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THE VERTEX CONNECTIVITY OF SOME CLASSES OF DIVISIBLE DESIGN GRAPHS

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Abstract. A k-regular graph is called a divisible design graph if its vertex set can be partitioned into m classes of size n, such that two distinct vertices from the same class have exactly λ_1 common neighbours, and two vertices from different classes have exactly λ_2 common neighbours. In this paper, we find the vertex connectivity of some classes of divisible design graphs, in particular, we present examples of divisible design graphs, whose vertex connectivity is less than k, where k is the degree of a vertex. We also show that the vertex connectivity a divisible design graphs may be less than k by any power of 2.

Keywords: Deza graph, divisible design graph, strongly regular graph, vertex connectivity

1. Introduction

Deza graphs were introduced in [7] as a generalisation of strongly regular graphs. A *strongly regular graph* (SRG for short) G with parameters (v, k, λ, μ) is a k-regular graph with v vertices such that any two adjacent vertices have λ common neighbours and any two non-adjacent vertices have μ common neighbours. A *Deza graph* Γ with parameters (v, k, b, a) is a k-regular graph with v vertices for which the number of common neighbours of two distinct vertices takes just two values, b or a, where $b \geqslant a$. A Deza graph of diameter 2 that is not a strongly regular graph is called a *strictly Deza graph*.

A k-regular graph is called a *divisible design graph* (DDG for short) if its vertex set can be partitioned into m classes of size n, such that two distinct vertices from

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the same class have exactly λ_1 common neighbours, and two vertices from different classes have exactly λ_2 common neighbours. The definition implies that all divisible design graphs are Deza graphs. Divisible design graphs were first studied in master's thesis by M.A. Meulenberg [14] and then studied in more detail in [6, 11].

The vertex connectivity of a k-regular Cayley graph is at least $\frac{2}{3}(k+1)$ (see [5]). The vertex connectivity of a strongly regular graph is equal to its valency, as was proved by A.E. Brouwer and D.M. Mesner in [4]. In [3] the same result was obtained in general for distance-regular graphs.

The vertex connectivity of Deza graphs obtained from strongly regular graphs by dual Seidel switching was studied in [8] by A.L. Gavrilyuk, S. Goryainov and V.V. Kabanov and in [10] by S. Goryainov and D. Panasenko. As a result of these studies, an infinite series of Deza graphs with vertex connectivity $k-1$ was found.

In this paper we find the vertex connectivity of some classes of divisible design graphs. We focus on the cases when the vertex connectivity is less than k. In particular, we present DDGs with vertex connectivity $k-1$ and, for any positive integer t, we present a DDG whose vertex connectivity equals $k-2^t$ ($k > 2^t$).

This paper is organised as follows. In Section 2, we give some definitions, notations and preliminary results on SRGs, DDGs and vertex connectivity. In Section 3, we present results on vertex connectivity. In Section 4, we discuss the problem of finding the vertex connectivity of DDGs in general.

2. Preliminaries

A strongly regular graph G is called *primitive* if both G and its complement are connected.

Lemma 1 ([1, Theorem 1.3.1]). *Let G be a primitive strongly regular graph with parameters (v, k, λ, μ). Then the following statements hold.*

(1) *G has three distinct eigenvalues k, r, s, where $k > r > 0 > s$. Moreover, r and s satisfy the quadratic equation $x^2 + (\mu - \lambda)x + (\mu - k) = 0$.*

(2) *If the eigenvalues r and s have equal multiplicities, then $r = (-1 + \sqrt{v})/2$ and $s = (-1 - \sqrt{v})/2$. Otherwise, r and s are integers.*

(3) *The equalities $\mu = k + rs$ and $\lambda = \mu + r + s$ hold.*

A strongly regular graph with $s = -2$ is called a *Seidel graph*. These graphs were characterised in [1, Theorem 3.12.4].

An incidence structure with v points and v blocks of constant size k is called a *symmetric* 2-(v, k, λ)-*design* if any pair of points occur together in exactly λ blocks and any two blocks intersect in exactly λ points.

An incidence structure on v points with constant block size k is a *(group) divisible design* whenever the point set can be partitioned into m classes of size n, such that two points from one class occur together in exactly λ_1 blocks, and two points from different classes occur together in exactly λ_2 blocks. A divisible design D is called *symmetric* if the dual of D (that is, the design with the transposed incidence matrix) is again a divisible design with the same parameters as D. Equivalently, DDGs can be defined as graphs whose adjacency matrix is the incidence matrix of a symmetric divisible design. A DDG with $m = 1$, $n = 1$ or $\lambda_1 = \lambda_2$ is called *improper* (these DDGs are (v, k, λ)-graphs), otherwise it is called *proper.* A (v, k, λ)-graph is a k-regular graph on v vertices with the property that any two distinct vertices have exactly λ common neighbors, that is, a strongly regular graph with $\lambda = \mu$, a clique or a coclique.

For a positive integer t, denote by I_t, O_t and J_t the identity matrix, the zero matrix and the all-ones matrix of size $t \times t$, respectively. For positive integers m and n, denote by $K_{(m,n)}$ the matrix $I_m \otimes J_n$. Note that $K_{(m,n)} = diag(J_n, \ldots, J_n)$. A graph Γ is a DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ if and only if its adjacency matrix A satisfies

$$A^2 = kI_v + \lambda_1(K_{(m,n)} - I_v) + \lambda_2(J_v - K_{(m,n)}).$$

The formula for A^2 also gives strong information about the eigenvalues of A and their multiplicities (see the following two lemmas).

Lemma 2 ([11, Lemma 2.1]). *A has at most five distinct eigenvalues* k, $\sqrt{k - \lambda_1}$, $-\sqrt{k - \lambda_1}$, $\sqrt{k^2 - \lambda_2 v}$ *and* $-\sqrt{k^2 - \lambda_2 v}$ *with corresponding multiplicities* $1, f_1, f_2, g_1$ *and* g_2*, where* $f_1 + f_2 = m(n-1)$ *and* $g_1 + g_2 = m - 1$.

Lemma 3 ([11, Theorem 2.2]). *Consider a proper DDG with parameters* $(v, k, \lambda_1, \lambda_2, m, n)$ *and eigenvalue multiplicities* f_1, f_2, g_1, g_2. *Then:*
(1) $k - \lambda_1$ *or* $k^2 - \lambda_2 v$ *is a nonzero square;*
(2) *if* $k - \lambda_1$ *is not a square, then* $f_1 = f_2 = m(n-1)/2$*;*
(3) *if* $k^2 - \lambda_2 v$ *is not a square, then* $g_1 = g_2 = (m-1)/2$.

