadratic homogeneous Boolean function in n variables is a quadratic part of some bent function in n variables, where n is even, $n \geqslant 6$.

To prove this fact, we need the following statements.

In [6] one can find

Proposition 3. *There exist exactly* 156 *nonisomorphic graphs with* 6 *vertices.*

In [7] all these graphs can be found. Let us prove first the following result.

Proposition 4. *An arbitrary quadratic homogeneous Boolean function in* 6 *variables is a quadratic part of some bent function in* 6 *variables.*

Proof. Let us put into the correspondence to an arbitrary quadratic homogeneous Boolean function f in 6 variables a graph G_f on 6 vertices by the following rule: vertices correspond to variables; there is an edge between two vertices if and only if the product of corresponding variables belongs to ANF of f.

Consider only those quadratic homogeneous Boolean functions that correspond to nonisomorphic graphs. It is clear that if a quadratic homogeneous function f is a quadratic part of some bent function then any quadratic homogeneous function f' with graph $G_{f'}$ isomorphic to G_f is also a quadratic part of some bent function. It holds since any permutation on vertices produce an affine transformation of variables and hence by Proposition 2 does not change a property of a function to be bent.

According to Proposition 3 there are exactly 156 nonisomorphic graphs with 6 variables. We prove the statement by listing in the table in Appendix 1 all 156

corresponding (to graphs) homogeneous quadratic Boolean functions and cubic parts that can be added to them in order to get a bent function in every case. So, the function equal to the sum of the quadratic function from the second column and cubic function from the third column of the table is always bent. Symbol | in both columns should be replaced by $\oplus$, and items like 12 and 123 by x_1x_2 and $x_1x_2x_3$ respectively. We use such short notation in the table for a compactness. Thus, we prove the statement. □

The following iterative construction was proposed by O. Rothaus (1966, 1976) and J. Dillon (1974), see [9].

Proposition 5. *Let f', f'', f''' be bent functions in n variables such that $f' \oplus f'' \oplus f'''$ is a bent function too. Then*

$$g(x, x_{n+1}, x_{n+2}) = f'(x)f''(x) \oplus f'(x)f'''(x) \oplus f''(x)f'''(x) \oplus$$
$$x_{n+1}f'(x) \oplus x_{n+1}f''(x) \oplus x_{n+2}f'(x) \oplus x_{n+2}f'''(x) \oplus x_{n+1}x_{n+2}$$

is a bent function in $n+2$ variables.

Now let us prove the main result.

Theorem 1. *An arbitrary quadratic homogeneous Boolean function in n variables is a quadratic part of some bent function in n variables, where n is even, $n \geqslant 6$.*

Proof. Let us prove it by induction. For $n = 6$ the result follows from Proposition 4. Suppose that it is proven for some n. Consider the case of $n+2$ variables. Let x be a vector of variables $(x_1, \ldots, x_n)$. Assume that $q(x, x_{n+1}, x_{n+2})$ is an arbitrary homogeneous quadratic Boolean function in $n+2$ variables. If q is identically zero, then by Proposition 1 there exists a cubic homogeneous bent function in every number of variables: it will be a bent function with an empty quadratic part.

Let us consider a nonzero q. Since it is nonzero, there exists at least one item in its ANF. W.l.o.g. suppose that ANF of q contains item $x_{n+1}x_{n+2}$. Otherwise by renumbering of variables we turn to this case. So, $q(x, x_{n+1}, x_{n+2})$ is of the form: $q(x, x_{n+1}, x_{n+2}) = h(x) \oplus a(x)x_{n+1} \oplus b(x)x_{n+2} \oplus x_{n+1}x_{n+2}$, where h is a homogeneous quadratic Boolean function in n variables, a, b are some linear functions in n variables.

Consider the quadratic homogeneous Boolean function $h(x) \oplus a(x)b(x)$ in n variables. By induction, there exists a cubic homogeneous Boolean function $c(x)$ such that $f'(x) = c(x) \oplus h(x) \oplus a(x)b(x)$ is a bent function in n variables. Let $f''(x) = f'(x) \oplus a(x)$ and $f'''(x) = f'(x) \oplus b(x)$. According to Proposition 2 functions f'', f''' are bent too. Note that $f' \oplus f'' \oplus f'''$ is also bent by the same reason.

