

СИБИРСКИЕ ЭЛЕКТРОННЫЕ МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

<http://semr.math.nsc.ru>

Том 19, стр. 144–144 (2022)

УДК 515.125

DOI 10.17377/semi.2015.12.xxx

MSC 54C35, 54D10

ON FUNCTION SPACES. II

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ABSTRACT. For certain properties \mathfrak{P} of topological T_0 -spaces, we prove that a T_0 -space \mathbb{Y} has property \mathfrak{P} if and only if the function space $C_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ endowed with a particular topology \mathcal{T} possesses \mathfrak{P} for some T_0 -space \mathbb{X} .

Keywords: A -space, core-compact space, d -space, essentially complete space, function space, injective space, sober space, T_0 -space.

1. INTRODUCTION

This paper continues [11]. In [11], the interplay of different topological properties for a T_0 -space \mathbb{Y} and its function space $C(\mathbb{X}, \mathbb{Y})$ (endowed with the pointwise convergence topology) was investigated. Here, we extend some results from [11] as well as from [10] considering different topologies on the set $C(\mathbb{X}, \mathbb{Y})$ of continuous functions from a T_0 -space \mathbb{X} to a T_0 -space \mathbb{Y} .

Our main results are Theorems 9, 12, 18, 20, 22, 30, 31, Propositions 27 and 32, and Corollaries 17, 21, and 23.

2. TOPOLOGIES ON $C(\mathbb{X}, \mathbb{Y})$

Given topological T_0 -spaces \mathbb{X} and \mathbb{Y} , let $C(\mathbb{X}, \mathbb{Y})$ denote the set of all continuous functions from \mathbb{X} to \mathbb{Y} . An arbitrary function $f \in C(\mathbb{X}, \mathbb{Y})$ can be considered as an element of the Cartesian power Y^X . The *pointwise convergence topology* on $C(\mathbb{X}, \mathbb{Y})$ is induced by the Tychonoff topology on Y^X . Therefore, the set

$$V_{x,U} = \{f \in C(\mathbb{X}, \mathbb{Y}) \mid f(x) \in U\}, \text{ where } x \in X \text{ and } \emptyset \neq U \in \mathcal{T}(\mathbb{Y}),$$

ERSHOV, YU. L., SCHWIDEFSKY, M. V., ON FUNCTION SPACES. II.

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THE RESEARCH WAS CARRIED OUT IN THE FRAMEWORK OF THE STATE CONTRACT OF SOBOLEV INSTITUTE OF MATHEMATICS, PROJECT NO. FWNF-2022-0012.

Received February 23, 2022, published ????

forms a subbasis of the pointwise convergence topology; we denote this topology by \mathcal{P} .

If \mathcal{T} is a topology on $C(\mathbb{X}, \mathbb{Y})$ then we denote the space $\langle C(\mathbb{X}, \mathbb{Y}), \mathcal{T} \rangle$ by $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ and by $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ when $\mathcal{T} = \mathcal{P}$. The specialization order on $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is denoted by $\leq_{\mathcal{T}}$ or just by \leq in case when $\mathcal{T} = \mathcal{P}$. For $f, g \in C(\mathbb{X}, \mathbb{Y})$, the relation $f \leq_{\mathcal{P}} g$ holds if and only if $f(x) \leq_{\mathbb{Y}} g(x)$ for all $x \in X$. We write $f \leq g$ instead of $f \leq_{\mathcal{P}} g$. It is clear that $f \leq_{\mathcal{T}} g$ implies that $f \leq g$ whenever $\mathcal{P} \subseteq \mathcal{T}$.

We also consider the mapping

$$\xi: \mathbb{Y} \rightarrow C(\mathbb{X}, \mathbb{Y}); \quad \xi: y \mapsto \xi_y, \quad \text{where } \xi_y(x) = y \text{ for all } x \in X.$$

Lemma 1. *Let \mathbb{X}, \mathbb{Y} be T_0 -spaces. If \mathcal{T} is a topology on $C(\mathbb{X}, \mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\leq_{\mathcal{P}})$ then $\leq_{\mathcal{T}}$ coincides with $\leq_{\mathcal{P}}$. In this case, the following holds.*

- (1) \mathbb{Y} contains a least element with respect to the specialization order $\leq_{\mathbb{Y}}$ if and only if $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ contains a least element with respect to $\leq_{\mathcal{T}}$ for some (equivalently, for each) T_0 -space \mathbb{X} .
- (2) \mathbb{Y} contains a greatest element with respect to $\leq_{\mathbb{Y}}$ if and only if $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ contains a greatest element with respect to $\leq_{\mathcal{T}}$ for some (equivalently, for each) T_0 -space \mathbb{X} .

Proof. According to Lemma 1.2.8 in [10], the pointwise convergence topology \mathcal{P} is finer than the Alexandroff topology $\mathcal{T}_A(\leq_{\mathcal{P}})$. As the specialization order of $\mathcal{T}_A(\leq_{\mathcal{P}})$ coincides with $\leq_{\mathcal{P}}$, we obtain that $\leq_{\mathcal{T}}$ coincides with $\leq_{\mathcal{P}}$; we denote this order simply by \leq .

(1) Let $\langle Y; \leq_{\mathbb{Y}} \rangle$ have a least element \perp . It is clear that $\xi_{\perp} \in C(\mathbb{X}, \mathbb{Y})$ is a least element in $\langle C(\mathbb{X}, \mathbb{Y}); \leq \rangle$. Conversely, suppose that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\leq)$ and $o \in C(\mathbb{X}, \mathbb{Y})$ is a least element in $\langle C(\mathbb{X}, \mathbb{Y}); \leq \rangle$. Choose arbitrary elements $x, x' \in X$ and suppose that $o(x) \in U \in \mathcal{T}(\mathbb{Y})$. Then $o \in V_{x,U} \in \mathcal{P} \subseteq \mathcal{T}$. If $y \in Y$ then $o \leq \xi_y$, whence $\xi_y \in V_{x,U}$ and $y \in U$. As we can choose y arbitrary, we conclude that $U = Y$ and $o(x') \in U$. Thus, we proved that $o(x) \leq_{\mathbb{Y}} o(x')$. Symmetrically, $o(x') \leq_{\mathbb{Y}} o(x)$, whence $o = \xi_{\perp}$ for some $\perp \in Y$. Since $\xi_{\perp} = o \leq \xi_y$, we conclude that $\perp \leq y$ for all $y \in Y$. Therefore, $\langle Y; \leq_{\mathbb{Y}} \rangle$ has a least element.

The proof of (2) is similar but simpler. \square

Given topological spaces $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$, let $M(\mathbb{X}, \mathbb{Y})$ denote the set of all functions from X to Y . We put

$$\begin{aligned} \lambda: M(\mathbb{Z} \times \mathbb{X}, \mathbb{Y}) &\rightarrow M(\mathbb{Z}, M(\mathbb{X}, \mathbb{Y})), & \lambda(f): z &\mapsto f(z, x); \\ \lambda^*: M(\mathbb{Z}, M(\mathbb{X}, \mathbb{Y})) &\rightarrow M(\mathbb{Z} \times \mathbb{X}, \mathbb{Y}), & \lambda^*(g): (z, x) &\mapsto [g(z)](x). \end{aligned}$$

It is not hard to verify that λ and λ^* are mutually inverse mappings. Hence they establish a one-to-one correspondence between sets $M(\mathbb{Z} \times \mathbb{X}, \mathbb{Y})$ and $M(\mathbb{Z}, M(\mathbb{X}, \mathbb{Y}))$.

For the following definition, we refer to [1]. A topology \mathcal{T} on $C(\mathbb{X}, \mathbb{Y})$ is *proper*, if $\lambda(C(\mathbb{Z} \times \mathbb{X}, \mathbb{Y})) \subseteq C(\mathbb{Z}, \mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y}))$ for an arbitrary space \mathbb{Z} . A topology \mathcal{T} on $C(\mathbb{X}, \mathbb{Y})$ is *admissible*, if $\lambda^*(C(\mathbb{Z}, \mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y}))) \subseteq C(\mathbb{Z} \times \mathbb{X}, \mathbb{Y})$ for an arbitrary space \mathbb{Z} . A topology \mathcal{T} on $C(\mathbb{X}, \mathbb{Y})$ is *exponential*, if \mathcal{T} is both proper and admissible.

The *compact-open topology* \mathcal{K} on $C(\mathbb{X}, \mathbb{Y})$ is defined by the subbasis of open sets $V_{K,U} = \{f \in C(\mathbb{X}, \mathbb{Y}) \mid f(K) \subseteq U\}$, where $K \subseteq X$ is compact and $\emptyset \neq U \in \mathcal{T}(\mathbb{Y})$,

The *Isbell topology* \mathcal{J} on $C(\mathbb{X}, \mathbb{Y})$ is defined by the subbasis of open sets

$$V_{\mathcal{H},U} = \{f \in C(\mathbb{X}, \mathbb{Y}) \mid f^{-1}(U) \in \mathcal{H}\},$$

where \mathcal{H} is a Scott-open set in $\langle \mathcal{T}(\mathbb{X}); \subseteq \rangle$ and $\emptyset \neq U \in \mathcal{T}(\mathbb{Y})$. The *core-open topology* \mathcal{C} on $C(\mathbb{X}, \mathbb{Y})$ is defined by the subbasis of open sets

$$V_{U,W} = \{f \in C(\mathbb{X}, \mathbb{Y}) \mid U \ll f^{-1}(W)\}, \text{ where } \emptyset \neq U \in \mathcal{T}(\mathbb{X}) \text{ and } \emptyset \neq W \in \mathcal{T}(\mathbb{Y}).$$

Let \mathcal{T} be a T_0 -topology on $\mathcal{T}(\mathbb{X})$. Consider the topology on $C(\mathbb{X}, \mathbb{Y})$ defined by the subbasis of open sets

$$V_{\mathcal{O},U} = \{f \in C(\mathbb{X}, \mathbb{Y}) \mid f^{-1}(U) \in \mathcal{O}\},$$

where $\mathcal{O} \in \mathcal{T}$ and $U \in \mathcal{T}(\mathbb{Y})$. We say that this topology is *induced by* \mathcal{T} and denote it by \mathcal{T}^\sharp . It is clear that $\mathcal{T}_0 \subseteq \mathcal{T}_1$ implies that $\mathcal{T}_0^\sharp \subseteq \mathcal{T}_1^\sharp$. Moreover, it follows from the definition above that \mathcal{J} is induced by the Scott topology on $\langle \mathcal{T}(\mathbb{X}); \subseteq \rangle$.

Lemma 2. *Let \mathbb{X}, \mathbb{Y} be T_0 -spaces. Then $\mathcal{T}_A(\subseteq)^\sharp \subseteq \mathcal{T}_A(\leq)$.*

Proof. It suffices to show that $V_{\mathcal{O},W} \in \mathcal{T}_A(\leq)$ for all upper cones \mathcal{O} in $\langle \mathcal{T}(\mathbb{X}); \subseteq \rangle$ and all open sets $W \in \mathcal{T}(\mathbb{Y})$. Indeed, let $f \in V_{\mathcal{O},W}$ and let $f \leq g$ in $C(\mathbb{X}, \mathbb{Y})$. We prove first that $f^{-1}(W) \subseteq g^{-1}(W)$. Indeed, let $x \in f^{-1}(W)$; then $f(x) \in W$. Since W is an upper cone with respect to $\leq_{\mathbb{Y}}$, we obtain that $g(x) \in W$ and thus, $x \in g^{-1}(W)$ which is our desired conclusion.

Since $f^{-1}(W) \in \mathcal{O}$ and $f^{-1}(W) \subseteq g^{-1}(W)$, we conclude that $g^{-1}(W) \in \mathcal{O}$ as \mathcal{O} is an upper cone with respect to \subseteq . This yields that $g \in V_{\mathcal{O},W}$ and $V_{\mathcal{O},W} \in \mathcal{T}_A(\leq)$. \square

In what follows, we consider also the *Scott topology* $\mathcal{S} = \mathcal{T}_S(\leq_{\mathcal{P}})$ on $C(\mathbb{X}, \mathbb{Y})$.