Let $V_1 \cup \ldots \cup V_t$ be a partition of the vertex set of a graph Γ with the property that, for any $i, j \in \{1, \ldots, t\}$, every vertex of V_i has exactly r_{ij} neighbours in V_j for some constant r_{ij} (depending on i and j). Then $V_1 \cup \ldots \cup V_t$ is called an *equitable t-partition* of Γ. The matrix $R = (r_{ij})_{t \times t}$ is called the *quotient matrix* of the equitable partition.

The vertex partition from the definition of a DDG is called the *canonical partition.*

Lemma 4 ([11, Theorem 3.1]). *The canonical partition of the vertex set of a DDG is equitable, and the quotient matrix R satisfies*

$$R^2 = RR^T = (k^2 - \lambda_2 v)I_m + \lambda_2 n J_m.$$

Moreover, the eigenvalues of R are k, $\sqrt{k^2 - \lambda_2 v}$ *and* $-\sqrt{k^2 - \lambda_2 v}$ *with corresponding multiplicities* 1, g_1 *and* g_2*, where* g_1 *and* g_2 *are given by Lemmas 2 and 3.*

A graph is called *walk-regular*, whenever for every $l \geqslant 2$ the number of closed walks of length l at a vertex x is independent of the choice of x.

Lemma 5 ([6, Theorem 4.3]). *A proper DDG is walk-regular if and only if the quotient matrix R has constant diagonal.*

2.1. **Constructions of DDGs.** The *incidence graph* of a design with incidence matrix N is the bipartite graph with adjacency matrix $\begin{bmatrix} O & N \\ N^\top & O \end{bmatrix}$.

Construction 1 ([11, Construction 4.1]). *The incidence graph of a symmetric* 2-(n, k, λ_1)*-design with* $1 < k \leqslant n$ *is a proper DDG with parameters* $(2n, k, \lambda_1, \lambda_2, 2, n)$*, where* $\lambda_2 = 0$.

Proposition 1 ([11, Proposition 4.3]). *For a proper connected DDG* Γ *with parameters* $(v, k, \lambda_1, \lambda_2, m, n)$*, the following statements are equivalent.*
(1) $\lambda_2 = 0$.
(2) Γ *comes from Construction 1.*

Construction 2 ([11, Construction 4.4]). *Let A' be the adjacency matrix of a connected (m, k', λ')-graph with $1 < k' < m$. Then, for any positive integer n, $n > 1$, the matrix $A' \otimes J_n$ is the adjacency matrix of a proper DDG with parameters $(mn, k, \lambda_1, \lambda_2, m, n)$, where $k = \lambda_1 = nk'$ and $\lambda_2 = n\lambda'$.*

Proposition 2 ([11, Proposition 4.5]). *For a proper connected DDG Γ with parameters $(v, k, \lambda_1, \lambda_2, m, n)$, the following statements are equivalent.*

(1) $\lambda_1 = k$.

(2) Γ *comes from Construction 2.*

Construction 3 ([11, Construction 4.6]). *Let $A_1, \dots, A_m$ $(m \geqslant 2)$ be the adjacency matrices of not necessary connected (n, k', λ')-graphs (possibly non-isomorphic, but having the same parameters) with $0 \leqslant k' \leqslant n-2$. Then the matrix $J_v - K_{(m,n)} + diag(A_1, \dots, A_m)$ is the adjacency matrix of a proper DDG with parameters $(mn, k, \lambda_1, \lambda_2, m, n)$, where $k = k' + n(m-1)$, $\lambda_1 = \lambda' + n(m-1)$ and $\lambda_2 = 2k - v$.*

Proposition 3 ([11, Proposition 4.7]). *For a proper DDG Γ with parameters $(v, k, \lambda_1, \lambda_2, m, n)$, the following statements are equivalent.*

(1) $\lambda_2 = 2k - v$.

(2) Γ *comes from Construction 3.*

The *lexicographic product* or *graph composition* $G[H]$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and adjacency defined by

$$(u_1, v_1) \sim (u_2, v_2) \text{ if and only if } u_1 \sim u_2, \text{ or } u_1 = u_2 \text{ and } v_1 \sim v_2.$$

Construction 4 ([11, Construction 4.10]). *Let G be a strongly regular graph with parameters $(v', k', \lambda, \lambda + 1)$. Then $G[K_2]$ is a DDG with parameters $(2v', 2k' + 1, 2k', 2\lambda + 2, v', 2)$.*

An automorphism of order 2 of a graph is called a *Seidel automorphism* if it interchanges only non-adjacent vertices. Permuting the rows (and not the columns) of the adjacency matrix of a graph according to Seidel automorphism is called *dual Seidel switching* (DSS for short).

Construction 5 ([9, Construction 2]). *Let Γ be a DDG obtained with Construction 4. Let A be the adjacency matrix of Γ, and P be a non-identity permutation matrix of the same size. If P represents a Seidel automorphism, then PA is the adjacency matrix of a DDG with the same parameters as Γ.*

Proposition 4 ([9, Theorem 2]). *For a proper DDG Γ with parameters $(v, k, \lambda_1, \lambda_2, m, n)$, where $\lambda_2 \notin \{0, 2k - v\}$, the following statements are equivalent.*

(1) $\lambda_1 = k - 1$.

(2) Γ *comes from Construction 4 or 5.*

An $m \times m$ matrix H is a *Hadamard matrix* if every entry is 1 or -1 and $HH^\top = mI_m$. A Hadamard matrix H is called *graphical* if H is symmetric with constant diagonal, and *regular* if all row and column sums are equal.

Construction 6 ([11, Construction 4.9]). *Consider a regular graphical Hadamard matrix H of order $l^2 \geqslant 4$ with diagonal entries -1 and row sum l. The graph with adjacency matrix*

$$A = \begin{bmatrix} M & N & O \\ N & O & M \\ O & M & N \end{bmatrix}, \text{ where}$$

$M = \frac{1}{2}\begin{bmatrix} J_{l^2}+H & J_{l^2}+H \\ J_{l^2}+H & J_{l^2}+H \end{bmatrix}$, $N = \frac{1}{2}\begin{bmatrix} J_{l^2}+H & J_{l^2}-H \\ J_{l^2}-H & J_{l^2}+H \end{bmatrix}$, *and* $O = O_{2l_2}$ *is a DDG with parameters* $(6l^2,\ 2l^2+l,\ l^2+l,\ (l^2+l)/2,\ 3,\ 2l^2)$.