Then, by Proposition 5 a Boolean function

$$g(x, x_{n+1}, x_{n+2}) = f'(x)f''(x) \oplus f'(x)f'''(x) \oplus f''(x)f'''(x)$$
$$\oplus x_{n+1}f'(x) \oplus x_{n+1}f''(x) \oplus x_{n+2}f'(x) \oplus x_{n+2}f'''(x) \oplus x_{n+1}x_{n+2}$$

is a bent function in $n+2$ variables. We see that

$$g(x, x_{n+1}, x_{n+2}) = f'(x)(f'(x) \oplus a(x)) \oplus f'(x)(f'(x) \oplus b(x)) \oplus (f'(x) \oplus a(x))(f'(x) \oplus b(x))$$
$$\oplus x_{n+1}f'(x) \oplus x_{n+1}(f'(x) \oplus a(x)) \oplus x_{n+2}f'(x) \oplus x_{n+2}(f'(x) \oplus b(x)) \oplus x_{n+1}x_{n+2} =$$
$$f'(x) \oplus a(x)b(x) \oplus a(x)x_{n+1} \oplus b(x)x_{n+2} \oplus x_{n+1}x_{n+2}.$$

Hence, we get a bent function

$$g(x, x_{n+1}, x_{n+2}) = c(x) \oplus h(x) \oplus a(x)x_{n+1} \oplus b(x)x_{n+2} \oplus x_{n+1}x_{n+2} = c(x) \oplus q(x, x_{n+1}, x_{n+2})$$

in $n+2$ variables with prescribed quadratic part $q(x, x_{n+1}, x_{n+2})$. □

4. Future remarks

Can a k-degree part of ANF of a bent function be any?

In particular, is it true that the cubic part of a bent function can be arbitrary?

- In case $n = 6$ the answer is **no**, since there exists only three classes of nonequivalent cubic bent functions: $123 \oplus 14 \oplus 25 \oplus 36$, $123 \oplus 245 \oplus 12 \oplus 14 \oplus 26 \oplus 35 \oplus 45$ and $123 \oplus 245 \oplus 346 \oplus 14 \oplus 26 \oplus 34 \oplus 35 \oplus 36 \oplus 45 \oplus 46$, but there are five classes of nonequivalent homogeneous cubic Boolean functions in 6 variables. So, we need to have items of the next degree in order to have a possibility to "put" all variants of the cubic part in a bent function. Here in notation we again use 123 for $x_1x_2x_3$ and so on.
- Case $n = 8$ is still open. The problem is that the existing classification of quartic bent functions in 8 variables (obtained by P. Langevin and G. Leander in 2011, see [5]) does not include the list of representatives of EA-classes.

We think it is a very interesting open problem to study in the general case.

In 2011 we have formulated the following hypothesis, see [12].

Hypothesis 1. *Any Boolean function in n variables of degree not more than $n/2$ can be represented as the sum of two bent functions in n variables (n is even, $n \geqslant 2$).*

The problem to prove or disprove this hypothesis is known now as the *Bent sum decomposition problem*. It is closely connected to the problem of asymptotic of the number of all bent functions.

For now the following is known in relation to this hypothesis.

- Hypothesis is confirmed for $n = 2, 4, 6$ (see [12] and [14]).
- Hypothesis was proved for quadratic Boolean functions, Maiorana—McFarland bent functions, partial spread functions, see [14].
- A weakened variant of the hypothesis was proved: any Boolean function of degree not more than $n/2$ can be represented as the sum of *constant* number of bent functions in n variables, see [11].

Hypothesis 1 can be reformulated like this: *an arbitrary ANF of degree not more than $n/2$ can be "divided" into two parts — every part gives the ANF of a bent function.*

Here we just give an idea that follows from Hypothesis 1 (assuming it holds): *k-degree part of the ANF of a bent function "tends" to be arbitrary.* It is necessary that at least $\sqrt{2^{\binom{n}{k}}}$ different variants of k-degree part of ANF should be realized in a bent function. Recall that the total number of all such variants is $2^{\binom{n}{k}}$.