Lemma 3. *Let \mathbb{X}, \mathbb{Y} be T_0 -spaces. If \mathcal{T} is a topology on $C(\mathbb{X}, \mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$ then ξ is a homeomorphic embedding of \mathbb{Y} into $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$.*

Proof. It is straightforward to verify that $\xi(Y) \cap V_{x,U} = \xi(U)$ for all $x \in X$ and $U \in \mathcal{T}(\mathbb{Y})$. Therefore, ξ is open. Moreover, for all $W \in \mathcal{T}(\mathbb{Y})$ and all $y \in Y$, we have

$$\xi_y^{-1}(W) = \begin{cases} X, & \text{if } y \in W; \\ \emptyset, & \text{if } y \notin W. \end{cases}$$

Furthermore, let $\mathcal{O} \in \mathcal{T}_A(\subseteq)$ and let $W \in \mathcal{T}(\mathbb{Y})$. Then we have

$$\begin{aligned} \xi^{-1}(V_{\mathcal{O},W}) &= \{y \in Y \mid \xi_y \in V_{\mathcal{O},W}\} = \{y \in Y \mid \xi_y^{-1}(W) \in \mathcal{O}\} = \\ &= \begin{cases} Y, & \text{if } \mathcal{O} = \mathcal{T}(\mathbb{X}); \\ W, & \text{if } \mathcal{O} \notin \{\mathcal{T}(\mathbb{X}), \emptyset\}; \\ \emptyset, & \text{if } \mathcal{O} = \emptyset. \end{cases} \end{aligned}$$

In any case, $\xi^{-1}(V_{\mathcal{O},W}) \in \mathcal{T}(\mathbb{Y})$. Therefore, $\xi^{-1}(V) \in \mathcal{T}(\mathbb{Y})$ for all $V \in \mathcal{T}_A(\subseteq)^\sharp$, whence the desired statement follows. \square

Let \mathbb{X} be a T_0 -space. If $\langle I; \leq \rangle$ is an up-directed poset and $\mathfrak{x} = \{x_i \mid i \in I\} \subseteq X$ then we say that \mathfrak{x} is a *net* in \mathbb{X} . If $A \subseteq X$, then \mathfrak{x} is *eventually in* A if there is $i \in I$ such that $x_j \in A$ for all $j \geq i$. Then \mathfrak{x} converges to some $x \in X$ if $x \in U \in \mathcal{T}(\mathbb{X})$ implies that \mathfrak{x} is eventually in U for all $U \in \mathcal{T}(\mathbb{X})$. Further, if \mathbb{Y} is also a T_0 -space and $\mathfrak{f} = \{f_j \mid j \in J\}$ is a net in $C(\mathbb{X}, \mathbb{Y})$ for some up-directed poset $\langle J; \leq \rangle$, then \mathfrak{f} *converges continuously to* $f \in C(\mathbb{X}, \mathbb{Y})$ if for arbitrary $x \in X$, the net $\{f_j(x_i) \mid (i, j) \in I \times J\}$ converges to $f(x)$ whenever $\mathfrak{x} = \{x_i \mid i \in I\}$ converges to x .

The *natural topology* \mathcal{T}_* on $C(\mathbb{X}, \mathbb{Y})$ consists of those sets $U \subseteq C(\mathbb{X}, \mathbb{Y})$ which satisfy the following condition for all nets \mathfrak{f} in $C(\mathbb{X}, \mathbb{Y})$ and all $f \in C(\mathbb{X}, \mathbb{Y})$:

if \mathfrak{f} converges continuously to $f \in U$ then \mathfrak{f} is eventually in U .

For the following facts, we refer to Chapters 6 and 15 in [10] as well as to Sections 5.3–5.4 in [14] and to Section II-4 in [13].

Proposition 4. *Let \mathbb{X}, \mathbb{Y} be T_0 -spaces.*

- (1) *If \mathcal{T}_0 is a proper and \mathcal{T}_1 is an admissible topology on $C(\mathbb{X}, \mathbb{Y})$ then $\mathcal{T}_0 \subseteq \mathcal{T}_1$.*
- (2) *If $\mathcal{T}_0 \subseteq \mathcal{T}_1$ and \mathcal{T}_1 is a proper topology on $C(\mathbb{X}, \mathbb{Y})$ then \mathcal{T}_0 is also a proper topology on $C(\mathbb{X}, \mathbb{Y})$.*
- (3) *If $\mathcal{T}_0 \subseteq \mathcal{T}_1$ and \mathcal{T}_0 is an admissible topology on $C(\mathbb{X}, \mathbb{Y})$ then \mathcal{T}_1 is also an admissible topology on $C(\mathbb{X}, \mathbb{Y})$.*
- (4) *$\mathcal{P} \subseteq \mathcal{K} \subseteq \mathcal{J} \subseteq \mathcal{T}_*$ and $\mathcal{K} \subseteq \mathcal{C}$.*
- (5) *The natural topology \mathcal{T}_* is the finest proper topology on $C(\mathbb{X}, \mathbb{Y})$.*

For the next statement, we refer to [5] and to [10, Theorem 6.2.1], see also [14, Theorem 5.4.4].

Theorem 5. *The following conditions are equivalent for a T_0 -space \mathbb{X} .*

- (1) *$\langle \mathcal{T}(\mathbb{X}); \subseteq \rangle$ is a continuous poset.*
- (2) *$\langle \mathcal{T}(\mathbb{X}), \mathcal{T}_S(\subseteq) \rangle$ is an α -space.*
- (3) *For an arbitrary T_0 -space \mathbb{Y} , there is an exponential topology on $C(\mathbb{X}, \mathbb{Y})$.*
- (4) *There is an exponential topology on $C(\mathbb{X}, \mathbb{S})$.*

A topological space \mathbb{X} which satisfies the equivalent conditions of Theorem 5 is called a *core-compact space*. A topological space \mathbb{X} is *locally compact*, if $\mathcal{T}(\mathbb{X})$ has a basis of compact open sets.

For the next statement, we refer to [2], [13, Lemma II-4.2], to [10, Theorem 6.3.3], and to [14, Proposition 5.4.20].

Proposition 6. *Let \mathbb{X}, \mathbb{Y} be T_0 -spaces.*

- (1) *If \mathbb{X} is core-compact then $\mathcal{C} = \mathcal{J} = \mathcal{T}_*$ and this topology is exponential on $C(\mathbb{X}, \mathbb{Y})$.*
- (2) *If \mathbb{X} is locally compact then $\mathcal{K} = \mathcal{C} = \mathcal{J} = \mathcal{T}_*$ and this topology is exponential on $C(\mathbb{X}, \mathbb{Y})$.*
- (3) *If \mathbb{X} is an α^* -space then $\mathcal{P} = \mathcal{K} = \mathcal{C} = \mathcal{J} = \mathcal{T}_*$ and this topology is exponential on $C(\mathbb{X}, \mathbb{Y})$.*

Corollary 7. *For a core-compact space \mathbb{X} and a T_0 -space \mathbb{Y} the exponential topology on $C(\mathbb{X}, \mathbb{Y})$ is defined by the subbasis of open sets*

$$V_{U,W} = \{f \in C(\mathbb{X}, \mathbb{Y}) \mid U \prec f^{-1}(W)\},$$

where $\emptyset \neq U \in \mathcal{T}(\mathbb{X})$ and $\emptyset \neq W \in \mathcal{T}(\mathbb{Y})$.

3. d -SPACES $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$

The proof of the following lemma repeats the proof of [11, Lemma 2].

Lemma 8. *Let \mathbb{X} and \mathbb{Y} be T_0 -spaces, let $\mathcal{P} \subseteq \mathcal{J} \subseteq \mathcal{T}_A(\leq_{\mathcal{P}})$, and let*

$$F = \{f_i \in C(\mathbb{X}, \mathbb{Y}) \mid i \in I\}$$

be an up-directed set with respect to $\leq_{\mathcal{P}}$. If a function $f: X \rightarrow Y$ is such that $f(x) = \sup_{\mathbb{Y}}\{f_i(x) \mid i \in I\}$ and $f(x) \in \text{cl}_{\mathbb{Y}}\{f_i(x) \mid i \in I\}$ for all $x \in X$, then $f = \sup F$ in $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$.

Theorem 9. *For T_0 -spaces \mathbb{X} and \mathbb{Y} , let \mathcal{T} be a proper topology on $C(\mathbb{X}, \mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T}$ or let \mathcal{T} be such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{S}$. Then $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is a d -space if and only if \mathbb{Y} is a d -space.*

Proof. According to [10, Lemma 6.1.9], $\mathcal{T}_A(\leq_{\mathcal{P}})$ is an admissible topology. By Proposition 4(1), we obtain that $\mathcal{T} \subseteq \mathcal{T}_A(\leq_{\mathcal{P}})$. Therefore, $\leq_{\mathcal{T}}$ and \leq agree by Lemma 1.

Suppose first that \mathbb{Y} is a d -space and consider the case when \mathcal{T} is a proper topology finer than \mathcal{P} . According to Lemma 8, each nonempty set $F = \{f_i \in C(\mathbb{X}, \mathbb{Y}) \mid i \in I\}$ which is up-directed with respect to \leq has a least upper bound f . It remains to prove that $f \in \text{cl}_{\mathcal{T}} F$.

We establish that F converges continuously to f . Let a net $\mathfrak{x} = \{x_j \mid j \in J\} \subseteq X$ with J being up-directed converge to some $x \in X$ and let $f(x) \in U \in \mathcal{T}(\mathbb{Y})$. Then $x \in f^{-1}(U) \in \mathcal{T}(\mathbb{X})$ and thus \mathfrak{x} is eventually in $f^{-1}(U)$. This means that $\{f(x_j) \mid j \in J\}$ is eventually in U . Since $f(x_j) \in \text{cl}_{\mathbb{Y}}\{f_i(x_j) \mid i \in I\}$ for all $j \in J$ by Lemma 8 and F is up-directed with respect to \leq , we conclude that $\{f_i(x_j) \mid (i, j) \in I \times J\}$ is eventually in U which proves our desired statement.

Let $f \in W \in \mathcal{T}$. By Proposition 4(5), we conclude that $W \in \mathcal{T}_*$. Since F converges continuously to f , we obtain that F is eventually in W . This yields that f is a limit point for F and $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is a d -space.

Suppose next that \mathbb{Y} is a d -space and consider the case when $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{S}$. In the notation above, we have to prove again that $f \in \text{cl}_{\mathcal{T}} F$. Indeed, if $f \in W \in \mathcal{T} \subseteq \mathcal{S}$ then by the definition of the Scott topology, we obtain that $f_i \in W$ for some $i \in I$ which is our desired conclusion.

Suppose now that $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is a d -space. In what follows, we use only the assumption that $\mathcal{P} \subseteq \mathcal{T}$. Let a set $D \subseteq Y$ be up-directed with respect to the specialization order. Then the set $\xi(D) = \{\xi_d \mid d \in D\}$ is also up-directed with respect to \leq . According to our assumption, there exists $f = \sup \xi(D) \in C(\mathbb{X}, \mathbb{Y})$ and $f \in \text{cl}_{\mathcal{T}} \xi(D)$. We show that $f \in \xi(Y)$. Indeed, let $f(x_0) \in U \in \mathcal{T}(\mathbb{Y})$. We also choose an arbitrary element $x_1 \in X$. Then $f \in V_{x_0, U} \in \mathcal{P} \subseteq \mathcal{T}$. Thus, there exists $d \in D$ such that $\xi_d \in V_{x_0, U}$, whence $f(x_1) \geq \xi_d(x_1) = d = \xi_d(x_0) \in U$. This yields that $f(x_1) \in U$ whence $f(x_0) \leq f(x_1)$. A similar argument shows that $f(x_1) \leq f(x_0)$. Therefore $f = \xi_y$ for some $y \in Y$. It is clear that $y = \sup_{\mathbb{Y}} D$. If $y \in U \in \mathcal{T}(\mathbb{Y})$, then $\xi_y \in V_{x, U} \in \mathcal{P} \subseteq \mathcal{T}$, where $x \in X$. As $\xi_y \in \text{cl}_{\mathcal{T}} \xi(D)$, we conclude that $\xi_d \in V_{x, U}$ for some $d \in D$. Thus, $d = \xi_d(x) \in U$ and $y \in \text{cl}_{\mathbb{Y}} D$. \square

The fact that $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is a d -space whenever \mathbb{Y} is a d -space was established in [6]. The fact that $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is a d -space whenever \mathbb{Y} is a d -space is established in [13, Lemma II-4-3(i)]. Theorem 3.3 in [3] shows that \mathbb{Y} is a d -space provided that $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is a d -space for some T_0 -space \mathbb{X} . For the case $\mathcal{T} = \mathcal{P}$, Theorem 9 was proved in [11], see also [10, Corollary 8.4.2].