2.2. **Vertex connectivity.** The *vertex connectivity* $\varkappa(G)$ of a graph G is the minimum number of vertices whose deletion from G disconnects it. Note that for a k-regular graph G the inequality $\varkappa(G) \leqslant k$ holds. Let x and y be two vertices in a graph G. Two simple paths connecting x and y are called *disjoint* if they have no common vertices different from x and y. A set of vertices S *disconnects* x and y if x and y belong to different connected components of the graph obtained from G by deleting S. A set S of vertices of a graph G is called *disconnecting* if it disconnects some two of its vertices. The following lemma is known as Menger's theorem.

Lemma 6 ([12, Theorem 5.9])**.** *The minimum cardinality of a set disconnecting non-adjacent vertices x and y is equal to the largest number of disjoint paths connecting these vertices.*

Lemma 7 ([16, Theorem 1])**.** *Let G_1 and G_2 be two graphs. If G_1 is non-complete and connected, then $\varkappa(G1[G2]) = \varkappa(G1) \cdot |V(G2)|$.*

Below we present known results on the vertex connectivity of Deza graphs.

Lemma 8 ([8, Theorem])**.** *Let Γ be a Deza graph obtained from a strongly regular graph G with non-principal eigenvalues r and s by dual Seidel switching. Then one of the following three cases holds.*

(1) *If $r > 2$ and $s < -2$, then the vertex connectivity of Γ is equal to its valency, and a disconnecting set of minimum cardinality is the neighbourhood of some vertex.*

(2) *$s = -2$ and the vertex connectivity of Γ is equal to its valency except for the case when G is 3×3-lattice graph and the vertex connectivity of Γ is 3.*

(3) *$r \leq 2$.*

Lemma 9 ([10, Theorems 1-2])**.** *Let Γ be a k-regular Deza graph obtained from a strongly regular graph G with $r = 1$ by dual Seidel switching. Then the vertex connectivity of Γ is equal to its valency except for the case when G is the complement of $n \times n$-lattice graph (in this case the vertex connectivity of Γ equals $k - 1$).*

3. The vertex connectivity of special classes of DDGs

3.1. DDGs with $\lambda_1 \in \{k-1, k\}$ or $\lambda_2 \in \{0, 2k-v\}$.

Proposition 5. *The vertex connectivity of a connected DDG Γ with parameters $(v, k, \lambda_1, \lambda_2, m, n)$, where $\lambda_2 = 0$, equals k.*

Proof. By Proposition 1, Γ is the incidence graph of a symmetric 2-(n, k, λ_1)-design. Such graphs are distance-regular graphs with diameter 3 (see [1, Theorem 1.6.1]), except for the case when $n = k$ (in this case, Γ is the complete bipartite graph $K_{n,n}$ and $\varkappa(\Gamma) = n = k$). Since the vertex connectivity of distance-regular graphs equals k (see [3]), the statement of the proposition is true. □

Proposition 6. *The vertex connectivity of a connected DDG Γ with parameters $(v, k, \lambda_1, \lambda_2, m, n)$, where $\lambda_1 = k$, equals k.*

Proof. By Proposition 2, Γ can only be obtained by Construction 2. Note that Construction 2 can be described as the lexicographic product of a connected (m, k', λ')-graph and a coclique of size n. Since (m, k', λ')-graphs with $1 < k' < m$ are strongly regular, their vertex connectivity equals k' (see [4]). So, by Lemma 7, $\varkappa(\Gamma) = k'n = k$. □

Proposition 7. *Let Γ be a DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n)$, where $\lambda_2 = 2k - v$. Then the following statements hold.*

(1) *If $\lambda_1 \neq k - 1$, then the vertex connectivity of Γ equals k.*
(2) *If $\lambda_1 = k - 1$, then the vertex connectivity of Γ equals $k - 1$.*

Proof. By Proposition 3, Γ can only be obtained by Construction 3. Recall that the adjacency matrix of Γ is $J_v - K_{(m,n)} + diag(A_1, \dots, A_m)$, where $A_1, \dots, A_m$ are the adjacency matrices of not necessary connected (n, k', λ')-graphs $\Gamma_1, \dots, \Gamma_m$ (possibly non-isomorphic, but having the same parameters).

Consider two non-adjacent vertices u_1 and u_2 in Γ. It follows from Construction 3 that u_1 and u_2 are two non-adjacent vertices in Γ_i for some $i \in \{1, \dots, m\}$. Vertices u_1 and u_2 have $\varkappa(\Gamma_i)$ disjoint paths connecting them in Γ_i and $(m-1)n$ disjoint paths connecting them in the rest of Γ. In total u_1 and u_2 have $\varkappa(\Gamma_i) + (m-1)n$ disjoint paths connecting them in Γ. So, by Lemma 6, the vertex connectivity of Γ equals $min(\varkappa(\Gamma_1), \dots, \varkappa(\Gamma_m)) + (m-1)n$.

An (n, k', λ')-graph is disconnected if and only if $k' = 0$ or $k' = 1$.

If $k' = 0$, then such an (n, k', λ')-graph is the coclique of size n, so Γ is the complete m-partite graph with parts of size n and $\varkappa(\Gamma) = 0 + (m-1)n = k$.

If $k' = 1$, then such an (n, k', λ')-graph is the union of $n/2$ edges, so Γ is the complete m-partite graph with parts of size n extended with a perfect matching of the complement (see [11, Section 4.1]) and $\varkappa(\Gamma) = 0 + (m-1)n = \lambda_1 = k - 1$. Note that $k' = 1$ is the only case when $\lambda_1 = k - 1$.

If $1 < k' \leq n - 2$, then such an (n, k', λ')-graph is a connected strongly regular graph and its vertex connectivity equals k'. Thus, $\varkappa(\Gamma) = k' + (m-1)n = k$. □

Proposition 8. *The vertex connectivity of a DDG Γ with parameters $(v, k, \lambda_1, \lambda_2, m, n)$, where $\lambda_1 = k - 1$ and $\lambda_2 \notin \{0, 2k - v\}$, equals $k - 1$.*

Proof. By Proposition 4, for such a DDG we have the following two cases.

Case 1: Γ is obtained with Construction 4, therefore Γ is $G[K_2]$, where G is a strongly regular graph with parameters $(v', k', \lambda, \lambda + 1)$. The vertex connectivity of G equals k', so, by Lemma 7, $\varkappa(\Gamma) = 2k' = k - 1$.

