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UNIFORM m-EQUIVALENCES AND NUMBERINGS OF CLASSICAL SYSTEMS

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Abstract. The paper considers the representability of algebraic structures (groups, lattices, semigroups, etc.) over equivalence relations on natural numbers. The concept of a (uniform) m-equivalence is studied. It is proved that the numbering equivalence of any numbered group is a uniform m-equivalence. On the other hand, we construct an example of a uniform m-equivalence over which no group is representable. Additionally we show that there exists a positive equivalence over which no upper (lower) semilattice is representable.

Keywords: uniform m-equivalence, group, lattice, field.

Introduction

The notion of a group, being one of the most fundamental concepts of mathematics and many other natural sciences is unsurprisingly very attractive to computability theory specialists who wish to study the possibility of presenting groups with certain algorithmic properties. One of the first classes which is fruitful to study in this regard is the class of computable groups which was initally considered by A. Maltsev (see [1, 2]). Among a multitude of papers on the topic we highlight the works concerning computable automorphism groups of computable systems [3, 4, 5, 6, 7, 8]. Wider classes of algorithmic representations of groups including negative and positive representations were studied in [9, 10]. See [11] as one of the works on arithmetic hierarchies of groups. One may also consider looking into the rather extensive bibliography in [12].

It should be noted that definitions of many other mathematical structures such as rings, fields, vector spaces, modules, algebras over a given field, etc, all contain

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the notion of a group, therefore when we consider the algorithmic representations of these structures we also have to study algorithmic representations of groups.

However it appears that the more general representations (numberings) of groups were not yet considered, since in its most general statement, the question seems quite hard to tackle. In this paper we introduce a notion of a uniform m-equivalence and show that a kernel of an algorithmic representation of any group is a uniform m-equivalence.

We additionally show that not every uniform m-equivalence is a kernel of an algorithmic representation of some group.

Also we study the questions of existence of algorithmic representations with their kernels being uniform m-equvalences for various algebraic systems including lattices (which are considered both as partial orders and as algebras, this distinction proved to be important) as well as semigroups and translationally precomplete algebras.

One section is devoted to structures of degrees of representability of rings and fields.

See [12, 13, 14] for notions that are not explicitly defined in this text.

For the rest of the paper we will, for the sake of brevity as well as for paying respect to the traditions of computability theory, we will call algorithmic representations numberings.

By "equivalence" we mean the equivalence relation on the set of natural numbers ω.

For the numbering ν by $ker(\nu)$ denote its numbering equivalence which we will call a kernel of ν.

A numbering ν of a universal algebra $\mathfrak{A} = \langle A; F_0, F_1, \dots, \rangle$ is a map from ω onto A such that any m-ary operation F_n on A is realised by a computable function f_n such that $\forall \overline{x} \in \omega^m[F_n(\nu(\overline{x}) = \nu(f_n(\overline{x}))]$, i.e. $\mathfrak{A}$ is isomorphic to the quotient $\langle \omega; f_0, f_1, \dots, \rangle / ker(\nu)$ of a computable algebra $\langle \omega; f_0, f_1, \dots, \rangle$ by the congruence $ker(\nu)$.

Definition 1. *A universal algebra is said to be representable over equivalence η if there exists a numbering of this algebra with η as its kernel.*

In this paper we mainly consider the existence questions concerning representations of classical algebraic systems over various types of equivalences.

Definition 2. *An equivalence η is called an m-equivalence (uniform m-equivalence) if there exists a family F of computable functions (an enumerable set F of computable functions) that induce the permutations of a quotient set ω/η such that for every pair of natural numbers x, y there exists a function $f \in F$ that m-reduces the class $\{x\}/\eta$ to the class $\{y\}/\eta$.*

Solvable equivalence relations are trivial examples of uniform m-equivalences.

1. Groups

Theorem 1. *If (G, ν) is numbered group group, then the kernel $ker(\nu)$ of its numbering is a uniform m-equivalence*

Proof. Let $a, b \in G, a = \nu(k), b = \nu(l)$. By η denote the kernel $ker(\nu)$. It is clear that the map $\pi_{a,b} = \lambda x.[ba^{-1}x]$ is a permutation on the main set of the group G, that moves a to b. Consider a computable function $f = \lambda x.[l * (k)^{-1} * x]$, where $*$ is a computable operation that represents the group operation in G in the numbering ν and $()^{-1}$ represents a computable operation, that realizes the operation of taking the inverse in G. It is clear that f realises a permutation $\pi_{a,b}$ on numbers of ν (i.e. $\pi_{a,b}\nu = \nu f$) and it also m-reduces the class $\{k\}/\eta$ to the class $\{l\}/\eta$. For showing uniformness it suffices to match every pair $\langle k, l\rangle \in \omega^2$ to a function. $f_{k,l} = \lambda x.[l * (k)^{-1} * x]$. □

Remark 1. The uniformity is provided by the fact that we are working in the signature $\langle \cdot, ^{-1} \rangle$. In the signature that consists only of the group operation $\langle \cdot \rangle$ the kernel of a numbering is still an m-equivalence, however not a uniform one.

We highlight that computable permutations, defined by the family F do not necessarliy form a group, since a inverse permutation may not be realised by a computable function. They do form a group if the numbering is positive. However it is possible to present computable permutations with non-computable inverses for some negative numberings, therefore the bound is tight in the Kleene-Mostovsky hierarchy (see [15]).

In [16] a question is posed: Does there exist a finitely-presented algebra with a precomplete equivalence of the standard numberings? Recall that the notion of the algebra contains the notion of the group.