4. SOBRIETY IN FUNCTION SPACES

The proof of the following statement almost repeats the proof of [11, Theorem 6]. Nonetheless, we give it here for the sake of completeness.

Proposition 10. *Let \mathbb{X}, \mathbb{Y} be T_0 -spaces and let \mathcal{T} be a topology on $C(\mathbb{X}, \mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^{\sharp}$. If $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is a sober space then \mathbb{Y} is also sober.*

Proof. Let \mathbb{Z} be a T_0 -space and let a subspace $\mathbb{Z}_0 \leq \mathbb{Z}$ be such that $Z = \text{sob}_{\mathbb{Z}} \mathbb{Z}_0$. Consider an arbitrary continuous function $f_0: \mathbb{Z}_0 \rightarrow \mathbb{Y}$. By Lemma 3, $\xi: \mathbb{Y} \rightarrow$

$\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is continuous. Thus, the function $\xi f_0: Z_0 \rightarrow \mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is continuous. As $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is a sober space, there is a continuous function $f: Z \rightarrow \mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ such that $f|_{Z_0} = \xi f_0$. We prove that $f(Z) \subseteq \xi(Y)$. Indeed, let $z \in Z$; then $z = \sup_{\mathbb{Z}}(\downarrow z \cap Z_0)$ and $z \in \text{cl}_{\mathbb{Z}}(\downarrow z \cap Z_0)$. It is straightforward to verify that $f(z) \in \text{cl}_{\mathbb{Y}} f(\downarrow z \cap Z_0)$ and $f(z) = \sup f(\downarrow z \cap Z_0)$, see Lemma 1.5.3 in [10]. Let $x_0, x_1 \in X$ and let $[f(z)](x_0) \in U \in \mathcal{J}(\mathbb{Y})$. Then $f(z) \in V_{x_0, U} \in \mathcal{P} \subseteq \mathcal{J}$, whence $z \in f^{-1}(V_{x_0, U}) \in \mathcal{T}(Z)$. Therefore there is $z_0 \in \downarrow z \cap Z_0$ such that $f(z_0) \in V_{x_0, U}$. Since the function $f(z_0)$ is constant, $f(z_0) \in V_{x_1, U} \in \mathcal{P} \subseteq \mathcal{J}$. But then $f(z) \in V_{x_1, U}$, as $f(z_0) \leq f(z)$. Therefore $[f(z)](x_1) \in U$. Similarly, $[f(z)](x_1) \in U$ implies that $[f(z)](x_0) \in U$; that is, $[f(z)](x_0) = [f(z)](x_1)$, which is our desired conclusion. Inclusion $f(Z) \subseteq \xi(Y)$ implies that $(\xi^{-1}f)|_{Z_0} = f_0$. In view of [10, Theorem 5.3.2], \mathbb{Y} is a sober space. \square

For the following statement, we refer to [11, Proposition 16] or to [10, Proposition 6.1.15].

Proposition 11. *Let \mathcal{T} be an exponential topology on $C(\mathbb{X}, \mathbb{Y})$.*

- (1) *If \mathbb{Y} is [densely] injective, then $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is [densely] injective.*
- (2) *If \mathbb{Y} is sober, then $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is sober.*

The following statement generalizes Theorem 6 in [11].

Theorem 12. *For a T_0 -space \mathbb{Y} , the following conditions are equivalent.*

- (1) *\mathbb{Y} is sober.*
- (2) *$\mathbb{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is sober for each core-compact space \mathbb{X} .*
- (3) *$\mathbb{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is sober for some core-compact space \mathbb{X} .*
- (4) *$\mathbb{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is sober for some T_0 -space \mathbb{X} .*
- (5) *$\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is sober for each T_0 -space \mathbb{X} .*
- (6) *$\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is sober for some T_0 -space \mathbb{X} .*
- (7) *$\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is sober for some [core-compact] T_0 -space \mathbb{X} and some topology \mathcal{T} such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^{\sharp}$.*

Proof. (1) implies (2) by Proposition 11(2). (2) implies (3) and (3) implies (4) in a trivial way. (4) implies (1) by Proposition 10. Statements (1), (5), and (6) are equivalent by [11, Theorem 6]. Furthermore, (3) obviously implies (7) and (7) implies (1) by Proposition 10. \square

5. ESSENTIAL COMPLETENESS IN FUNCTION SPACES

As in [7], we consider the following properties of a T_0 -space \mathbb{X} .

- (H₀) \mathbb{X} has a least element 0 with respect to the specialization order \leq .
- (H₁) \mathbb{X} is a join semilattice with respect to \leq ; \vee denotes the join operation in X .
- (H₂) The join operation $\vee: \mathbb{X}^2 \rightarrow \mathbb{X}$ is continuous; that is, $\langle X, \vee, \mathcal{T}(\mathbb{X}) \rangle$ is a topological join-semilattice.

In [7], the following statement was proved, see also [10, Corollary 10.4.2].

Theorem 13. [10, Corollary 10.4.2] *A T_0 -space \mathbb{X} is essentially complete if and only if \mathbb{X} is a d-space with properties (H₀)–(H₂).*

Proposition 14. *Let \mathbb{X}, \mathbb{Y} be T_0 -spaces and let \mathcal{T} be a topology on $C(\mathbb{X}, \mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^{\sharp}$. If $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ has the properties (H₁)–(H₂) then \mathbb{Y} also possesses the properties (H₁)–(H₂).*

Proof. According to our assumption, for arbitrary $y_0, y_1 \in Y$, there is $f = \xi_{y_0} \vee \xi_{y_1} \in C(\mathbb{X}, \mathbb{Y})$. We prove that f is a constant function. Indeed, let $f(x_0) \in U \in \mathcal{T}(\mathbb{Y})$; then $f \in V_{x_0, U} \in \mathcal{P} \subseteq \mathcal{T}$. We may assume without loss of generality that $U \neq Y$. We choose an arbitrary element $x_1 \in X$. Since \vee is a continuous function, there are $V_0, V_1 \in \mathcal{T}$ such that $\xi_{y_0} \in V_0$, $\xi_{y_1} \in V_1$, and $V_0 \cap V_1 \subseteq V_{x_0, U}$. As $U \neq Y$, we conclude that $V_{x_0, U} \neq C(\mathbb{X}, \mathbb{Y})$. There are three possible cases.

Case 1: $V_{1-i} = C(\mathbb{X}, \mathbb{Y})$ for some $i < 2$. In this case, $\xi_{y_i} \in V_i = V_0 \cap V_1 \subseteq V_{x_0, U}$ and thus $y_i \in U$. Therefore, $\xi_{y_i} \in V_{x_1, U}$. As $\xi_{y_i} \leq f$, we obtain that $f \in V_{x_1, U}$ and $f(x_1) \in U$.

Case 2: $V_0, V_1 \neq C(\mathbb{X}, \mathbb{Y})$. Since $\mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$ and $V_0, V_1 \in \mathcal{T}$, we conclude that there are $k_0 > 0$ and $k_1 > 0$, there are upper cones $\mathcal{H}_{i_0}, \dots, \mathcal{H}_{i_{k_i}}$ in $\langle \mathcal{T}(\mathbb{X}); \subseteq \rangle$, and there are open sets $W_{i_0}, \dots, W_{i_{k_i}} \in \mathcal{T}(\mathbb{Y})$, $i < 2$, such that

$$\xi_{y_i} \in V_{\mathcal{H}_{i_0}, W_{i_0}} \cap \dots \cap V_{\mathcal{H}_{i_{k_i}}, W_{i_{k_i}}} \subseteq V_i \quad \text{for all } i < 2.$$

Therefore, $\xi_{y_i}^{-1}(W_{ij}) \in \mathcal{H}_{ij}$ for all $i < 2$ and all $j \leq k_i$. We put $U_0 = W_{00} \cap \dots \cap W_{0k_0}$ and $U_1 = W_{10} \cap \dots \cap W_{1k_1}$; then $U_0, U_1 \in \mathcal{T}(\mathbb{Y})$. As $\emptyset \notin \{V_0, V_1\}$, we conclude that $\mathcal{H}_{ij} \neq \emptyset$ for all $i < 2$ and all $j \leq k_i$. As $V_0, V_1 \neq C(\mathbb{X}, \mathbb{Y})$, we may assume that $\mathcal{H}_{ij} \neq \mathcal{T}(\mathbb{X})$ for all $i < 2$ and all $j \leq k_i$. This yields that $\xi_{y_i}^{-1}(W_{ij}) \neq \emptyset$ whence $\xi_{y_i}^{-1}(W_{ij}) = X$ for all $i < 2$ and all $j \leq k_i$. Hence, $y_i \in W_{ij}$ for all $i < 2$ and all $j \leq k_i$ and therefore, $y_0 \in U_0$ and $y_1 \in U_1$.

Since ξ_{y_0}, ξ_{y_1} are constant functions, we have $\xi_{y_0} \in V_{x_1, U_0}$ and $\xi_{y_1} \in V_{x_1, U_1}$. Therefore $f \in V_{x_1, U_0} \cap V_{x_1, U_1} = V_{x_1, U_0 \cap U_1}$, whence $f(x_1) \in U_0 \cap U_1 \in \mathcal{T}(\mathbb{Y})$. Moreover, if $y' \in U_0 \cap U_1$ then

$$\xi_{y'} \in V_{\mathcal{H}_{00}, W_{00}} \cap \dots \cap V_{\mathcal{H}_{0k_0}, W_{0k_0}} \cap V_{\mathcal{H}_{10}, W_{10}} \cap \dots \cap V_{\mathcal{H}_{1k_1}, W_{1k_1}} \subseteq V_0 \cap V_1 \subseteq V_{x_0, U},$$

whence $y' \in U$. We have therefore proved that $f(x_1) \in U_0 \cap U_1 \subseteq U$. This means that $f(x_0) \leq f(x_1)$. A symmetric argument shows that $f(x_1) \leq f(x_0)$. We obtain that $f = \xi_y$ for some $y \in Y$, whence $y = y_0 \vee y_1$ in \mathbb{Y} . Thus, \mathbb{Y} has the property (H₁).