Case 2: Γ is obtained with Construction 5. Γ can be viewed as follows (see [9, Construction 2]). Consider $G'[K_2]$, where G' is a Deza graph obtained from a strongly regular graph G with parameters $(v', k', \lambda, \lambda + 1)$ by dual Seidel switching with respect to Seidel automorphism φ. By the definition, the vertices of $G'[K_2]$ can be viewed as pairs $\{(u, v) : u \in V(G'), v \in V(K_2)\}$. Modify $G'[K_2]$ as follows: for any transposition $(u_1\, u_2)$ of φ, take the corresponding two pairs of vertices $(u_1, v_1), (u_1, v_2)$ and $(u_2, v_1), (u_2, v_2)$ in Γ', delete the edges $\{(u_1, v_1), (u_1, v_2)\}$ and $\{(u_2, v_1), (u_2, v_2)\}$, and insert the edges $\{(u_1, v_1), (u_2, v_2)\}$ and $\{(u_1, v_2), (u_2, v_1)\}$. The resulting graph is isomorphic to Γ.

Let r and s be the non-principal eigenvalues of G. Consider the cases according to Lemma 8.

(1) If $r > 2$ and $s < -2$, then by Lemma 8(1) the vertex connectivity of G' equals k.

Note that Γ can be viewed as two copies of G' connected by additional edges. Also note that since $n = 2$, each entry on the main diagonal of the quotient matrix of Γ can only be 0 (if two vertices forming this part are non-adjacent) or 1 (if two vertices forming this part are adjacent).

Consider two vertices of Γ that form a part of the canonical partition. These vertices can be written as (u_1, v_1) and (u_1, v_2), where $v_1 \neq v_2$. The vertices (u_1, v_1) and (u_1, v_2) are adjacent in Γ if and only if u_1 is fixed by φ. Γ is not walk-regular (see [9, Section 5]), so, by Lemma 5, the main diagonal of the quotient matrix is not constant, so it contains both 0 and 1. Thus, there exists a part of the canonical partition of Γ consisting of two adjacent vertices. Consider such two vertices. They have $2k'$ common neighbours. If we remove all $2k'$ their common neighbours, we separate the edge formed by these vertices from the rest. So, $\varkappa(\Gamma) \leqslant 2k'$.

Next, we show that, for any pair of non-adjacent vertices in Γ, there are $2k'$ disjoint paths connecting them.

There are two types of non-adjacent vertices in Γ:

(1.1) Vertices (u_1, v_1) and (u_2, v_2), where u_1 and u_2 are two non-adjacent vertices from G' such that the transposition $(u_1\, u_2)$ is not in φ (v_1 can be equal to v_2).

Consider two non-adjacent vertices (u_1, v_1) and (u_2, v_1) from the same copy of G', where u_1 and u_2 are two non-adjacent vertices from G'. Since the vertex connectivity of G' equals k', (u_1, v_1) and (u_2, v_1) have k' disjoint paths connecting them in their copy of G'. Also the vertices (u_1, v_2) and (u_2, v_2) have k' disjoint paths connecting them in their copy ($v_1 \neq v_2$). Since the vertices (u_1, v_1) and (u_1, v_2) (as well as (u_2, v_1) and (u_2, v_2)) form a part of the canonical partition, they have exactly k' common neighbours in one copy of G' and exactly k' common neighbours in the other copy. Therefore, each of k' disjoint paths connecting (u_1, v_1) and (u_2, v_1) in their copy of G' also connects (u_1, v_1) and (u_2, v_2), (u_1, v_2) and (u_2, v_1), and (u_1, v_2) and (u_2, v_2). A similar argument applies to each of k' disjoint paths connecting (u_1, v_2) and (u_2, v_2) in their copy of G'.

So, any two non-adjacent vertices from the case (1.1) have k' disjoint paths in each copy of G' connecting them, which in total gives $2k'$ disjoint paths connecting them.

(1.2) Vertices (u_1, v_1) and (u_1, v_2), where u_1 is moved by φ, $v_1 \neq v_2$. These vertices form a part of the canonical partition, so they have $2k'$ common neighbours, so they have $2k'$ disjoint paths connecting them.

Any pair of non-adjacent vertices in Γ has $2k'$ disjoint paths connecting them, therefore, by Lemma 6, the vertex connectivity of Γ is equal to $2k'$.

(2) If $r \leqslant 2$ and r is not an integer, then G has at most 25 vertices (see [8, Conclusion]). The only strongly regular graphs satisfying this condition are Paley graphs with parameters $(13, 6, 2, 3)$ and $(17, 8, 3, 4)$. By computer calculations using SageMath, these graphs do not have Seidel automorphisms.

(3) If $r = 1$ (or, equivalently, $s = -2$ by Lemma 1(1)), then G is a Seidel graphs. There are three Seidel graphs with $\lambda = \mu - 1$: 3×3-lattice graph with parameters $(9, 4, 1, 2)$, Petersen graph with parameters $(10, 3, 0, 1)$ and triangular graph $T(5)$ with parameters $(10, 6, 3, 4)$. By computer calculations using SageMath, $T(5)$ does not have Seidel automorphisms, one graph can be obtained by DSS from 3×3-lattice graph and one graph can be obtained by DSS from Petersen graph. The vertex connectivity of DDGs obtained from these two graphs equals $k - 1$.

(4) If $r = 2$ (or, equivalently, $s = -3$ by Lemma 1(1)), then there exist 26 SRGs with $\lambda = \mu - 1$ (see [13]): 15 graphs with parameters $(25, 12, 5, 6)$, 10 graphs with parameters $(26, 10, 3, 4)$ and one graph with parameters $(50, 7, 0, 1)$. By computer

calculations using SageMath, the vertex connectivity of DDGs obtained from these graphs equals $k-1$.

Thus, all DDGs obtained by Construction 5 have vertex connectivity equal to $k-1$. □

3.2. **DDGs obtained with Construction 6.** Let Γ be a DDG with parameters $(6l^2,\, 2l^2+l,\, l^2+l,\, (l^2+l)/2,\, 3,\, 2l^2)$ obtained with Construction 6 with positive l. Consider the subgraph induced by the first $2l^2$ vertices of Γ (in terms of Construction 6 this subgraph has adjacency matrix M) and the subgraph induced by the last $2l^2$ vertices of Γ (in terms of Construction 6 this subgraph has adjacency matrix N). Denote by Γ_1 and Γ_2 the first and the second subgraph, respectively.

Lemma 10. *Γ_1 is an (l^2+l)-regular graph and Γ_2 is an l^2-regular graph.*

Proof. Consider notations from Construction 6.

Let x and y be the numbers of 1s and -1s in each row of H, respectively (since H is a regular graphical Hadamard matrix, x and y do not depend on the choice of a row). Then $x-y=l$ and $x+y=l^2$.

Note that the number of 1s in each row of $\frac{1}{2}(J+H)$ is x, and the number of 1s in each row of $\frac{1}{2}(J-H)$ is y. Thus the number of 1s in each row of M is $2x$, which equals l^2+l, and the number of 1s in each row of N is $x+y$, which equals l^2. So Γ_1 is an (l^2+l)-regular graph and Γ_2 is an l^2-regular graph. □

Lemma 11. *The vertex connectivity of Γ is at most $2l^2$.*

Proof. Consider notations from Construction 6.