5. Conclusion

It is very interesting to study if it is possible to define a bent function through the conditions on ANF. Of course, these questions are interesting in respect to an arbitrary class of cryptographic Boolean functions, not only to bent functions. The author is very grateful to E. Ponomareva for valuable contribution in proving of Theorem 1 and to V. Idrisova for kind help and remarks.

References

[1] C. Charnes, U. Dempwolff and J. Pieprzyk, *The eight variable homogeneous degree three bent functions*, Journal of Discrete Algorithms, **6**:1 (2008), 66–72.

[2] C. Charnes, M. Rotteler, T. Beth, *Homogeneous bent functions, invariants, and designs*, Designs, Codes and Cryptography, **26**:1–3 (2002), 139–154.

[3] T. Cusick, P. Stanica, *Cryptographic Boolean Functions and Applications*, Elsevier (2009), ISBN: 9780123748904, 248 pages.

[4] M. Matsui, *Linear Cryptanalysis Method for DES Cipher*, In: Helleseth T. (eds) Advances in Cryptology EUROCRYPT '93. Lecture Notes in Computer Science, **765** (1994), 386–397.

[5] P. Langevin, G. Leander, *Counting all bent functions in dimension eight 99270589265934370305785861242880*, Designs, Codes and Cryptography, **59** (2011), 193–205.

[6] *List of all* 156 *nonisomorphic graphs on* 6 *vertices*, **https://users.cecs.anu.edu.au/ bdm/data/graphs.html**.

[7] *The On-Line Encyclopedia of Integer Sequences*, Edited by N.J.A.Sloane, **https://oeis.org/**.

[8] O. Rothaus, *On bent functions*, Journal of Combinatorial Theory Series A, **20**:3 (1976), 300–305.

[9] N. Tokareva, *Bent functions: results and applications to cryptography*, Elsevier (2015), ISBN-10: 012802318X. ISBN-13: 978-0128023181, 220 pages.

[10] N. N. Tokareva, *Duality between bent functions and affine functions*, Discrete Mathematics, **312**:3 (2012), 666–670.

[11] N. N. Tokareva, *On decomposition of a Boolean function into sum of bent functions*, Siberian Electronic Mathematical Reports, **11** (2012), 745–751.

[12] N. N. Tokareva, *On the number of bent functions from iterative constructions: lower bounds and hypotheses*, Advances in Mathematics of Communications, **5**:4 (2011), 609–621.

[13] N. N. Tokareva, *The group of automorphisms of the set of bent functions*, Discrete Mathematics and Applications, **20**:5–6 (2010), 655–664.

[14] L. Qu, S. Fu, Q. Dai, C. Li, *When a Boolean Function can be Expressed as the Sum of two Bent Functions*, Cryptology ePrint Archive, Report 2014/048, available on **http://eprint.iacr.org/**.

[15] C. Qu, J. Seberry, J. Pieprzyk, *Homogeneous bent functions*, Discrete Applied Mathematics, **102**:1–2 (2000), 133–139