Finally, we prove that \mathbb{Y} has the property (H₂). Suppose that $y = y_0 \vee y_1 \in U \in \mathcal{T}(\mathbb{Y})$. Then $\xi_{y_0} \vee \xi_{y_1} = \xi_y \in V_{x, U}$ for each $x \in X$. Since \vee is a continuous function on $C_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$, there are open sets $V_0, V_1 \in \mathcal{T}$ such that $\xi_{y_0} \in V_0$, $\xi_{y_1} \in V_1$, and $V_0 \cap V_1 \subseteq V_{x_0, U}$. As $\mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$, we conclude that there are $k_0 > 0$ and $k_1 > 0$, there are upper cones $\mathcal{H}_{i_0}, \dots, \mathcal{H}_{i_{k_i}}$ in $\langle \mathcal{T}(\mathbb{X}); \subseteq \rangle$, and there are open sets $W_{i_0}, \dots, W_{i_{k_i}} \in \mathcal{T}(\mathbb{Y})$, $i < 2$, such that

$$\xi_{y_i} \in V_{\mathcal{H}_{i_0}, W_{i_0}} \cap \dots \cap V_{\mathcal{H}_{i_{k_i}}, W_{i_{k_i}}} \subseteq V_i \quad \text{for all } i < 2.$$

Therefore, $\xi_{y_i}^{-1}(W_{ij}) \in \mathcal{H}_{ij}$ for all $i < 2$ and all $j \leq k_i$. In particular, $\mathcal{H}_{ij} \neq \emptyset$ for all $i < 2$ and all $j \leq k_i$. We put

$$U_0 = \bigcap \{W_{0j} \mid j \leq k_0, \mathcal{H}_{0j} \neq \mathcal{T}(\mathbb{X})\} \quad \text{and} \quad U_1 = \bigcap \{W_{1j} \mid j \leq k_1, \mathcal{H}_{1j} \neq \mathcal{T}(\mathbb{X})\}.$$

Then we have that $U_0, U_1 \in \mathcal{T}(\mathbb{Y})$. Moreover, $\xi_{y_i}^{-1}(W_{ij}) \neq \emptyset$ whence $\xi_{y_i}^{-1}(W_{ij}) = X$ for all $i < 2$ and all $j \leq k_i$ such that $\mathcal{H}_{ij} \neq \mathcal{T}(\mathbb{X})$. As above, we obtain that $y_0 \in U_0$ and $y_1 \in U_1$ and $U_0 \cap U_1 \subseteq U$. Therefore, \mathbb{Y} has the property (H₂). \square

The following statement has quite a straightforward proof and is to find as Proposition 3.2 in [4].

Proposition 15. *Let \mathbb{X} be a core-compact space and let $\mathcal{B} \subseteq \mathcal{J}(\mathbb{X})$ be an additive basis of $\mathcal{J}(\mathbb{X})$. If open sets $U, U', V_0, \dots, V_n \in \mathcal{B}$, $n < \omega$, are such that $U \prec U' \prec V_0 \cup \dots \cup V_n$ then there are $W_0, \dots, W_n \in \mathcal{B}$ such that*

$$W_i \prec V_i \text{ for all } i \leq n \text{ and } U \subseteq W_0 \cup \dots \cup W_n \subseteq U'.$$

Proof. For each $x \in U'$, there is $i \leq n$ such that $x \in U' \cap V_i \in \mathcal{J}(\mathbb{X})$. As \mathbb{X} is core-compact and as \mathcal{B} is a basis, there is $W_x \in \mathcal{B}$ such that $x \in W_x \prec U' \cap V_i$. Hence, $U \prec U' \subseteq \bigcup_{x \in U'} W_x$. This yields that $U \subseteq \bigcup_{j \leq k} W_{x_j} \subseteq U'$ for some $x_0, \dots, x_k \in U'$. For each $i \leq n$, we put $W_i = \bigcup \{W_{x_j} \mid j \leq k, W_{x_j} \prec U' \cap V_i\}$. Then it is clear that $W_i \prec V_i$ for all $i \leq n$ and that $U \subseteq W_0 \cup \dots \cup W_n \subseteq U'$. \square

Proposition 16. *Let \mathbb{X} be a core-compact space and let \mathbb{Y} possess the properties (H₁)–(H₂). Then $\mathbb{C}_J(\mathbb{X}, \mathbb{Y})$ also possesses the properties (H₁)–(H₂).*

Proof. Let $f_0, f_1 \in C(\mathbb{X}, \mathbb{Y})$. For each $x \in X$, we put

$$g(x) = f_0(x) \vee f_1(x).$$

We prove that $g \in C(\mathbb{X}, \mathbb{Y})$. Indeed, let $V \in \mathcal{J}(\mathbb{Y})$ and let $x \in g^{-1}(V)$. As $f_0(x) \vee f_1(x) \in V$ and \vee is continuous in \mathbb{Y} , we conclude that there are open sets $V_0, V_1 \in \mathcal{J}(\mathbb{Y})$ such that $f_0(x) \in V_0$, $f_1(x) \in V_1$, and $V_0 \cap V_1 \subseteq V$. But then

$$x \in U = f_0^{-1}(V_0) \cap f_1^{-1}(V_1) \in \mathcal{J}(\mathbb{X}).$$

If $z \in U$ then $g(z) = f_0(z) \vee f_1(z) \geq f_0(z) \in V_0$ and $g(z) \geq f_1(z) \in V_1$, whence $g(z) \in V_0 \cap V_1 \subseteq V$ which implies that $x \in U \subseteq g^{-1}(V)$. The fact that $U \in \mathcal{J}(\mathbb{X})$ proves that $g^{-1}(V) \in \mathcal{J}(\mathbb{Y})$. Therefore, g is continuous and $g = f_0 \vee f_1$ in $\mathbb{C}_J(\mathbb{X}, \mathbb{Y})$. Hence, $\mathbb{C}_J(\mathbb{X}, \mathbb{Y})$ has the property (H₁).

To prove that $\mathbb{C}_J(\mathbb{X}, \mathbb{Y})$ has the property (H₂), we consider two functions $f_0, f_1 \in C(\mathbb{X}, \mathbb{Y})$ and open sets $U \in \mathcal{J}(\mathbb{X})$, $W \in \mathcal{J}(\mathbb{Y})$ such that

$$g = f_0 \vee f_1 \in V_{U, W}.$$

This means that $U \prec g^{-1}(W)$. Since \mathbb{Y} possesses (H₂), for each element $x \in g^{-1}(W)$, there are open sets $V_{0x}, V_{1x} \in \mathcal{J}(\mathbb{Y})$ such that $f_0(x) \in V_{0x}$, $f_1(x) \in V_{1x}$, and $V_{0x} \cap V_{1x} \subseteq W$. We claim that

$$g^{-1}(W) = \bigcup \{f_0^{-1}(V_{0x}) \cap f_1^{-1}(V_{1x}) \mid x \in g^{-1}(W)\}.$$

Indeed it is clear that $g^{-1}(W) \subseteq \bigcup \{f_0^{-1}(V_{0x}) \cap f_1^{-1}(V_{1x}) \mid x \in g^{-1}(W)\}$. To prove the reverse inclusion, we consider an arbitrary element $x \in g^{-1}(W)$ and establish that $U(x) = f_0^{-1}(V_{0x}) \cap f_1^{-1}(V_{1x}) \subseteq g^{-1}(W)$. Indeed, let $z \in U(x)$. Then $g(z) \geq f_0(z) \in V_{0x}$ and $g(z) \geq f_1(z) \in V_{1x}$, whence $g(z) \in V_{0x} \cap V_{1x} \subseteq W$ which implies that $z \in g^{-1}(W)$.

Furthermore, since $U \prec g^{-1}(W) = \bigcup \{U(x) \mid x \in g^{-1}(W)\}$, we conclude that there are $x_0, \dots, x_k \in g^{-1}(W)$ such that $U \prec U(x_0) \cup \dots \cup U(x_k)$. As \mathbb{X} is core-compact, there is $V \in \mathcal{J}(\mathbb{X})$ such that

$$U \prec V \prec U(x_0) \cup \dots \cup U(x_k).$$

Taking $\mathcal{B} = \mathcal{J}(\mathbb{X})$ and applying Proposition 15, we obtain that there are open sets $S_0, \dots, S_k, T_0, \dots, T_k \in \mathcal{J}(\mathbb{X})$ such that

$$S_i \prec T_i \prec U(x_i) \text{ for all } i \leq k \text{ and } U \subseteq S_0 \cup \dots \cup S_n \subseteq V.$$

For each $i \leq k$, we have $T_i \prec U(x_i) = f_0^{-1}(V_{0x_i}) \cap f_1^{-1}(V_{1x_i})$ whence

$$\begin{aligned} f_0 &\in W_0 = V_{T_0, V_{0x_0}} \cap \dots \cap V_{T_k, V_{0x_k}} \in \mathcal{J}; \\ f_1 &\in W_1 = V_{T_0, V_{1x_0}} \cap \dots \cap V_{T_k, V_{1x_k}} \in \mathcal{J}. \end{aligned}$$

To establish (H_2) , it suffices to prove that $W_0 \cap W_1 \subseteq V_{U, W}$. Indeed, if $h \in W_0 \cap W_1$ then $h \in V_{T_i, V_{0x_i}} \cap V_{T_i, V_{1x_i}}$ for each $i \leq k$. This implies that

$$S_i \prec T_i \subseteq h^{-1}(V_{0x_i}) \cap h^{-1}(V_{1x_i}) = h^{-1}(V_{0x_i} \cap V_{1x_i}) \subseteq h^{-1}(W)$$

for all $i \leq k$. Hence, $U \subseteq S_0 \cup \dots \cup S_k \prec h^{-1}(W)$ yielding $h \in V_{U, W}$, which is our desired conclusion. \square

Corollary 17. *For a T_0 -space \mathbb{Y} , the following statements are equivalent.*

- (1) \mathbb{Y} is a topological semilattice with respect to $\leq_{\mathbb{Y}}$.
- (2) $\mathcal{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is a topological semilattice with respect to \leq for each core-compact space \mathbb{X} .
- (3) $\mathcal{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is a topological semilattice with respect to \leq for some core-compact space \mathbb{X} .
- (4) $\mathcal{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is a topological semilattice with respect to \leq for some T_0 -space \mathbb{X} .
- (5) $\mathcal{C}(\mathbb{X}, \mathbb{Y})$ is a topological semilattice with respect to \leq for each T_0 -space \mathbb{X} .
- (6) $\mathcal{C}(\mathbb{X}, \mathbb{Y})$ is a topological semilattice with respect to \leq for some T_0 -space \mathbb{X} .
- (7) $\mathcal{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is a topological semilattice with respect to \leq for some [core-compact] T_0 -space \mathbb{X} and some topology \mathcal{T} such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^{\sharp}$.

Proof. (1) implies (2) by Proposition 16. (2) implies (3) and (3) implies (4) in a trivial way. (4) implies (1) by Proposition 14. Statements (1), (5), and (6) are equivalent by [11, Proposition 9]. Furthermore, (3) obviously implies (7) and (7) implies (1) by Proposition 14. \square

The next theorem is a generalization of Theorem 10 from [11].

Theorem 18. *For a T_0 -space \mathbb{Y} , the following statements are equivalent.*

- (1) \mathbb{Y} is essentially complete.
- (2) $\mathcal{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is essentially complete for each core-compact space \mathbb{X} .
- (3) $\mathcal{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is essentially complete for some core-compact space \mathbb{X} .
- (4) $\mathcal{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is essentially complete for some T_0 -space \mathbb{X} .
- (5) $\mathcal{C}(\mathbb{X}, \mathbb{Y})$ is essentially complete for each T_0 -space \mathbb{X} .
- (6) $\mathcal{C}(\mathbb{X}, \mathbb{Y})$ is essentially complete for some T_0 -space \mathbb{X} .
- (7) $\mathcal{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is essentially complete for some [core-compact] T_0 -space \mathbb{X} and some topology \mathcal{T} such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^{\sharp}$.

Proof. If \mathbb{X} is a core-compact space and \mathbb{Y} is an essentially complete space, then \mathbb{Y} is a d -space and possesses the properties (H_0) – (H_2) by Theorem 13. By Theorem 9, $\mathcal{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is a d -space. By Proposition 16, $\mathcal{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ has the properties (H_1) – (H_2) . By Lemma 1 and Lemma 2, $\mathcal{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ has the property (H_0) . Applying Theorem 13 again, we obtain that $\mathcal{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is essentially complete. Therefore, (1) implies (2).

It is straightforward that (2) implies (3), (3) implies (4), and (3) implies (7).

Suppose now that $\mathcal{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is an essentially complete space for some T_0 -spaces \mathbb{X} and \mathbb{Y} . According to Theorem 13, $\mathcal{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is a d -space and has the properties (H_0) – (H_2) . By the proof of Theorem 9, \mathbb{Y} is also a d -space. By Proposition 14, \mathbb{Y} has the properties (H_1) – (H_2) . By Lemma 1 and Lemma 2, \mathbb{Y} has the property (H_0) .