Denote by A' the matrix $\begin{bmatrix} M & O \\ O & N \end{bmatrix}$, which can be obtained by removing $2l^2$ rows and columns from the middle of A. The matrix A', considered as an adjacency matrix, defines a disconnected graph with two components: the (l^2+l)-regular graph Γ_1 and the l^2-regular graph Γ_2. Thus, vertex connectivity of Γ is at most $2l^2$. □

Lemma 12. *If $\varkappa(\Gamma_1) \geq l^2, \varkappa(\Gamma_2) \geq l^2-l$ and $\varkappa(\Gamma_1)+\varkappa(\Gamma_2) \geq 2l^2$, then the vertex connectivity of Γ equals $2l^2$.*

Proof. Denote by Γ_0 the coclique induced by the $2l^2$ vertices of Γ that correspond to the middle rows and columns of A. Note that any vertex from Γ_1 has exactly l^2 neighbours in Γ_0 and any vertex from Γ_2 has exactly l^2+l neighbours in Γ_0.

Let t_1, t_2 and t_0 be non-negative integers, such that $t_1+t_2+t_0 < 2l^2$. If $t_1 \geq \varkappa(\Gamma_1)$ and $t_2 \geq \varkappa(\Gamma_2)$, then $t_1+t_2 \geq 2l^2$, which contradicts to the inequality above. Denote by $\hat{\Gamma}_1, \hat{\Gamma}_2$ and $\hat{\Gamma}_0$ the graphs obtained by deletion of t_1, t_2 and t_0 vertices from Γ_1, Γ_2 and Γ_0, respectively (also denote by $\hat{\Gamma}$ the graph obtained by deletion of $t_1+t_2+t_0$ vertices from Γ).

Consider three possible cases:

Case 1: $t_1 < \varkappa(\Gamma_1)$ and $t_2 < \varkappa(\Gamma_2)$. Then $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are connected. Thus, if there is at least one vertex in $\hat{\Gamma}_0$ that has neighbours in both $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$, then $\hat{\Gamma}$ is connected. Let us show that such a vertex exists.

Since Γ_2 is an l^2-regular graph, we have the inequality $\varkappa(\Gamma_2) \leq l^2$ and, consequently, $t_2 < l^2$. Since l^2 is less than l^2+l, which is the number of neighbours in Γ_0 for any vertex from Γ_2, we conclude that any vertex from Γ_0 is adjacent to at least one vertex from $\hat{\Gamma}_2$. This means that any vertex from $\hat{\Gamma}_0$ is adjacent to at least one vertex from $\hat{\Gamma}_2$.

Consider two subcases: $t_1 < l^2$ and $t_1 \geq l^2$.

If $t_1 < l^2$, an argument similar to the argument above implies that any vertex from $\hat{\Gamma}_0$ is adjacent to at least one vertex from $\hat{\Gamma}_1$. So there exists a vertex from $\hat{\Gamma}_0$ that has neighbours in both $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$.

If $t_1 \geq l^2$, then the inequalities $t_1 + t_2 + t_0 < 2l^2$ and $t_2 \geq 0$ imply that t_0 is less than l^2, which is the number of neighbours in Γ_0 for any vertex from Γ_1. Therefore any vertex from $\hat{\Gamma}_1$ is adjacent to at least one vertex from $\hat{\Gamma}_0$. So there exists a vertex from $\hat{\Gamma}_0$ that has neighbours in both $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$.

Case 2: $t_1 \geq \varkappa(\Gamma_1)$ and $t_2 < \varkappa(\Gamma_2)$. Then $\hat{\Gamma}_1$ is disconnected and $\hat{\Gamma}_2$ is connected. Thus, if, for each connected component of $\hat{\Gamma}_1$, there is at least one vertex in $\hat{\Gamma}_0$ having neighbours in both this component and $\hat{\Gamma}_2$, then $\hat{\Gamma}$ is connected. Let us show that such vertices exist.

Since $t_2 < \varkappa(\Gamma_2) \leq l^2$, any vertex from $\hat{\Gamma}_0$ is adjacent to at least one vertex from $\hat{\Gamma}_2$ (see Case 1).

Since $t_1 \geq \varkappa(\Gamma_1) \geq l^2$ and $t_2 \geq 0$, the inequality $t_0 < l^2$ holds. So any vertex from $\hat{\Gamma}_1$ is adjacent to at least one vertex from $\hat{\Gamma}_0$ (see Case 1). So, for each connected component of $\hat{\Gamma}_1$, there is a vertex from $\hat{\Gamma}_0$ having neighbours in both this component and $\hat{\Gamma}_2$.

Case 3: $t_1 < \varkappa(\Gamma_1)$ and $t_2 \geq \varkappa(\Gamma_2)$. This case is similar to Case 2.

So, the vertex connectivity of Γ equals $2l^2$. □

3.3. **DDGs with parameters** $(6 \cdot 4^t, 2 \cdot 4^t + 2^t, 4^t + 2^t, 2 \cdot 4^{t-1} + 2^{t-1}, 3, 2 \cdot 4^t)$**.** If H_1 and H_2 are Hadamard matrices, then so is the Kronecker product $H_1 \otimes H_2$. Moreover, if H_1 and H_2 are regular with row sums l_1 and l_2, respectively, then $H_1 \otimes H_2$ is regular with row sum $l_1 l_2$. Similarly, the Kronecker product of two graphical Hadamard matrices is graphical again.

Consider regular graphical Hadamard matrices H and H', where

$$H = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \text{ and } H' = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

Denote by H_1 the matrix H. For any integer t such that $t > 1$, denote by H_t the Kronecker product $H_{t-1} \otimes H'$. The matrix H_t is a regular graphical Hadamard matrices of order 4^t with diagonal entries -1 and row sum 2^t (see [2, Section 10.5.1]).

Applying Construction 6 to H_t, we obtain a DDG with parameters $(6 \cdot 4^t$, $2 \cdot 4^t + 2^t$, $4^t + 2^t$, $2 \cdot 4^{t-1} + 2^{t-1}$, 3, $2 \cdot 4^t)$. The smallest example is a DDG with parameters $(24, 10, 6, 3, 3, 8)$ and adjacency matrix

$$\begin{bmatrix} D & D & D & I & O & O \\ D & D & I & D & O & O \\ D & I & O & O & D & D \\ I & D & O & O & D & D \\ O & O & D & D & D & I \\ O & O & D & D & I & D \end{bmatrix},$$

where $D = J - I, J = J_4, I = I_4$ and $O = O_4$.