Appendix 1n

N	homogeneous quadratic function	homogeneous cubic function
1	–	123|125|126|134|136|145|146|156|234|235|245|246|256|345|346|356
2	12	125|126|134|136|145|146|156|234|235|245|246|256|345|346|356
3	12|13|14	123|125|126|134|136|146|156|234|235|245|246|256|345|346|356
4	12|13|14|15	123|125|126|134|136|145|146|156|234|235|245|246|256|345|346|356
5	12|13|14|15|16	123|125|134|145|234|235|245|246|256|345|346|356
6	12|13|14|15|16|23|24|25|26|34|35|36|45|46|56	–
7	12|13|14|15|16|23|24|26|34|35|45|46|56	123|135|136|234|235|236|245|246
8	12|13|14|15|16|23|24|34	125|145|235|256|356
9	12|13|14|15|16|23|24|34|45|46	123|124|126|136|145|245|345
10	12|13|14|15|16|23|25|26|34|35|36|45|46|56	124|125|126|134|146|234|245
11	12|13|14|15|16|23|25|26|34|36|45|56	123|135|136|156|235
12	12|13|14|15|16|23|25|26|34|56	124|125|145
13	12|13|14|15|16|23|25|26|35|36	125|135|136|156|245|246|256
14	12|13|14|15|16|23|25|34|35|36|45|56	123|124|134|145|234|235|245
15	12|13|14|15|16|23|25|34|36|45|56	123|124|134|135|145|234|235|245
16	12|13|14|15|16|23|26|34|35|45|56	123|125|135
17	12|13|14|15|16|23|26|34|35|56	123|125|235
18	12|13|14|15|16|23|26|35|56	123|124|136|234|236|246
19	12|13|14|15|16|23|34	124|125|146|235|245|246|256|356
20	12|13|14|15|16|23|34|35	124|134|135|145|245|246|256
21	12|13|14|15|16|23|34|35|36|45|56	124|134|145|234|245
22	12|13|14|15|16|23|34|45|46	135|145|345
23	12|13|14|15|16|23|34|45|56	–
24	12|13|14|15|16|24|34|45|46	124|125|126|136|145|234|235|236|256|345
25	12|13|14|15|16|26|34|45	123|134|234
26	12|13|14|15|16|56	124|134|234|235|236|246|345
27	12|13|14|15|23|24|25|26|34|35|36|45|46	–
28	12|13|14|15|23|24|25|26|34|35|36|45|46|56	125|126|134|136|145
29	12|13|14|15|23|24|25|34|35|45	146|156|236|256|346
30	12|13|14|15|23|25|34|45	123|126|135|136|156|235|236|256
31	12|13|14|15|23|34|35|36|45|46|56	–
32	12|13|14|16|23|24|26|34|35|45|56	–
33	12|13|14|16|23|24|34|45	–
34	12|13|14|16|23|24|34|56	–
35	12|13|14|16|23|25|34|36|45|56	123|124|134|145|234|235|245
36	12|13|14|16|23|26|34|35|45|56	–
37	12|13|14|16|23|26|34|36|45|56	–
38	12|13|14|16|23|34|45|46|56	–
39	12|13|14|16|23|34|45|56	–
40	12|13|14|16|23|34|56	123|125|235
41	12|13|14|16|23|45|46|56	124|126|134|135|145|146|156
42	12|13|14|16|24|25|26|34|35|36|45	123|124|234
43	12|13|14|23	125|126|134|136|146|156|234|235|245|246|256|345|346|356
44	12|13|14|23|24|25|26|34|35|36|46|56	–
45	12|13|14|23|24|25|34|35|45	123|126|134|136|146|236|246
46	12|13|14|23|24|34	125|126|136|156|235|256|356
47	12|13|14|23|24|34|35|36|45|46|56	–
48	12|13|14|23|24|34|56	–
49	12|13|14|23|34	124|125|126|136|145|156|235|245|246|256|356
50	12|13|14|23|34|56	123|135|145|146|156|235|236
51	12|13|14|45|56	234|236|246
52	12|13|14|56	234|235|236|246|345
53	12|13|15|16|23|24|26|34|35|45|46|56	123|124|134|135|136|145|146|234|235|236|245|246
54	12|13|15|16|23|24|26|34|36	123|245|246|256
55	12|13|15|16|23|24|34|45	–
56	12|13|15|16|23|24|34|45|46	123|124|126|136|145|245|345
57	12|13|15|16|23|24|34|45|46|56	124|125|145
58	12|13|15|16|23|25|26|34|36|45|56	123|134|135|234|345
59	12|13|15|16|23|25|26|45|46|56	134|135|145|146|156
60	12|13|15|16|23|34|35|36|45|56	–