By Theorem 13, \mathbb{Y} is essentially complete and (4) implies (1). Then (7) implies (1) in a similar way.

Statements (1) and (5)–(6) are equivalent by [11, Theorem 10]. \square

6. A -SPACES

Definition 1. A T_0 -space \mathbb{X} is an A -space [an f -space] if it has a basic subspace \mathbb{X}_0 [consisting of compact elements, respectively] which is a *parus* with respect to the specialization order $\leq_{\mathbb{X}}$; that is, if $x, y \in X_0$ and $x, y \leq a$ for some $a \in X$, then there exists $x \vee y \in X_0$. In this case, \mathbb{X}_0 is called an A -basic subspace of \mathbb{X} [an f -basic subspace of \mathbb{X} , respectively].

The following theorem was established in [10].

Theorem 19. [10, Theorem 7.3.4] *Let \mathbb{X} be an α^* -space and let \mathbb{Y} be an A -space with a least element. Then $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is also an A -space with a least element.*

The next statement shows that the requirement in Theorem 19 on \mathbb{Y} to have a least element can be weakened under certain circumstances. The proof of Theorem 20 follows the same lines as the one of Theorem 7.3.4 in [10]. In the proof, we also use some ideas from [15].

Theorem 20. *Let \mathbb{X} be a core-compact space and let \mathbb{Y} be a T_0 -space such that $\downarrow y$ has a least element for all $y \in Y$. Then $\downarrow f$ has a least element with respect to $\leq_{\mathcal{P}}$ for each $f \in C(\mathbb{X}, \mathbb{Y})$. Moreover, \mathbb{Y} is an A_d -space then $\mathbb{C}_s(\mathbb{X}, \mathbb{Y})$ is an A_d -space.*

Proof. As usual, we write \leq instead of $\leq_{\mathcal{P}}$ for the sake of simplicity. For each $y \in Y$, let \perp_y denote the least element of the set $\downarrow y$ in \mathbb{Y} and let \mathbb{Y}_0 be an A -basic subspace in \mathbb{Y} .

According to Theorem 9, $\mathbb{C}_s(\mathbb{X}, \mathbb{Y})$ is a d -space.

Claim 1. Let \mathbb{Z} be an arbitrary α -space and let $z \in Z$ be such that $\downarrow_{\mathbb{Z}} z$ has a least element \perp_z . Then $\perp_z \prec_{\mathbb{Z}} \perp_z$.

Proof of Claim. Indeed, let $a \in \uparrow \perp_z$. Since $a \in Z$ and Z is a basic subspace in \mathbb{Z} , we conclude that there is $b \in Z$ such that $b \prec a$. But then $b \in \downarrow a$ and thus $\perp_z \leq_{\mathbb{Z}} b \prec a$. This implies that $a \in \text{int } \uparrow b \subseteq \uparrow \perp_z$ which yields that $\uparrow \perp_z \in \mathcal{T}(\mathbb{Z})$. \square

It follows from Claim 1 that $\perp_y \in Y_0$ for all $y \in Y$.

Claim 2. $\downarrow f$ has a least element with respect to \leq for each $f \in C(\mathbb{X}, \mathbb{Y})$.

Proof of Claim. Consider the mapping

$$\perp_f: X \rightarrow Y, \quad \perp_f: x \mapsto \perp_{f(x)}.$$

It is straightforward that $\perp_f \leq g$ for each $g \leq f$. So, it remains to prove that \perp_f is continuous. Indeed, let $V \in \mathcal{T}(\mathbb{Y})$; then applying Claim 1 and using the continuity of f , we have

$$\perp_f^{-1}(V) = \{x \in X \mid \perp_{f(x)} \in V\} = \bigcup \{f^{-1}(\uparrow \perp_y) \mid y \in Y, \perp_y \in V\} \in \mathcal{T}(\mathbb{X}),$$

which completes the proof. \square

Similarly to the proof of [10, Theorem 7.3.4], we consider the set

$$\begin{aligned} \mathcal{W} = & \left\{ \langle f, \emptyset, \emptyset \mid f \in C(\mathbb{X}, \mathbb{Y}) \right\} \cup \\ & \cup \left\{ \langle f, \{U_0, \dots, U_n\}, \{y_0, \dots, y_n\} \mid \right. \\ & \left. f \in C(\mathbb{X}, \mathbb{Y}), n < \omega, U_0, \dots, U_n \in \mathcal{T}(\mathbb{X}) \setminus \{\emptyset\}, y_0, \dots, y_n \in Y_0 \right\}, \end{aligned}$$

where each $\langle f, \{U_0, \dots, U_n\}, \{y_0, \dots, y_n\} \rangle \in \mathcal{W}$ possesses the following properties:

- (*) the set $\{U_0, \dots, U_n\}$ is closed under nonempty intersections;
- (**) if $U_i \subseteq U_j$ for some $i, j \leq n$ then $y_j \leq_{\mathbb{Y}} y_i$.
- (***) $\perp_{f(x)} \leq y_i$ for all $x \in \bigcup_{j \leq n} U_j$, where $i \leq n$ is such that $U_i = \bigcap \{U_j \mid j \leq n, x \in U_j\}$.

For each $W = \langle f, \{U_0, \dots, U_n\}, \{y_0, \dots, y_n\} \rangle \in \mathcal{W}$, we define a mapping $f_W: X \rightarrow Y$ by the following rule:

$$f_W(x) = \begin{cases} y_i, & \text{if } U_i = \bigcap \{U_j \mid j \leq n, x \in U_j\}; \\ \perp_{f(x)}, & \text{if } x \notin \bigcup_{j \leq n} U_j. \end{cases}$$

It follows from the definition that $f_W(X) \subseteq Y_0$.

Claim 3. $f_W \in C(\mathbb{X}, \mathbb{Y})$ for each $W \in \mathcal{W}$.

Proof of Claim. Let $W = \langle f, \{U_0, \dots, U_n\}, \{y_0, \dots, y_n\} \rangle$. Since Y_0 is a basic subspace of \mathbb{Y} , it suffices to show that $f_W^{-1}(\text{int } \uparrow a) \in \mathcal{T}(\mathbb{X})$ for all $a \in Y_0$. Indeed, for a fixed element $a \in Y_0$, the following three cases are possible.

Case 1: $\perp_{f(x)} \in \text{int } \uparrow a$ for some $x \in X$. We have in this case that $a \prec \perp_{f(x)}$ whence $a = \perp_{f(x)}$ as $\perp_{f(x)}$ is a least element in $\downarrow f(x)$. We claim that $f_W^{-1}(\text{int } \uparrow a) = f^{-1}(\text{int } \uparrow a)$.

Indeed, let $x' \in f_W^{-1}(\uparrow \perp_{f(x)}) = f_W^{-1}(\text{int } \uparrow a)$. This means that $\perp_{f(x)} \leq_{\mathbb{Y}} f_W(x')$. There are two subcases.

Case 1.1: $x' \in \bigcup_{i \leq n} U_i$. Let $j \leq n$ be such that $U_j = \bigcap \{U_i \mid i \leq n, x' \in U_i\}$. Then $\perp_{f(x)} \leq f_W(x') = y_j$. As $\perp_{f(x')} \leq y_j$ by (***) , we conclude that $a = \perp_{f(x)} = \perp_{f(x')} \leq f(x')$. This implies that $x' \in f^{-1}(\text{int } \uparrow a)$.

Case 1.2: $x' \notin \bigcup_{i \leq n} U_i$. In this case, $\perp_{f(x)} \leq f_W(x') = \perp_{f(x')}$ whence $a = \perp_{f(x)} \prec \perp_{f(x')} = \perp_{f(x')} \leq f(x')$. Therefore, $x' \in f^{-1}(\text{int } \uparrow a)$ again.

Thus, $f_W^{-1}(\text{int } \uparrow a) \subseteq f^{-1}(\text{int } \uparrow a)$. To prove the reverse inclusion, suppose that $x' \in f^{-1}(\text{int } \uparrow a)$. This yields that $a = \perp_{f(x)} \leq f(x')$ whence $\perp_{f(x)} = \perp_{f(x')}$. If $x' \in \bigcup_{i \leq n} U_i$ then there is $j \leq n$ such that $U_j = \bigcap \{U_i \mid i \leq n, x' \in U_i\}$. Thus, $a = \perp_{f(x)} = \perp_{f(x')} \leq y_j = f_W(x')$ by (***) . If $x' \notin \bigcup_{i \leq n} U_i$ then $a = \perp_{f(x)} = \perp_{f(x')} = f_W(x')$ and $x' \in f_W^{-1}(\text{int } \uparrow a)$ in any case.

Therefore, $f^{-1}(\text{int } \uparrow a) \subseteq f_W^{-1}(\text{int } \uparrow a)$ and $f_W^{-1}(\text{int } \uparrow a) = f^{-1}(\text{int } \uparrow a) \in \mathcal{T}(\mathbb{X})$ in Case 1.

Case 2: $\{x \in X \mid a \prec \perp_{f(x)}\} = \emptyset$ and $I = \{i \leq n \mid a \prec y_i\} \neq \emptyset$. We put $U = \bigcup_{i \in I} U_i$; then $U \in \mathcal{T}(\mathbb{X})$. We show that $U = f_W^{-1}(\text{int } \uparrow a)$. Indeed, if $x \in U$ then $x \in U_j$ for some $j \in I$. But then $f_W(x) = y_i$ for some $i \leq n$ such that $x \in U_i \subseteq U_j$. Applying (**), we obtain that $a \prec y_j \leq y_i = f_W(x)$. This means that $x \in f_W^{-1}(\text{int } \uparrow a)$ and $U \subseteq f_W^{-1}(\text{int } \uparrow a)$. Conversely, let $x \in f_W^{-1}(\text{int } \uparrow a)$. As

$\{x \in X \mid a \prec \perp_{f(x)}\} = \emptyset$, we conclude that $y_i = f_W(x) \in \text{int } \uparrow a$ whence $i \in I$. It follows that $x \in U_i \subseteq U$. Therefore, $f_W^{-1}(\text{int } \uparrow a) \subseteq U$ which is our desired conclusion.

Case 3: $\{x \in X \mid a \prec \perp_{f(x)}\} = \{i \leq n \mid a \prec y_i\} = \emptyset$. In this case, $f_W^{-1}(\text{int } \uparrow a) = \emptyset \in \mathcal{T}(\mathbb{X})$. \square

The following statement is a corollary of the definition of f_W .

Claim 4. For $f \in C(\mathbb{X}, \mathbb{Y})$ and $W = \langle f, \{U_0, \dots, U_n\}, \{y_0, \dots, y_n\} \rangle \in \mathcal{W}$, we have $f_W \leq f$ if and only if $y_i \leq f(x)$ for all $x \in \bigcup_{j \leq n} U_j$, where $i \leq n$ is such that $U_i = \bigcap \{U_j \mid j \leq n, x \in U_j\}$.

Claim 5. Let $f \in C(\mathbb{X}, \mathbb{Y})$ and let $W = \langle f, \{U\}, \{y\} \rangle$ be such that $y \in Y_0$, $U \in \mathcal{T}(\mathbb{X})$, and $U \prec f^{-1}(\text{int } \uparrow y)$. Then $f_W \ll f$.

Proof of Claim. First of all, we notice that $W \in \mathcal{W}$ provided that $y \prec f(x)$ for all $x \in U$. Indeed, for each $x \in U$, we have $\perp_{f(x)} = \perp_y \leq y \prec f(x)$ which implies that W possesses the property (**). The conditions (*)–(**) are satisfied in a trivial way. Moreover, it is straightforward to see that $f_W \leq f$.