By replacing

$$D \to \begin{bmatrix} D & D & D & I \\ D & D & I & D \\ D & I & D & D \\ I & D & D & D \end{bmatrix}, \; I \to \begin{bmatrix} I & I & I & D \\ I & I & D & I \\ I & D & I & I \\ D & I & I & I \end{bmatrix}, \; O \to \begin{bmatrix} O & O & O & O \\ O & O & O & O \\ O & O & O & O \\ O & O & O & O \end{bmatrix}, \qquad (1)$$

we get a recursive construction for a DDG with parameters $(6 \cdot 4^t,\ 2 \cdot 4^t + 2^t,\ 4^t + 2^t,\ 2 \cdot 4^{t-1} + 2^{t-1},\ 3,\ 2 \cdot 4^t)$. Denote by Γ^t this DDG.

Consider the subgraph formed by the first $2 \cdot 4^t$ vertices of Γ^t (in terms of Construction 6 this subgraph has adjacency matrix M) and the subgraph formed by the last $2 \cdot 4^t$ vertices of Γ^t (in terms of Construction 6 this subgraph has adjacency matrix N). Let Γ^t_1 denote the first subgraph and Γ^t_2 denote the second subgraph.

Let us describe one of the ways to count the number of common neighbours in Γ^t; this approach is used in Lemmas 16 and 17 to count the number of common neighbours in Γ^t_1 and Γ^t_2. Each vertex from Γ^t can be associated with the corresponding block-row of the adjacency block-matrix. Two blocks from distinct block-rows are called a *pair* if they lie in the same block-column.

Lemma 13. *Let x and y be two distinct vertices from Γ^t. The following cases hold.*

(1) *If the vertices x and y are associated with the same block-row, then, for this block-row and the vertices x and y, each block D gives* 2 *common neighbours and each block I or O gives* 0 *common neighbours.*

(2) *If the vertices x and y are associated with distinct block-rows, then the following subcases hold.*

(2.1) *If the vertices x and y correspond to the rows with the same index within their block-rows then, for these block-rows and the vertices x and y, each pair D and D gives* 3 *common neighbours; each pair I and I gives* 1 *common neighbour; each pair D and I or pair I and D gives* 0 *common neighbours; any pair with block O gives* 0 *common neighbours.*

(2.2) *If the vertices x and y correspond to rows with different indices within their block-rows, then, for these block-rows and the vertices x and y, each pair D and D gives* 2 *common neighbours; each pair I and I gives* 0 *common neighbours; each pair D and I or pair I and D gives* 1 *common neighbour; any pair with block O gives* 0 *common neighbours.*

Proof. It follows from the definition of blocks D, I and O. □

Next, we investigate how the replacement of blocks in the recursive construction of Γ^t affects the adjacency block-matrix. In this investigation, we exclude all the cases related to the block O.

Lemma 14. *For a block-row in the recursive construction of Γ^t, each block D gives* 3 *new blocks D and* 1 *new block I in each block-row of the adjacency block-matrix after replacing; each block I gives* 1 *new block D and* 3 *new blocks I in each block-row of the adjacency block-matrix after replacing.*

Proof. It follows from the description of the recursive construction (see (1)). □

Lemma 15. *For two distinct block-rows in the recursive construction of Γ^t,the following cases hold.*

(1) *If two distinct block-rows are obtained from one block-row, then each block D gives* 2 *pairs D and D,* 1 *pair I and D and* 1 *pair D and I after replacing; each block I gives* 2 *pairs I and I,* 1 *pair I and D and* 1 *pair D and I after replacing.*

(2) *If two distinct block-rows are obtained from two distinct block-rows and have the same index within replaceable block-row, then each pair D and D gives* 3 *pairs D and D and* 1 *pair I and I; each pair I and I gives* 3 *pairs I and I and* 1 *pair D and D; each pair D and I gives* 3 *pairs D and I and* 1 *pair I and D; each pair I and D gives* 3 *pairs I and D and* 1 *pair D and I.*

(3) *If two distinct block-rows are obtained from two distinct block-rows and have different indices within replaceable block-row, then each pair D and D gives* 2 *pairs D and D,* 1 *pair D and I and* 1 *pair I and D; each pair I and I gives* 2 *pairs I and I,* 1 *pair D and I and* 1 *pair I and D; each pair D and I gives* 3 *pairs D and I and* 1 *pair I and D; each pair I and D gives* 3 *pairs I and D and* 1 *pair D and I.*

Proof. It follows from the definition of blocks D and I and the description of the recursive construction (see (1)). □

In the following two lemmas, we present a structural description of Γ_1^t and Γ_2^t, respectively.

Lemma 16. *The graph Γ_1^t is a DDG with parameters* $(2 \cdot 4^t, 4^t + 2^t, 4^t + 2^t, 2 \cdot (4^{t-1} + 2^{t-1}), 4^t, 2)$.

Proof. The graph Γ_1^1 has adjacency matrix $\begin{bmatrix} D & D \\ D & D \end{bmatrix}$. For Γ_1^t, a recursive construction similar to (1) can be applied. Denote by A_1^t the adjacency block-matrix of Γ_1^t. Note that A_1^t does not contain a block O.

The following table shows the number of blocks in each block-row of A_1^t according to Lemma 14.

t	D	I
1	2	0
2	6	2
3	20	12
...	...	...
t	$4^{t-1} + 2^{t-1}$	$4^{t-1} - 2^{t-1}$
$t+1$	$4^t + 2^t$	$4^t - 2^t$
...	...	...

TABLE 1. The number of blocks in each block-row of A_1^t

Note that, given a block-raw in A_1^t, there is exactly one block-row that is equal to the given one (that is, the block-row with the same sequence of blocks).

The next table shows the number of pairs of blocks in two distinct block-rows of A_1^t according to Lemma 15 (the case of equal block-rows is excluded, except for $t = 1$). Note that this table represents each of the cases from Lemma 15.

t	D, D	I, I	D, I	I, D
1	2	0	0	0
2	4	0	2	2
3	12	4	8	8
...	...	...	...	...
t	$2 \cdot (4^{t-2} + 2^{t-2})$	$2 \cdot (4^{t-2} - 2^{t-2})$	$2 \cdot 4^{t-2}$	$2 \cdot 4^{t-2}$
$t+1$	$2 \cdot (4^{t-1} + 2^{t-1})$	$2 \cdot (4^{t-1} - 2^{t-1})$	$2 \cdot 4^{t-1}$	$2 \cdot 4^{t-1}$
...	...	...	...	...