N	homogeneous quadratic function	homogeneous cubic function
61	12|13|15|16|23|34|45|56	124|126|146
62	12|13|15|16|26|34|45	123|134|234
63	12|13|15|16|34|35	123|245|246|256
64	12|13|15|16|34|45|46	123|124|134|135|145|234|235|236|256|345
65	12|13|15|16|45	123|136|234|235|236|256|346
66	12|13|15|23|26|34|35|36|45|46	–
67	12|13|15|23|34|35|45	126|134|136|146|236|246
68	12|13|16|23|24|34|45	–
69	12|13|16|23|24|34|45|56	–
70	12|13|16|23|25|34|35|45|56	–
71	12|13|16|23|25|34|45|56	–
72	12|13|16|23|34|35|45|46|56	–
73	12|13|23	125|126|134|136|145|146|156|234|235|245|246|256|345|346|356
74	12|13|23|24|26|34|35|56	123|125|135
75	12|13|23|34|35	124|126|134|136|145|146|156|245|246|256|346
76	12|13|23|34|35|36|45|46|56	–
77	12|13|23|34|35|45	136|146|346
78	12|13|23|34|45|56	–
79	12|13|23|34|56	–
80	12|13|23|45|56	124|134|145|146|234|246|345
81	12|14|14|16|23|34|45|46|56	124|125|145
82	12|14|15|16|23|24|34	123|145|235|256|356
83	12|14|15|16|23|26|34|35	124|134|234
84	12|14|15|16|23|26|34|45|56	123|125|135
85	12|14|15|16|23|34	123|124|126|135|145|235|245|246|256|356
86	12|14|16|23|24|34|56	123|124|125|126|134|136|145
87	12|14|16|23|24|45|56	134|136|146
88	12|14|16|23|25|34|36|45|56	123|124|134|135|145|234|235|245
89	12|14|16|23|26|34|35|45|56	123|125|135
90	12|14|16|23|34|45	–
91	12|14|16|23|34|45|56	–
92	12|14|16|34|45	123|124|135|235|236|256
93	12|14|23|25|34|45	123|126|135|136|156|235|236|256
94	12|14|23|26|35|45|56	123|135|235
95	12|14|23|34	123|124|125|126|134|135|136|145|156|235|245|246|256|356
96	12|14|23|34|45|56	124|126|146
97	12|14|23|34|56	123|135|145|146|156|235|236
98	12|15|16|23|24|34|45|56	123|125|126|136|145
99	12|15|16|23|24|45|46	123|125|135|136|156|256
100	12|15|16|23|24|45|46|56	134|135|145
101	12|15|23|24|25|34|35|45	126|136|156|236|256
102	12|15|23|24|25|35|45	123|126|134|135|136|146|156|234|235|236|246|256
103	12|15|23|24|34|35|45	123|126|136|146|234|236|246
104	12|15|23|24|34|45	126|146|246
105	12|15|23|34|45	126|146|246
106	12|16|23|24|25|34|36|45|46|56	123|134|136|234|236|246
107	12|16|23|24|25|34|36|45|56	123|135|235
108	12|16|23|24|26|34|35|45|46|56	123|124|234|235|245
109	12|16|23|24|26|34|45|46|56	124|126|146
110	12|16|23|25|26|34|36|45|56	123|134|136|234|236
111	12|16|23|25|34|36|45|56	123|124|135|234|235|245
112	12|16|23|26|34|35	124|145|245
113	12|16|23|26|34|35|45	–
114	12|16|23|26|34|45|56	123|125|135
115	12|16|23|34|45|56	124|126|146
116	12|16|26|34|35|45	123|125|134|135|136|145|146
117	12|23	123|125|126|134|136|145|146|156|234|235|245|246|256|345|346|356
118	12|23|24|25|26|34|35|36|45	–
119	12|23|24|25|26|34|35|36|45|46|56	–
120	12|23|24|25|26|34|45|56	–