Suppose that $f \leq g = \bigvee_{i \in I} g_i$ for some family $\{g_i \in C(\mathbb{X}, \mathbb{Y}) \mid i \in I\}$ which is up-directed with respect to $\leq_{\mathcal{P}}$. According to Theorem 9, $h(x) = \bigvee_{i \in I} g_i(x)$ exists for all $x \in X$. It is clear that $h(x) \leq g(x)$ for all $x \in X$. Conversely, if $g(x) \in U \in \mathcal{T}(\mathbb{Y})$ then $g \in V_{x,U} \in \mathcal{P} \subseteq \mathcal{S}$. Since $\mathbb{C}_{\mathcal{S}}(\mathbb{X}, \mathbb{Y})$ is a d -space, we conclude that $g_i \in V_{x,U}$ for some $i \in I$, whence $h(g) \geq g_i(x) \in U$. Therefore, $h(x) \in U$ and $g(x) \leq h(x)$; that is, $g(x) = \bigvee_{i \in I} g_i(x)$ for all $x \in X$. Hence, we have by our assumption that

$$U \prec f^{-1}(\text{int } \uparrow y) \subseteq g^{-1}(\text{int } \uparrow y) = \bigcup_{i \in I} g_i^{-1}(\text{int } \uparrow y).$$

This yields that $U \subseteq g_i^{-1}(\text{int } \uparrow y)$ and $g_i(U) \subseteq \uparrow y$ for some $i \in I$. We claim that $f_W \leq g_i$. Indeed, if $x \in U$ then $f_W(x) = y \leq g_i(x)$. Suppose now that $x \notin U$. Since $f(x) \leq g(x)$ and $g_i(x) \leq g(x)$, we conclude that $\perp_{f(x)} = \perp_{g(x)} = \perp_{g_i(x)}$. Therefore, $f_W(x) = \perp_{f(x)} = \perp_{g_i(x)} \leq g_i(x)$ and $f_W \leq g_i$ which proves that $f_W \ll f$. \square

Claim 6. We have

$$f = \bigvee \left\{ f_{\langle f, \{U\}, \{y\} \rangle} \mid y \prec f(x) \text{ for some } x \in X, U \prec f^{-1}(\text{int } \uparrow y) \right\}$$

for each function $f \in C(\mathbb{X}, \mathbb{Y})$. In particular, $\langle C(\mathbb{X}, \mathbb{Y}); \leq_{\mathcal{P}} \rangle$ is a continuous poset.

Proof of Claim. According to Claim 5, f is an upper bound for the family

$$\mathcal{F}_f = \left\{ f_{\langle f, \{U\}, \{y\} \rangle} \mid y \prec f(x) \text{ for some } x \in X, U \prec f^{-1}(\text{int } \uparrow y) \right\}.$$

Let g be another upper bound for \mathcal{F}_f . It suffices to establish that $f \leq g$. Indeed, let $x \in X$. As Y_0 is a basic subspace of \mathbb{Y} , we conclude that $f(x) = \bigvee \{y \in Y_0 \mid y \prec f(x)\}$. Fix an element $y \in Y_0$ such that $y \prec f(x)$. Then we have $x \in f^{-1}(\text{int } \uparrow y) \in \mathcal{T}(\mathbb{X})$. As \mathbb{X} is core-compact, the poset $\langle \mathcal{T}(\mathbb{X}); \subseteq \rangle$ is continuous. Therefore, there is a nonempty set $U_y \in \mathcal{T}(\mathbb{X})$ such that $x \in U_y \prec f^{-1}(\text{int } \uparrow y)$. We put $W(y) = \langle f, \{U_y\}, \{y\} \rangle$. Then we have $f_{W(y)} \in \mathcal{F}_f$ and thus, $y = f_{W(y)}(x) \leq g(x)$. This implies that $f(x) = \bigvee \{y \in Y_0 \mid y \prec f(x)\} \leq g(x)$. Hence, $f \leq g$ which is our desired conclusion.

The last statement follows from Claim 5. \square

From Claim 6 and Lemma 1.10.5 in [10], we obtain the following statement.

Claim 7. Let $f \in C(\mathbb{X}, \mathbb{Y})$ and let $W = \langle f, \{U\}, \{y\} \rangle$ be such that $y \in Y_0$, $U \in \mathcal{T}(\mathbb{X})$, and $U \prec f^{-1}(\text{int } \uparrow y)$. Then $f_W \prec_s f$.

We put $C_0 = \{f_W \mid W \in \mathcal{W}\}$.

Claim 8. If Y_0 is a parus in $\langle Y; \leq_Y \rangle$ then \mathbb{C}_0 is a parus in $\langle C(\mathbb{X}, \mathbb{Y}); \leq_{\mathcal{P}} \rangle$.

Proof of Claim. We consider arbitrary sequences

$$\begin{aligned} W_0 &= \langle g_0, \{U_0, \dots, U_m\}, \{a_0, \dots, a_m\} \rangle \in \mathcal{W}, \\ W_1 &= \langle g_1, \{V_0, \dots, V_n\}, \{b_0, \dots, b_n\} \rangle \in \mathcal{W} \end{aligned}$$

and assume that $f_0 = f_{W_0}$, $f_1 = f_{W_1} \leq h$ for some $h \in C(\mathbb{X}, \mathbb{Y})$. As \mathbb{Y}_0 is a parus in \mathbb{Y} , for all $x \in X$, there is $f_0(x) \vee f_1(x) \in Y_0$. We put

$$\begin{aligned} U &= \{U_0, \dots, U_m\} \cup \{V_0, \dots, V_n\} \cup \{U_i \cap V_j \mid i \leq m, j \leq n, U_i \cap V_j \neq \emptyset\} = \\ &= \{Z_0, \dots, Z_k\}; \\ F &= \{y_0, \dots, y_k\}; \\ W &= \langle h, U, F \rangle, \end{aligned}$$

where y_i , $i \leq k$, is defined in the following way. For each $i \leq k$, we have $\bigcap \{U_j \mid j \leq m, Z_i \subseteq U_j\} \neq \emptyset$. Hence, $\bigcap \{U_j \mid j \leq m, Z_i \subseteq U_j\} = U_{m_i}$ for some $m_i \leq m$. Symmetrically, $\bigcap \{V_j \mid j \leq n, Z_i \subseteq V_j\} \neq \emptyset$, whence $\bigcap \{V_j \mid j \leq n, Z_i \subseteq V_j\} = V_{n_i}$ for some $n_i \leq n$. Thus for each element $x \in Z_i$, we have $a_{m_i} \leq f_0(x) \leq h(x)$ and $b_{n_i} \leq f_1(x) \leq h(x)$. As \mathbb{Y}_0 is a parus in \mathbb{Y} , there is an element $y_i = a_{m_i} \vee b_{n_i} \in Y_0$.

By our definition, W possesses the property (*). Suppose that $Z_{i_0} \subseteq Z_{i_1}$ for some $i_0, i_1 \leq k$. For each $j < 2$, let $m_j \leq m$ and $n_j \leq n$ be such that

$$U_{m_j} = \bigcap \{U_t \mid t \leq m, Z_{i_j} \subseteq U_t\} \quad \text{and} \quad V_{n_j} = \bigcap \{V_t \mid t \leq n, Z_{i_j} \subseteq V_t\}.$$

Since $Z_{i_0} \subseteq Z_{i_1}$, we obtain that $U_{m_0} \subseteq U_{m_1}$ and $V_{n_0} \subseteq V_{n_1}$. We have therefore by our definition that $y_{i_0} = a_{m_0} \vee b_{n_0} \geq a_{m_1} \vee b_{n_1} = y_{i_1}$ as W_0 and W_1 have the property (**). This yields that W possesses the property (**). To establish that W also possesses the property (***), we consider an arbitrary element $x \in \bigcup_{i \leq k} Z_i$. Let $j \leq k$ be such that $Z_j = \bigcap \{Z_t \mid t \leq k, x \in Z_t\}$ and let

$$U_{m_j} = \bigcap \{U_t \mid t \leq m, Z_j \subseteq U_t\}; \quad V_{n_j} = \bigcap \{V_t \mid t \leq n, Z_j \subseteq V_t\}.$$

If $x \in U_s$ for some $s \leq m$, then $Z_j \subseteq U_s$ as $U_s \in U$. This implies that $x \in Z_j \subseteq U_{m_j} \subseteq U_s$. Similarly, $x \in V_s$ for some $s \leq n$ implies that $x \in Z_j \subseteq V_{n_j} \subseteq V_s$. Therefore, we obtain that

$$U_{m_j} = \bigcap \{U_t \mid t \leq m, x \in U_t\}; \quad V_{n_j} = \bigcap \{V_t \mid t \leq n, x \in V_t\}.$$

As W_0 and W_1 possess the property (***), we obtain

$$\begin{aligned} y_j &= a_{m_j} \vee b_{n_j} \geq a_{m_j} = f_0(x) \geq \perp_{g_0(x)}; \\ y_j &= a_{m_j} \vee b_{n_j} \geq b_{n_j} = f_1(x) \geq \perp_{g_1(x)}; \\ h(x) &\geq f_0(x) = a_{m_j}; \\ h(x) &\geq f_1(x) = b_{n_j}. \end{aligned}$$

Hence $\perp_{g_0(x)} = \perp_{h(x)} = \perp_{g_1(x)}$ and

$$h(x) \geq y_j = f_W(x) \geq \perp_{h(x)} = \perp_{g_0(x)} = \perp_{g_1(x)}.$$

This proves that W also has the property (***) .

Finally, we prove that $f_W = f_0 \vee f_1$ in $\mathbb{C}(\mathbb{X}, \mathbb{Y})$. We prove first that $f_0, f_1 \leq f_W$. Indeed, let $x \in X$. Two cases are possible. In what follows, we use the notation introduced above.

Case 1: $x \in \bigcup_{i \leq k} Z_k$. In this case, $f_0(x) = a_{m_j}$, $f_1(x) = b_{n_j}$, $f_W(x) = y_j = a_{m_j} \vee b_{n_j} \geq a_{m_j} = f_0(x)$. Similarly, $f_W(x) \geq b_{n_j} = f_1(x)$.

Case 2: $x \notin \bigcup_{i \leq k} Z_i$. This implies that $x \notin \bigcup_{i \leq m} U_i \cup \bigcup_{i \leq n} V_i$. Therefore,

$$\perp_{g_0(x)} = f_0(x) \leq h(x); \quad \perp_{g_1(x)} = f_1(x) \leq h(x),$$

whence $f_0(x) = f_1(x) = \perp_{g_0(x)} = \perp_{g_1(x)} = \perp_{h(x)} = f_W(x)$.

Therefore, f_W is indeed an upper bound of f_0 and f_1 . Suppose that $f_0, f_1 \leq f$ for some $f \in C(\mathbb{X}, \mathbb{Y})$; we have to establish that $f_W \leq f$. Indeed, for $x \in X$, two cases are possible.

Case 1: $x \in \bigcup_{i \leq k} Z_k$. According to the considerations in *Case 1* above, we have $f_W(x) = y_j = a_{m_j} \vee b_{n_j} = f_0(x) \vee f_1(x) \leq f(x)$ in this case.

Case 2: $x \notin \bigcup_{i \leq k} Z_i$. This implies as in *Case 2* above that $f_W(x) = \perp_{g_0(x)} = \perp_{g_1(x)} = f_0(x) = f_1(x) \leq f(x)$.

Therefore, $f_W = f_0 \vee f_1$ and the proof of Claim is complete. \square

Claim 9. \mathbb{C}_0 is a basic subspace in $\mathbb{C}_s(\mathbb{X}, \mathbb{Y})$.