TABLE 2. The number of pairs of blocks in two different block-rows of A_1^t

The degree of each vertex from Γ_1^t (the number of 1s in the corresponding row of the adjacency matrix) is $3 \cdot (4^{t-1} + 2^{t-1}) + 4^{t-1} - 2^{t-1}$, which is equal to $4^t + 2^t$.

Next, we find the number of common neighbours for each pair of distinct vertices from Γ_1^t.

For each block-row in the adjacency block-matrix of Γ_1^t, there is exactly one equal block-row, so, for each row in the adjacency matrix Γ_1^t, there is exactly one equal row. The number of common neighbours for the vertices corresponding to such equal rows is $4^t + 2^t$.

Consider two vertices associated with the same block-row (see Lemma 13(1)). They have $2 \cdot (4^{t-1} + 2^{t-1})$ common neighbours. Note that this case is similar to the case of two unequal rows associated with two equal block-rows of the adjacency block-matrix.

Consider two vertices associated with distinct block-rows (see Lemma 13(2)). They have $3 \cdot 2 \cdot (4^{t-2} + 2^{t-2}) + 2 \cdot (4^{t-2} - 2^{t-2})$ common neighbours, which is equal to $2 \cdot (4^{t-1} + 2^{t-1})$ (see Lemma 13(2.1)), or $2 \cdot 2 \cdot (4^{t-2} + 2^{t-2}) + 2 \cdot 4^{t-2} + 2 \cdot 4^{t-2}$ common neighbours, which is equal to $2 \cdot (4^{t-1} + 2^{t-1})$ (see Lemma 13(2.2)).

Thus, all the vertices, except the vertices corresponding to equal rows of the adjacency matrix, have exactly $2 \cdot (4^{t-1} + 2^{t-1})$ common neighbours.

So, Γ_1^t is a DDG with parameters $(2 \cdot 4^t, 4^t + 2^t, 4^t + 2^t, 2 \cdot (4^{t-1} + 2^{t-1}), 4^t, 2)$. In particular, any pair of vertices corresponding to equal rows of the adjacency matrix forms a block of size 2 of the canonical partition. □

Lemma 17. *The graph Γ_2^t is a DDG with parameters* $(2 \cdot 4^t, 4^t, 0, 2 \cdot 4^{t-1}, 4^t, 2)$.

Proof. The graph Γ_2^1 has adjacency matrix $\begin{bmatrix} D & I \\ I & D \end{bmatrix}$. For Γ_2^t, a recursive construction similar to (1) can be applied. Denote by A_2^t the adjacency block-matrix of Γ_2^t. Note that A_2^t does not contain a block O.

The following table shows the number of blocks in each block-row of A_2^t according to Lemma 14.

t	D	I
1	1	1
2	4	4
3	16	16

t	D	I
...	...	...
t	4^{t-1}	4^{t-1}
$t+1$	4^t	4^t
...	...	...

TABLE 3. The number of blocks in each block-row of A_2^t

Note that, given a block-raw in A_1^t, there is exactly one block-row that is opposite to the given one (that is, the block-row with opposite blocks ($D \leftrightarrow I$ or $I \leftrightarrow D$)).

The next table shows the number of pairs of blocks in two distinct block-rows of A_2^t according to Lemma 15 (the case of equal block-rows is excluded, except for $t = 1$). Note that this table represents each of the cases from Lemma 15.

t	D, D	I, I	D, I	I, D
1	0	0	1	1
2	2	2	2	2
3	8	8	8	8
...	...	...	...	...
t	$2 \cdot 4^{t-2}$	$2 \cdot 4^{t-2}$	$2 \cdot 4^{t-2}$	$2 \cdot 4^{t-2}$
$t+1$	$2 \cdot 4^{t-1}$	$2 \cdot 4^{t-1}$	$2 \cdot 4^{t-1}$	$2 \cdot 4^{t-1}$
...	...	...	...	...

TABLE 4. The number of pairs of blocks in two different block-rows of A_2^t

The degree of each vertex from Γ_2^t (the number of 1s in the corresponding row of the adjacency matrix) is $3 \cdot 4^{t-1} + 4^{t-1}$, which is equal 4^t.

Next, we find the number of common neighbours for each pair of distinct vertices from Γ_2^t.

For each block-row in the adjacency block-matrix of Γ_2^t, there is exactly one opposite block-row, so, for each row in the adjacency matrix Γ_1^t, there is exactly one opposite row. The number of common neighbours for the vertices corresponding to such opposite rows is 0.

Consider two vertices corresponding to two non-opposite rows associated with opposite block-rows of the adjacency block-matrix. There are 0 pairs D and D or I and I in two opposite block-rows, so such vertices have 1 common neighbour in each pair D and I or I and D. Thus, they have $2 \cdot 4^{t-1}$ common neighbours.

Consider two vertices associated with the same block-row (see Lemma 13(1)). They have $2 \cdot 4^{t-1}$ common neighbours.

Consider two vertices associated with distinct block-rows (see Lemma 13(2)). They have $3 \cdot 2 \cdot 4^{t-2} + 2 \cdot 4^{t-2}$ common neighbours, which is equal to $2 \cdot 4^{t-1}$ (see Lemma 13(2.1)), or $2 \cdot 2 \cdot 4^{t-2} + 2 \cdot 4^{t-2} + 2 \cdot 4^{t-2}$ common neighbours, which is equal to $2 \cdot 4^{t-1}$ (see Lemma 13(2.2)).

Thus, all the vertices, except the vertices corresponding to opposite rows of the adjacency matrix, have exactly $2 \cdot 4^{t-1}$ common neighbours.

So, Γ_2^t is a DDG with parameters $(2\cdot 4^t, 4^t, 0, 2\cdot 4^{t-1}, 4^t, 2)$. In particular, any pair of vertices corresponding to opposite rows of the adjacency matrix forms a block of size 2 of the canonical partition. Note, that since these two vertices correspond to opposite rows, they are adjacent. Thus, Γ_2^t has diameter 2. □

There are two known DDGs of diameter 2 with $\lambda_1 = 0$ (more generally, there are two known strictly Deza graphs with $a = 0$), one on 8 vertices and one on 32 vertices. The series Γ_2^t covers both cases and gives an infinite series of DDGs of diameter 2 with $\lambda_1 = 0$ (more generally, strictly Deza graphs with $a = 0$).