N	homogeneous quadratic function	homogeneous cubic function
121	12|23|24|25|26|34|56	–
122	12|23|24|25|34|35|45	123|126|134|136|146|236|246
123	12|23|24|25|34|45	123|126|135|136|156|235|236|256
124	12|23|24|26|34|35|36|45|46|56	–
125	12|23|24|26|34|36|45|56	123|124|134|135|145|234|245
126	12|23|24|26|34|45|46|56	–
127	12|23|24|26|34|45|56	–
128	12|23|24|26|34|46|56	–
129	12|23|24|26|34|56	–
130	12|23|24|34|35|36|45|46|56	–
131	12|23|24|34|45	123|126|135|136|156|236|256
132	12|23|24|34|45|46|56	–
133	12|23|25|26|34|45|46	123|124|134|135|136|156|234|235|245|345
134	12|23|25|34	123|126|135|145|146|156|234|246|356
135	12|23|25|34|35|36|45|56	123|124|134|135|136|145|146|234|235|236|245|246
136	12|23|25|34|35|45	123|126|134|136|146|234|236|246
137	12|23|25|34|35|56	–
138	12|23|25|34|45	123|126|134|135|136|146|156|234|235|236|246|256
139	12|23|26|34|35|36|45|56	135|136|156
140	12|23|26|34|35|45|56	–
141	12|23|26|34|45|56	–
142	12|23|26|34|56	–
143	12|23|26|35|45|56	125|126|134|135|136|146|234|235|345
144	12|23|34	125|126|135|145|146|156|235|245|246|256|356
145	12|23|34|35|45	124|126|145|146|156|246|256
146	12|23|34|35|45|56	–
147	12|23|34|45	124|126|135|136|145|146|156|236|246|256
148	12|23|34|45|56	–
149	12|23|34|56	–
150	12|23|45	125|135|136|146|156|234|256|346
151	12|23|45|56	124|125|134|136|145|146|156|246
152	12|34	125|126|136|145|146|156|235|245|246|256|356
153	12|34|56	–
154	13|23|45	124|126|134|135|145|146|156|246|256
155	16|23|24|25|26|34|36|45|56	123|134|135|136|145|156|234|235|245
156	16|23|26|34|35|36|56	123|124|125|126|145|156|235

Appendix 1

N	homogeneous quadratic function	homogeneous cubic function
1	–	123|125|126|134|136|145|146|156|234|235|245|246|256|345|346|356
2	16|23|26|34|35|36|56	123|124|125|126|145|156|235
3	16|23|24|25|26|34|36|45|56	123|134|135|136|145|156|234|235|245
4	13|23|45	124|126|134|135|145|146|156|246|256
5	12	125|126|134|136|145|146|156|234|235|245|246|256|345|346|356
6	12|34	125|126|136|145|146|156|235|245|246|256|356
7	12|34|56	–
8	12|23	123|125|126|134|136|145|146|156|234|235|245|246|256|345|346|356
9	12|23|45	125|135|136|146|156|234|256|346
10	12|23|45|56	124|125|134|136|145|146|156|246
11	12|23|34	125|126|135|145|146|156|235|245|246|256|356
12	12|23|34|56	–
13	12|23|34|45	124|126|135|136|145|146|156|236|246|256
14	12|23|34|45|56	–
15	12|23|34|35|45	124|126|145|146|156|246|256
16	12|23|34|35|45|56	–
17	12|23|26|35|45|56	125|126|134|135|136|146|234|235|345
18	12|23|26|34|56	–
19	12|23|26|34|45|56	–
20	12|23|26|34|35|45|56	–
21	12|23|26|34|35|36|45|56	135|136|156
22	12|23|25|34	123|126|135|145|146|156|234|246|356
23	12|23|25|34|45	123|126|134|135|136|146|156|234|235|236|246|256
24	12|23|25|34|35|56	–
25	12|23|25|34|35|45	123|126|134|136|146|234|236|246
26	12|23|25|34|35|36|45|56	123|124|134|135|136|145|146|234|235|236|245|246
27	12|23|25|26|34|45|46	123|124|134|135|136|156|234|235|245|345
28	12|23|24|34|45	123|126|135|136|156|236|256
29	12|23|24|34|45|46|56	–
30	12|23|24|34|35|36|45|46|56	–
31	12|23|24|26|34|56	–
32	12|23|24|26|34|46|56	–
33	12|23|24|26|34|45|56	–
34	12|23|24|26|34|45|46|56	–
35	12|23|24|26|34|36|45|56	123|124|134|135|145|234|245
36	12|23|24|26|34|35|36|45|46|56	–
37	12|23|24|25|34|45	123|126|135|136|156|235|236|256
38	12|23|24|25|34|35|45	123|126|134|136|146|236|246
39	12|23|24|25|26|34|56	–
40	12|23|24|25|26|34|45|56	–
41	12|23|24|25|26|34|35|36|45	–
42	12|23|24|25|26|34|35|36|45|46|56	–
43	12|16|26|34|35|45	123|125|134|135|136|145|146
44	12|16|23|34|45|56	124|126|146
45	12|16|23|26|34|45|56	123|125|135
46	12|16|23|26|34|35	124|145|245
47	12|16|23|26|34|35|45	–
48	12|16|23|25|34|36|45|56	123|124|135|234|235|245
49	12|16|23|25|26|34|36|45|56	123|134|136|234|236
50	12|16|23|24|26|34|45|46|56	124|126|146
51	12|16|23|24|26|34|35|45|46|56	123|124|234|235|245
52	12|16|23|24|25|34|36|45|56	123|135|235
53	12|16|23|24|25|34|36|45|46|56	123|134|136|234|236|246
54	12|15|23|34|45	126|146|246
55	12|15|23|24|34|45	126|146|246
56	12|15|23|24|34|35|45	123|126|136|146|234|236|246
57	12|15|23|24|25|35|45	123|126|134|135|136|146|156|234|235|236|246|256
58	12|15|23|24|25|34|35|45	126|136|156|236|256
59	12|15|16|23|24|45|46	123|125|135|136|156|256
60	12|15|16|23|24|45|46|56	134|135|145