Proof of Claim. Let $f \in V \in \mathfrak{S}$. According to Claim 6, $f = \bigvee \mathcal{F}_f$, where

$$\mathcal{F}_f = \left\{ f_{\langle f, \{U\}, \{y\} \rangle} \mid y \prec f(x) \text{ for some } x \in X, U \prec f^{-1}(\text{int } \uparrow y) \right\}.$$

According to Claim 8, for each finite nonempty subfamily $\mathcal{G} \subseteq \mathcal{F}_f$, there is $W(\mathcal{G}) \in \mathbb{W}$ such that $f_{\mathcal{G}} = f_{W(\mathcal{G})} = \bigvee \mathcal{G} \leq f$. Thus, the family

$$\mathcal{D} = \{f_{\mathcal{G}} \mid \mathcal{G} \subseteq \mathcal{F}_f, 0 < |\mathcal{G}| < \omega\}$$

is up-directed with respect to $\leq_{\mathcal{P}}$ and $f = \bigvee \mathcal{D}$. This implies that $f_{\mathcal{G}} \in V$ for some finite nonempty family $\mathcal{G} \subseteq \mathcal{F}_f$. Moreover, $f \in \text{int}_{\mathfrak{S}} \uparrow g$ for all $g \in \mathcal{G}$ according to Claim 6. Therefore,

$$f \in \bigcap \{\text{int}_{\mathfrak{S}} \uparrow g \mid g \in \mathcal{G}\} = \text{int}_{\mathfrak{S}} \uparrow f_{\mathcal{G}},$$

which implies that $f_{\mathcal{G}} \prec_s f$. \square

The desired statements follow from Claims 2, 8, and 9. \square

Corollary 21. *The following statements are equivalent for a T_0 -space \mathbb{Y} .*

- (1) \mathbb{Y} is an A_d -space such that $\downarrow y$ has a least element for each $y \in Y$.
- (2) For each core-compact space \mathbb{X} , $\mathbb{C}_s(\mathbb{X}, \mathbb{Y})$ is an A_d -space such that $\downarrow f$ has a least element for each $f \in C(\mathbb{X}, \mathbb{Y})$.

Proof. (1) implies (2) by Theorem 20. Let \mathbb{T} be a trivial (one-element) topological space; \mathbb{T} is obviously core-compact. If (2) holds, then $\mathbb{Y} \cong \mathbb{C}_s(\mathbb{T}, \mathbb{Y})$ is an A_d -space such that $\downarrow f$ has a least element for each $f \in C(\mathbb{X}, \mathbb{Y})$. Hence, (1) also holds. \square

Theorem 22. *If \mathbb{X} is a core-compact space and \mathbb{Y} is an A -space with a least element then $\mathbb{C}_j(\mathbb{X}, \mathbb{Y})$ is an A -space with a least element.*

Proof. The proof is very close to the one of Theorem 7.3.4 in [10]; we use certain parts of it here.

Let \perp denote the least element in \mathbb{Y} and let \mathbb{Y}_0 be an A -basic subspace in \mathbb{Y} ; in particular, $\perp \in Y_0$. It follows from Lemma 1 that $\mathbb{C}_j(\mathbb{X}, \mathbb{Y})$ has a least element.

Similarly to the proof of [10, Theorem 7.3.4], we consider the set

$$\mathcal{W} = \{ \langle \emptyset, \emptyset \rangle \} \cup \left\{ \langle \{U_0, \dots, U_n\}, \{y_0, \dots, y_n\} \rangle \mid \right. \\ \left. n < \omega, U_0, \dots, U_n \in \mathcal{T}(\mathbb{X}) \setminus \{ \emptyset \}, y_0, \dots, y_n \in Y_0 \right\},$$

where each $\langle \{U_0, \dots, U_n\}, \{y_0, \dots, y_n\} \rangle \in \mathcal{W}$ possesses the following properties:

- (*) the set $\{U_0, \dots, U_n\}$ is closed under nonempty intersections;
- (**) if $U_i \subseteq U_j$ for some $i, j \leq n$ then $y_j \leq_{\mathbb{Y}} y_i$.

For each $W = \langle \{U_0, \dots, U_n\}, \{y_0, \dots, y_n\} \rangle \in \mathcal{W}$, we define a mapping $f_W: X \rightarrow Y$ by the following rule:

$$f_W(x) = \begin{cases} y_i, & \text{if } U_i = \bigcap \{U_j \mid j \leq n, x \in U_j\}; \\ \perp, & \text{if } x \notin \bigcup_{j \leq n} U_j. \end{cases}$$

We put $C_0 = \{f_W \mid W \in \mathcal{W}\}$.

Claims 1–3 were established in the proof of [10, Theorem 7.3.4]

Claim 1. $f_W \in C(\mathbb{X}, \mathbb{Y})$ for each $W \in \mathcal{W}$.

Claim 2. For $f \in C(\mathbb{X}, \mathbb{Y})$ and $W = \langle \{U_0, \dots, U_n\}, \{y_0, \dots, y_n\} \rangle \in \mathcal{W}$, we have $f_W \leq f$ if and only if $y_i \leq f(x)$ for all $x \in \bigcup_{j \leq n} U_j$, where $i \leq n$ is such that $U_i = \bigcap \{U_j \mid j \leq n, x \in U_j\}$.

Claim 3. If Y_0 is a parus in $\langle Y; \leq_{\mathbb{Y}} \rangle$ then C_0 is a parus in $\langle C(\mathbb{X}, \mathbb{Y}); \leq_{\mathcal{P}} \rangle$.

Claim 4. C_0 is a basic subspace in $\mathbb{C}_j(\mathbb{X}, \mathbb{Y})$.

Proof of Claim. Suppose first that $U \in \mathcal{T}(\mathbb{X})$, $W \in \mathcal{T}(\mathbb{Y})$, and $f \in C(\mathbb{X}, \mathbb{Y})$ are such that $f \in V_{U,W} \in \mathcal{J}$; then $U \prec f^{-1}(W)$. Since Y_0 is a basic subspace in \mathbb{Y} , $f^{-1}(W) = \bigcup_{y \in Y_0 \cap W} f^{-1}(\text{int } \uparrow y)$. Therefore, $U \prec f^{-1}(\text{int } \uparrow y_0) \cup \dots \cup f^{-1}(\text{int } \uparrow y_n)$ for some $y_0, \dots, y_n \in Y_0 \cap W$. Since \mathbb{X} is core-compact, there is $U' \in \mathcal{T}(\mathbb{X})$ with $U \prec U' \prec f^{-1}(\text{int } \uparrow y_0) \cup \dots \cup f^{-1}(\text{int } \uparrow y_n)$. We put $\mathcal{B} = \mathcal{T}(\mathbb{X})$ and apply Proposition 15. According to it and the fact that \mathbb{X} is core-compact, there are $W_0, \dots, W_n, S_0, \dots, S_n \in \mathcal{T}(\mathbb{X})$ such that

$$W_i \prec S_i \prec f^{-1}(\text{int } \uparrow y_i) \text{ for all } i \leq n \text{ and } U \subseteq W_0 \cup \dots \cup W_n \subseteq U'.$$

We claim that $\bigcap_{i \leq n} V_{W_i, W} \subseteq V_{U, W}$. Indeed, if $g \in \bigcap_{i \leq n} V_{W_i, W}$ then $W_i \prec g^{-1}(W)$. Hence, $U \subseteq W_0 \cup \dots \cup W_n \prec g^{-1}(W)$ which implies that $g \in V_{U, W}$.

For each $i \leq n$, we put $f_i = f_{\langle \{S_i\}, \{y_i\} \rangle}$; then $f_i \in C_0$. Moreover, if $x \in S_i$ then $f_i(x) = y_i \prec f(x)$ as $S_i \prec f^{-1}(\text{int } \uparrow y_i)$. Furthermore, $y_i \in W$ whence $W_i \prec S_i \subseteq f_i^{-1}(W)$ and $f_i \in V_{W_i, W}$. Therefore, $f_i \leq f$ for all $i \leq n$ by Claim 2. According to Claim 3, there is $h = f_0 \vee \dots \vee f_n \leq f$. As $f_i \in V_{W_i, W}$ for all $i \leq n$, we conclude that $h \in \bigcap_{i \leq n} V_{W_i, W} \subseteq V_{U, W}$.

Finally, $f \in \bigcap_{i \leq n} V_{S_i, \text{int } \uparrow y_i}$ as $S_i \prec f^{-1}(\text{int } \uparrow y_i)$ for each $i \leq n$. Suppose that $g \in \bigcap_{i \leq n} V_{S_i, \text{int } \uparrow y_i}$; then for all $i \leq n$ and all $x \in S_i$, one has $f_i(x) = y_i \prec g(x)$, whence $f_i \leq g$ by Claim 2. This implies that $h \leq g$. Summarizing, we obtain that $f \in \bigcap_{i \leq n} V_{S_i, \text{int } \uparrow y_i} \subseteq \uparrow h$; that is, $h \prec_j f$.

To complete the proof, we assume that $f \in V \in \mathcal{J}$. Then there is $k < \omega$ and there are open sets $U_0, \dots, U_k \in \mathcal{T}(\mathbb{X})$ and $W_0, \dots, W_k \in \mathcal{T}(\mathbb{Y})$ such that

$$f \in V_{U_0, W_0} \cap \dots \cap V_{U_k, W_k} \subseteq V.$$

According to what we have just proved, there are $h_0, \dots, h_k \in C(\mathbb{X}, \mathbb{Y})$ such that $h_i \prec_{\mathcal{J}} f$ and $h_i \in V_{U_i, W_i}$ for all $i \leq k$. By Claim 3, there is $h = h_0 \vee \dots \vee h_k \leq f$. In particular, $h \in V_{U_0, W_0} \cap \dots \cap V_{U_k, W_k} \subseteq V$ and $h \prec_{\mathcal{J}} f$. \square

The desired statement follows from Claims 3–4. \square

We notice that Theorem 22 generalizes Theorem 7.3.4 in [10] (see Theorem 19) as $\mathcal{J} = \mathcal{P}$ on $C(\mathbb{X}, \mathbb{Y})$ by Proposition 6(3) whenever \mathbb{X} is an α^* -space.

Corollary 23. *The following statements are equivalent for a T_0 -space \mathbb{Y} .*

- (1) \mathbb{Y} is an A -space with a least element.
- (2) For each core-compact space \mathbb{X} , $\mathbb{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is an A -space with a least element.
- (3) For each α^* -space \mathbb{X} , $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is an A -space with a least element.

Proof. (1) implies (2) by Theorem 22. (1) implies (3) as $\mathcal{J} = \mathcal{P}$ for an α^* -space \mathbb{X} . A trivial (one-element) topological space \mathbb{T} is obviously an α^* -space and thus core-compact. If (2) holds, then $\mathbb{Y} \cong \mathbb{C}_{\mathcal{J}}(\mathbb{T}, \mathbb{Y})$ is an A -space with a least element. Thus, (1) also holds and each of (2) and (3) implies (1). \square

7. INJECTIVITY IN FUNCTION SPACES

We will make use of the following results.

Theorem 24. [10, Theorem 4.2.3] *A topological T_0 -space \mathbb{X} is injective if and only if the following conditions hold:*

- (1) \mathbb{X} is a d -space;
- (2) \mathbb{X} is an α -space;
- (3) $\langle X; \leq_{\mathbb{X}} \rangle$ is a complete lattice.

Theorem 25. [10, Theorem 4.2.4] *A topological T_0 -space \mathbb{X} is densely injective if and only if the following conditions hold:*

- (1) \mathbb{X} is a d -space;
- (2) \mathbb{X} is an α -space;
- (3) \mathbb{X} is a bc -domain.