Lemma 18. *Let Γ be a connected Deza graph with parameters (v, k, b, a) with the second largest eigenvalue q. If Γ has a disconnecting set of minimum cardinality, that is not the neighbourhood of some vertex, then the following equality holds: $k - 2q \leq b$.*

Proof. In this proof we reinterpret the main idea of the proof of [8, Proposition 5]. Let S be a disconnecting set of minimum cardinality in Γ, $|S| = \varkappa(\Gamma) \leq k$. Let A and B be the connected components that remain after removing S from Γ. Assume that $|A| > 1$ and $|B| > 1$, so S is not the neighbourhood of some vertex. Since the spectrum of a disconnected graph is the union of the spectra of connected components, the spectrum $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_{v-|S|}$ of the graph $A \cup B$ is the union of the spectra $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{|A|}$ and $\omega_1 \geq \omega_2 \geq \ldots \geq \omega_{|B|}$ of the graphs A and B, respectively.

The graph $A \cup B$ is an induced subgraph of the graph Γ and, by the theorem on spectrum interlacing [1, Theorem 3.3.1], the second largest eigenvalue θ_2 of the graph $A \cup B$ does not exceed the second largest eigenvalue q of the graph Γ. Evidently, $\theta_2 \geq min(\sigma_1, \omega_1)$. Since the largest eigenvalue in any graph is greater or equal to its mean vertex degree (that is, the arithmetic mean of degrees of its vertices; see [1, Lemma 3.2.1]), we can assume without loss of generality that the mean vertex degree in the graph B is at most q.

For a vertex $x \in B$, we set $B(x) := B \cap \Gamma(x)$ and $S(x) := S \cap \Gamma(x)$, where $\Gamma(x)$ is the neighbourhood of the vertex x in Γ. Then $|B(x)| + |S(x)| = k$. Let us estimate the mean vertex degree in the subgraphs B and S. Since

$$\sum_{x \in B} \frac{|B(x)|}{|B|} \leq q,$$

the inequality

$$\sum_{x \in B} \frac{|S(x)|}{|B|} \geq k - q$$

holds.

Let us estimate the mean number of common neighbours in S for an arbitrary pair of different vertices $x, y \in B$:

$$\sum_{x \in B} \sum_{y \in B \setminus \{x\}} (|S(x)| + |S(y)|) = \sum_{x \in B} ((|B| - 1)|S(x)| + \sum_{y \in B \setminus \{x\}} |S(y)|)$$

$$\sum_{x \in B} \sum_{y \in B \setminus \{x\}} (|S(x)| + |S(y)|) = (|B| - 1) \sum_{x \in B} |S(x)| + \sum_{x \in B} \sum_{y \in B \setminus \{x\}} |S(y)|$$

$$\sum_{x \in B} \sum_{y \in B \setminus \{x\}} (|S(x)| + |S(y)|) = (|B| - 1) \sum_{x \in B} |S(x)| + (|B| - 1) \sum_{z \in B} |S(z)|$$

$$\frac{\sum\limits_{x\in B}\sum\limits_{y\in B\setminus\{x\}}(|S(x)|+|S(y)|)}{|B|(|B|-1)}=\frac{(\sum\limits_{x\in B}|S(x)|)(|B|-1)\cdot 2}{|B|(|B|-1)}\geq 2(k-q)$$

Since $|B|>1$, the subgraph B contains a pair of vertices x and y with the property $|S(x)|+|S(y)|\geq 2(k-q)$.

Let α, β and γ be integers such that $\beta=|S(x)\cap S(y)|$, $\alpha+\beta=|S(x)|$ and $\beta+\gamma=|S(y)|$. Then $\alpha+\beta+\gamma\leq|S|\leq k$. Further, $\alpha+\gamma=|S(x)|+|S(y)|-2\beta$ and $|S(x)\cap S(y)|=\beta\leq k-(\alpha+\gamma)=k-(|S(x)|+|S(y)|-2\beta)$. Hence $|S(x)|+|S(y)|\leq\beta+k$ and, therefore, $\beta+k\geq 2(k-q)$. Thus, $|S(x)\cap S(y)|=\beta\geq k-2q$.

On the other hand, $|S(x)\cap S(y)|\leq b$, which gives the inequality $k-2q\leq b$. □

Lemma 19. *The vertex connectivity of* Γ_2^t *equals* 4^t.

Proof. Let us calculate the spectrum of Γ_2^t as a DDG:

$$\begin{aligned}\{k,\pm\sqrt{k-\lambda_1},\pm\sqrt{k^2-\lambda_2 v}\}&=\{4^t,\pm\sqrt{4^t-0},\pm\sqrt{(4^t)^2-2\cdot 4^{t-1}\cdot 2\cdot 4^t}\}\\&=\{4^t,\pm 2^t,0\}.\end{aligned}$$

The second largest eigenvalue of Γ_2^t is 2^t. Considering Γ_2^t as a Deza graph with the parameters (v,k,b,a), we get $b=4^{t-1}$. So, the inequality from Lemma 18 becomes $4^t-2\cdot 2^t\leq 4^{t-1}$, which holds only for $t=1$. Thus, for any $t>1$, the vertex connectivity of Γ_2^t equals k, where $k=4^t$. If $t=1$, then Γ_2^t is a DDG with parameters $(8,4,0,2,4,2)$. By computer calculations using SageMath, the vertex connectivity of this graph equals 4. □

Lemma 20. *The vertex connectivity of* Γ^t *equals* $2\cdot 4^t$.

Proof. By Proposition 6, the vertex connectivity of Γ_1^t equals 4^t+2^t, By Lemma 19 the vertex connectivity of Γ_2^t equals 4^t. Thus, by Lemma 12, the vertex connectivity of Γ^t equals $2\cdot 4^t$, which is 2^t less than the degree of a vertex. □

4. Conclusion

Computations in SageMath show that, among connected proper DDGs on at most 39 vertices found in [15], there are 32 DDGs with vertex connectivity less than k, where k is the degree of a vertex. For these 32 DDGs, one graph is a DDG with parameters $(24,10,6,3,3,8)$ obtained with Construction 6, and the other 31 graphs are DDGs with $\lambda_1=k-1$ (this case is described in Propositions 7 and 8).

There are more constructions of regular graphical Hadamard matrices with positive l (see [2, Section 10.5.1]), so in view of Construction 6, there are more DDGs whose vertex connectivity is less than k, where k is the degree of a vertex. In this paper we focused on the smallest graph from Construction 6 and its generalisation (in the sense of the recursive construction). We are interested if there exist examples of of DDGs, whose vertex connectivity is less than k.

There are some approaches for obtaining more general results about the vertex connectivity of DDGs. For example, the approach that was used in [4] and [8], or the approach that was used in [3]. Since the spectrum of a DDG is not completely determined by its parameters, the first approach can only be used in specific cases like Lemma 19. The second approach requires more detailed consideration and possible development of new tools to apply it to DDGs. We are interested if general results will be obtained for DDGs, using both known approaches.

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