N	homogeneous quadratic function	homogeneous cubic function
61	12|15|16|23|24|34|45|56	123|125|126|136|145
62	12|14|23|34	123|124|125|126|134|135|136|145|156|235|245|246|256|356
63	12|14|23|34|56	123|135|145|146|156|235|236
64	12|14|23|34|45|56	124|126|146
65	12|14|23|26|35|45|56	123|135|235
66	12|14|23|25|34|45	123|126|135|136|156|235|236|256
67	12|14|16|34|45	123|124|135|235|236|256
68	12|14|16|23|34|45	–
69	12|14|16|23|34|45|56	–
70	12|14|14|16|23|34|45|46|56	124|125|145
71	12|14|16|23|26|34|35|45|56	123|125|135
72	12|14|16|23|25|34|36|45|56	123|124|134|135|145|234|235|245
73	12|14|16|23|24|45|56	134|136|146
74	12|14|16|23|24|34|56	123|124|125|126|134|136|145
75	12|14|15|16|23|34	123|124|126|135|145|235|245|246|256|356
76	12|14|15|16|23|26|34|45|56	123|125|135
77	12|14|15|16|23|26|34|35	124|134|234
78	12|14|15|16|23|24|34	123|145|235|256|356
79	12|13|23	125|126|134|136|145|146|156|234|235|245|246|256|345|346|356
80	12|13|23|45|56	124|134|145|146|234|246|345
81	12|13|23|34|56	–
82	12|13|23|34|45|56	–
83	12|13|23|34|35	124|126|134|136|145|146|156|245|246|256|346
84	12|13|23|34|35|45	136|146|346
85	12|13|23|34|35|36|45|46|56	–
86	12|13|23|24|26|34|35|56	123|125|135
87	12|13|16|23|34|35|45|46|56	–
88	12|13|16|23|25|34|45|56	–
89	12|13|16|23|25|34|35|45|56	–
90	12|13|16|23|24|34|45	–
91	12|13|16|23|24|34|45|56	–
92	12|13|15|23|34|35|45	126|134|136|146|236|246
93	12|13|15|23|26|34|35|36|45|46	–
94	12|13|15|16|45	123|136|234|235|236|256|346
95	12|13|15|16|34|45|46	123|124|134|135|145|234|235|236|256|345
96	12|13|15|16|34|35	123|245|246|256
97	12|13|15|16|26|34|45	123|134|234
98	12|13|15|16|23|34|45|56	124|126|146
99	12|13|15|16|23|34|35|36|45|56	–
100	12|13|15|16|23|25|26|45|46|56	134|135|145|146|156
101	12|13|15|16|23|25|26|34|36|45|56	123|134|135|234|345
102	12|13|15|16|23|24|34|45	–
103	12|13|15|16|23|24|34|45|46	123|124|126|136|145|245|345
104	12|13|15|16|23|24|34|45|46|56	124|125|145
105	12|13|15|16|23|24|26|34|36	123|245|246|256
106	12|13|15|16|23|24|26|34|35|45|46|56	123|124|134|135|136|145|146|234|235|236|245|246
107	12|13|14	123|125|126|134|136|146|156|234|235|245|246|256|345|346|356
108	12|13|14|56	234|235|236|246|345
109	12|13|14|45|56	234|236|246
110	12|13|14|23	125|126|134|136|146|156|234|235|245|246|256|345|346|356
111	12|13|14|23|34	124|125|126|136|145|156|235|245|246|256|356
112	12|13|14|23|34|56	123|135|145|146|156|235|236
113	12|13|14|23|24|34	125|126|136|156|235|256|356
114	12|13|14|23|24|34|56	–
115	12|13|14|23|24|34|35|36|45|46|56	–
116	12|13|14|23|24|25|34|35|45	123|126|134|136|146|236|246
117	12|13|14|23|24|25|26|34|35|36|46|56	–
118	12|13|14|16|24|25|26|34|35|36|45	123|124|234
119	12|13|14|16|23|45|46|56	124|126|134|135|145|146|156
120	12|13|14|16|23|34|56	123|125|235