Corollary 26. *A topological T_0 -space \mathbb{X} is [densely] injective if and only if \mathbb{X} is a A_d -space with a least and a greatest element [with a least element].*

Proposition 27. *Let \mathbb{X}, \mathbb{Y} be T_0 -spaces and let \mathcal{T} be a topology on $C(\mathbb{X}, \mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$. If $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is an α^* -space [an α -space] for some space \mathbb{X} , then \mathbb{Y} is also an α^* -space [an α -space].*

Proof. Let $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ be an α^* -space. Suppose that $y \in U \in \mathcal{T}(\mathbb{Y})$ and fix an element $x \in X$; then $\xi_y \in V_{x, U} \in \mathcal{P} \subseteq \mathcal{T}$. According to our assumption, there are continuous functions $f_0, \dots, f_m \in V_{x, U}$ and a set $V \in \mathcal{T}$ such that $\xi_y \in V \subseteq \uparrow f_0 \cup \dots \cup \uparrow f_m$. Since $V \in \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$, there are upper cones $\mathcal{H}_0, \dots, \mathcal{H}_n$ in $\langle \mathcal{T}(\mathbb{X}); \subseteq \rangle$ and open sets $W_0, \dots, W_n \in \mathcal{T}(\mathbb{Y})$ such that

$$\xi_y \in V_{\mathcal{H}_0, W_0} \cap \dots \cap V_{\mathcal{H}_n, W_n} \subseteq V \subseteq \uparrow f_0 \cup \dots \cup \uparrow f_m \subseteq V_{x, U}.$$

Therefore, $\xi_i^{-1}(W_j) \in \mathcal{H}_j$ for all $j \leq n$. In particular, $\mathcal{H}_j \neq \emptyset$ for all $j \leq n$. We put

$$W = \bigcap \{W_j \mid j \leq n, \mathcal{H}_j \neq \mathcal{T}(\mathbb{X})\}.$$

Then we have that $W \in \mathcal{T}(\mathbb{Y})$. Moreover, $\xi_y^{-1}(W_j) \neq \emptyset$ whence $\xi_y^{-1}(W_j) = X$ for all $j \leq n$ such that $\mathcal{H}_j \neq \mathcal{T}(\mathbb{X})$. This implies that $y \in W$. If $y' \in W$, then $\xi_{y'}^{-1}(W_j) = X$ for all $j \leq n$ such that $\mathcal{H}_j \neq \mathcal{T}(\mathbb{X})$. Therefore,

$$\xi_{y'} \in V_{\mathcal{H}_0, W_0} \cap \dots \cap V_{\mathcal{H}_n, W_n} \subseteq V \subseteq \uparrow f_0 \cup \dots \cup \uparrow f_m \subseteq V_{x, U}$$

and thus $y' \in U$. We established therefore that $y \in W \subseteq \uparrow f_0(x) \cup \dots \cup \uparrow f_m(x)$ and $f_0(x), \dots, f_m(x) \in U$. This implies that \mathbb{Y} is an α^* -space.

If $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is an α -space, we use in the argument above with $m = 0$. \square

Proposition 28. *Let \mathbb{X}, \mathbb{Y} be T_0 -spaces and let \mathcal{T} be a topology on $C(\mathbb{X}, \mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$. If $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is a densely injective space then \mathbb{Y} is also densely injective.*

Proof. According to Theorem 25, $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is an α -space, a d -space, and a bc-domain. According to Proposition 27, \mathbb{Y} is an α -space. According to the proof of Theorem 9, \mathbb{Y} is a d -space. In view of Theorem 25, in order to prove the dense injectivity of \mathbb{Y} , it suffices to show that \mathbb{Y} is a partial join-semilattice with respect to the specialization order $\leq_{\mathbb{Y}}$.

Let $y_0, y_1 \leq y$ in \mathbb{Y} . This means that $\xi_{y_0}, \xi_{y_1} \leq \xi_y$ in $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$. As $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is a bc-domain, there is a continuous function $f = \xi_{y_0} \vee \xi_{y_1}$. We prove that f is constant. Indeed, let $f(x_0) \in U \in \mathcal{T}(\mathbb{Y})$ and let $x_1 \in X$ be an arbitrary element. Then $f \in V_{x_0, U} \in \mathcal{P} \subseteq \mathcal{T}$. By Corollary 26, $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is an A_d -space. By [11, Lemma 7], there are open sets $V_0, V_1 \in \mathcal{T}$ such that $\xi_{y_0} \in V_0, \xi_{y_1} \in V_1$, and $V_0 \cap V_1 \subseteq V_{x_0, U}$. Using the assumption that $\mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$ and applying the same argument as in the proof of Proposition 14, we obtain that $f = \xi_y$ for some $y \in Y$, whence $y = y_0 \vee y_1$ in \mathbb{Y} .

Applying Theorem 25 again, we conclude that \mathbb{Y} is densely injective. \square

Proposition 29. *Let \mathbb{X}, \mathbb{Y} be T_0 -spaces and let \mathcal{T} be a topology on $C(\mathbb{X}, \mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$. If $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is an injective space then \mathbb{Y} is also injective.*

Proof. According to Theorem 24 $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is an α -space, a d -space, and a complete lattice with respect to the specialization order \leq . By Proposition 27, \mathbb{Y} is an α -space. By the proof of Theorem 9, \mathbb{Y} is a d -space. Since $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is an A_d -space having a least and a greatest element by Corollary 26, it possesses the properties (H_0) – (H_2) by [11, Lemma 7]. By Lemma 1 and Proposition 14, the space \mathbb{Y} also possesses the properties (H_0) – (H_2) . This means that \mathbb{Y} is a complete lattice with respect to the specialization order $\leq_{\mathbb{Y}}$. Applying Theorem 24 again, we conclude that \mathbb{Y} is an injective space. \square

The following statement generalizes Theorem 19 in [11].

Theorem 30. *For a T_0 -space \mathbb{Y} , the following conditions are equivalent.*

- (1) \mathbb{Y} is densely injective.
- (2) $\mathbb{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is densely injective for each core-compact space \mathbb{X} .
- (3) $\mathbb{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is densely injective for some core-compact space \mathbb{X} .
- (4) $\mathbb{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is densely injective for some T_0 -space \mathbb{X} .
- (5) $\mathbb{C}_{\mathcal{S}}(\mathbb{X}, \mathbb{Y})$ is densely injective for each core-compact space \mathbb{X} .

- (6) $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is densely injective for each α^* -space \mathbb{X} .
- (7) $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is densely injective for some α^* -space \mathbb{X} .
- (8) $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is densely injective for some T_0 -space \mathbb{X} .
- (9) $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is densely injective for some [core-compact] T_0 -space \mathbb{X} and some topology \mathcal{T} such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$.

Proof. (1) implies (2) by Proposition 11(1). (2) implies (3) implies (4), (6) implies (7) implies (8), and (3) implies (9) in a trivial way. (4) implies (1) and each of (8) and (9) implies (1) by Proposition 28. Furthermore, (2) implies (5) as $\mathcal{J} = \mathcal{S}$ whenever $\mathbb{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is densely injective, see [10, Lemma 1.8.6].

Let \mathbb{T} be a trivial (one-element) topological space; \mathbb{T} is obviously core-compact. If (5) holds, then $\mathbb{Y} \cong \mathbb{C}_{\mathcal{S}}(\mathbb{T}, \mathbb{Y})$ is a densely injective space. Hence, (1) also holds, and (5) implies (1).

Statements (1), (6), and (7) are equivalent by [11, Theorem 19]. □

The following statement generalizes Theorem 20 in [11].

Theorem 31. *For a T_0 -space \mathbb{Y} , the following conditions are equivalent.*

- (1) \mathbb{Y} is injective.
- (2) $\mathbb{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is injective for each core-compact space \mathbb{X} .
- (3) $\mathbb{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is injective for some core-compact space \mathbb{X} .
- (4) $\mathbb{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is injective for some T_0 -space \mathbb{X} .
- (5) $\mathbb{C}_{\mathcal{S}}(\mathbb{X}, \mathbb{Y})$ is injective for each core-compact space \mathbb{X} .
- (6) $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is injective for each α^* -space \mathbb{X} .
- (7) $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is injective for some α^* -space \mathbb{X} .
- (8) $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is injective for some T_0 -space \mathbb{X} .
- (9) $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is injective for some [core-compact] T_0 -space \mathbb{X} and some topology \mathcal{T} such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$.

Proof. (1) implies (2) by Proposition 11(1). (2) implies (3) implies (4), (6) implies (7) implies (8), and (3) implies (9) in a trivial way. (4) implies (1) and each of (8) and (9) implies (1) by Proposition 29. Furthermore, (2) implies (5) as $\mathcal{J} = \mathcal{S}$ whenever $\mathbb{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is injective.

Let \mathbb{T} be a trivial (one-element) topological space; \mathbb{T} is obviously core-compact. If (5) holds, then $\mathbb{Y} \cong \mathbb{C}_{\mathcal{S}}(\mathbb{T}, \mathbb{Y})$ is an injective space, which yields (1).

Statements (1), (6), and (7) are equivalent by [11, Theorem 20]. □

The fact that (1) implies (2) in Theorems 30 and 31 was established in [13, Proposition II-4.6]. The fact that (3) implies (1) in Theorems 30 and 31 was established in [3, Theorems 4.6, 4.7].

8. Δ -SPACES $\mathbb{C}(\mathbb{X}, \mathbb{Y})$

Definition 2. [8] A continuous function $\delta: \mathbb{X} \rightarrow \mathbb{X}$ from a topological space \mathbb{X} into itself is a *deflation*, if the set $\delta(X)$ is finite and $\delta(x) \leq_{\mathbb{X}} x$ for all $x \in X$.

A T_0 -space \mathbb{X} is a Δ -space, if there is an up-directed family $\{\delta_i: \mathbb{X} \rightarrow \mathbb{X} \mid i \in I\}$ of deflations of \mathbb{X} with the property that for every $U \in \mathcal{J}(\mathbb{X})$ and every $x \in U$, there is $i \in I$ such that $\delta_i(x) \in U$.

A space \mathbb{X} is a Δ_d -space, if \mathbb{X} is a Δ -space and a d -space simultaneously.

The following partial generalization of Theorem 23 from [11] holds.

Proposition 32. *Let \mathbb{X}, \mathbb{Y} be T_0 -spaces and let \mathcal{T} be a topology on $C(\mathbb{X}, \mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$. If $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is a Δ -space [Δ_d -space] then \mathbb{Y} is also a Δ -space [Δ_d -space].*

Proof. Let D denote an up-directed family of deflations of the space $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ which satisfies all the requirements of Definition 2. We fix an element $a \in X$. For each $\delta \in D$, consider the mapping

$$\kappa_\delta: \mathbb{Y} \rightarrow \mathbb{Y}, \quad \kappa_\delta: y \mapsto \delta(\xi_y)(a).$$

The set $\kappa_\delta(Y)$ is finite, as the set $\{\delta(\xi_y) \mid y \in Y\}$ is finite. Moreover, for every $y \in Y$, we have $\kappa_\delta(y) = \delta(\xi_y)(a) \leq \xi_y(a) = y$, since δ is a deflation of $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$. Let $U \in \mathcal{T}(\mathbb{X})$; then

$$\begin{aligned} \kappa_\delta^{-1}(U) &= \{y \in Y \mid \delta(\xi_y)(a) \in U\} = \{y \in Y \mid \delta(\xi_y) \in V_{a,U} \in \mathcal{T}\} = \\ &= \{y \in Y \mid \xi_y \in \delta^{-1}(V_{a,U}) \in \mathcal{T}\} = \xi^{-1}\delta^{-1}(V_{a,U}) \in \mathcal{T}(\mathbb{Y}), \end{aligned}$$

as ξ is continuous by Lemma 3 and δ is continuous by Definition 2. Thus, κ_δ is continuous, whence it is a deflation of \mathbb{Y} . Moreover, if $\delta, \delta' \in D$ are such that $\delta \leq \delta'$, then $\kappa_\delta(y) = \delta(\xi_y)(a) \leq \delta'(\xi_y)(a) = \kappa_{\delta'}(y)$ for all $y \in Y$. Therefore $\{\kappa_\delta \mid \delta \in D\}$ is an up-directed family of deflations of $\mathbb{C}_{\mathcal{T}}(\mathbb{Y}, \mathbb{Y})$ by Lemma 1. Finally, if $y \in U \in \mathcal{T}(\mathbb{Y})$ then $\xi_y \in V_{a,U} \in \mathcal{P} \subseteq \mathcal{T}$. In view of the choice of D , there is a deflation $\delta \in D$ such that $\delta(\xi_y) \in V_{a,U}$. Then $\kappa_\delta(y) = \delta(\xi_y)(a) \in U$, and the proof is complete.

The statement about Δ_d -spaces follows with the use of Theorem 9. \square

Proposition 32 was established for the case $\mathcal{T} = \mathcal{J}$ in [3, Theorem 5.5].

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