N	homogeneous quadratic function	homogeneous cubic function
121	12|13|14|16|23|34|45|56	–
122	12|13|14|16|23|34|45|46|56	–
123	12|13|14|16|23|26|34|36|45|56	–
124	12|13|14|16|23|26|34|35|45|56	–
125	12|13|14|16|23|25|34|36|45|56	123|124|134|145|234|235|245
126	12|13|14|16|23|24|34|56	–
127	12|13|14|16|23|24|34|45	–
128	12|13|14|16|23|24|26|34|35|45|56	–
129	12|13|14|15	123|125|126|134|136|145|146|156|234|235|245|246|256|345|346|356
130	12|13|14|15|23|34|35|36|45|46|56	–
131	12|13|14|15|23|25|34|45	123|126|135|136|156|235|236|256
132	12|13|14|15|23|24|25|34|35|45	146|156|236|256|346
133	12|13|14|15|23|24|25|26|34|35|36|45|46	–
134	12|13|14|15|23|24|25|26|34|35|36|45|46|56	125|126|134|136|145
135	12|13|14|15|16	123|125|134|145|234|235|245|246|256|345|346|356
136	12|13|14|15|16|56	124|134|234|235|236|246|345
137	12|13|14|15|16|26|34|45	123|134|234
138	12|13|14|15|16|24|34|45|46	124|125|126|136|145|234|235|236|256|345
139	12|13|14|15|16|23|34	124|125|146|235|245|246|256|356
140	12|13|14|15|16|23|34|45|56	–
141	12|13|14|15|16|23|34|45|46	135|145|345
142	12|13|14|15|16|23|34|35	124|134|135|145|245|246|256
143	12|13|14|15|16|23|34|35|36|45|56	124|134|145|234|245
144	12|13|14|15|16|23|26|35|56	123|124|136|234|236|246
145	12|13|14|15|16|23|26|34|35|56	123|125|235
146	12|13|14|15|16|23|26|34|35|45|56	123|125|135
147	12|13|14|15|16|23|25|34|36|45|56	123|124|134|135|145|234|235|245
148	12|13|14|15|16|23|25|34|35|36|45|56	123|124|134|145|234|235|245
149	12|13|14|15|16|23|25|26|35|36	125|135|136|156|245|246|256
150	12|13|14|15|16|23|25|26|34|56	124|125|145
151	12|13|14|15|16|23|25|26|34|36|45|56	123|135|136|156|235
152	12|13|14|15|16|23|25|26|34|35|36|45|46|56	124|125|126|134|146|234|245
153	12|13|14|15|16|23|24|34	125|145|235|256|356
154	12|13|14|15|16|23|24|34|45|46	123|124|126|136|145|245|345
155	12|13|14|15|16|23|24|26|34|35|45|46|56	123|135|136|234|235|236|245|246
156	12|13|14|15|16|23|24|25|26|34|35|36|45|46|56	–

Natalia Nikolaevna Tokareva
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia
Email address: tokareva@math.nsc.ru