Corollary 1. *No group is representable over any precomplete equivalence relation.*

Proof. A permutation $\pi_{a,b}$ over G from Theorem 1 has a fixed point iff $a = b$ (in that case it will be a trivial permutation), i.e. if $a \neq b$, a computable function corresponding to this permutation does not have non-kernel fixed points and thus a kernel may not be a precomplete equivalence. □

Therefore there is no finitely-presented group with a precomplete positive kernel (since the positivity of the standard numbering of an algebra is a necessary condition for the positivity of an algebra itself).

Theorem 1 raises two natural questions:

– How big is the class of uniform m-equivalence relations

– Is the uniformity of an m-equivalence relation sufficent for a group to be presented over this relation?

We will call a family F of computable functions m-reducing for the equivalence relation η if it satisfies the conditions of Definition 2.

Let f be a computable permutation on ω. Let $F_f = \{f^n | n \in Z\}$, where Z is the set of all integers and $f^0(x) = x$ $(x \in \omega)$, $f^{-k} = (f^k)^{-1}$ $(k \in \omega)$. We will now show that the class of uniform m-equivalences is quite big.

Proposition 1. *For every computable permutation f that has no finite cycles and splits ω into an infinite number of orbits there exists a set of uniform m-equivalence relations for which a computable family F_f is m-reducing. This set has continuum cardinality.*

Proof. Assume f splits ω into orbits $O_f(x) = \{f^k(x) | k \in Z\}$. Define $a_0 = 0, a_{n+1} = \min\{y | y \notin O_f(a_0) \cup \dots \cup O_f(a_n)\}$ (i.e. a_k is the smallest number from its orbit) and we will order the f-orbits with the relation $\leqslant_f$ in such a way that all elements of the orbit a_i are preceding elements of the orbit a_j if $a_i < a_j$. Inside each orbit we assume that $x \leqslant_f y \Leftrightarrow \exists k \geq 0(f^k(x) = y)$. Let $A = \{a_0 < a_1 < \dots\}$. We construct an equivalence relation η_0 by defining its classes as follows: $M_0 = A, M_{n+1} = f^{n+1}M_0, M_{-n-1} = f^{-n-1}M_0, n \in \omega$. It is clear that the familty F_f of computable functions is m-reducing for η_0. We now construct a family of equivalence relations by using a binary tree with η_0 at its root. Informally, the equivalence relations $\eta_{01}, \eta_{10}, \eta_{11}$ are constructed from $\eta_0 = \eta_{00}$ by «moving» all f-orbits of a_0 to the right if inex of η contains 1 i.e. $a_0 = a_1 \wedge a_1 = f^{-1}(a_2) \pmod{\eta_{01}}$ and by «gluing» of all images and preimages of f. Respectively $a_0 = f^{-1}(a_1)a_1 \wedge a_1 = a_2 \pmod{\eta_{10}}$ and $a_0 = f^{-1}(a_1) \wedge a_1 = f^{-1}(a_2) \pmod{\eta_{11}}$. Therefore if an orbit $O_f(a_k)$ moves one position right with respect to a_{k-1}, then all other orbits $O_f(a_l), l > k$ also move one position right. We will now describe this construction formally.

Let $E = \{\overline{\varepsilon} = \varepsilon_0\varepsilon_1 \ldots | \varepsilon_i \in \{0, 1\}\}$ be a set of all countably-infinite sequences of zeros and ones and $\overline{\varepsilon} \in E$. Construct a mapping from $\varphi : E \longrightarrow \Theta$, where Θ is a set of all equivalences on ω such that for $i \in \omega$

$$\begin{cases} \varepsilon_i = 0 \Rightarrow a_i = a_{i+1} \pmod{\varphi(\overline{\varepsilon})}, \\ \varepsilon_i = 1 \Rightarrow a_i = f^{-1}(a_{i+1}) \pmod{\varphi(\overline{\varepsilon})}. \end{cases}$$

Additionally, if $x = y \pmod{\varphi(\overline{\varepsilon})}$, then the following f-images and preimages of x, y will also be equal modulo $\varphi(\overline{\varepsilon})$ and $\varphi(\overline{\varepsilon})$ is the smallest equivalence satisfying that property.

In the language of ε-sequences we have that $\eta_0 = \varphi(00\ldots), \eta_1 = \varphi(10\ldots)$, etc. For the equivalence relation $\varphi(11\ldots)$ (the sequence consists of 1s only) holds $\forall i \in \omega(a_i = f^{-1}(a_{i+1}) \pmod{\varphi(11\ldots)}))$.

We now show that φ is injective. If $\overline{\varepsilon}^1 \neq \overline{\varepsilon}^2$ then there exists the smallest k for which $\varepsilon^1_k \neq \varepsilon^2_k$. For the sake of clarity, let $\varepsilon^1_k = 0, \varepsilon^2_k = 1$. Then $\langle a_k, a_{k+1}\rangle \in \overline{\varepsilon}^1 \setminus \overline{\varepsilon}^2$. The case $\varepsilon^1_k = 1, \varepsilon^2_k = 0$ can be dealt with in a similar way.

By construction the family F_f is m-reducing for any equivalence from $\{\varphi(\overline{\varepsilon}) | \overline{\varepsilon} \in E\}$. It follows from injectiveness of φ that this set has cardinality of the continuum. Thus the statement is proved.

□

Proposition 2. *If all equivalences classes of η are decidable, then η is an m-equivalence.*

Proof. If classes $\alpha = \{p\}/\eta$ and $\beta = \{q\}/\eta$ are the same then m-reduction is performed by an identity function. If they are different, then the computable function mapping α to the point q and β to the point p that acts as an identity on the set $\omega \setminus (\alpha \cup \beta)$ is the required function. □

Proposition 3. *If a uniform m-equivalence relation has at least one decidable class, then the relation itself is decidable.*

Proof. Let η be a uniform m-equivalence and assume that an η-class β is decidable. Then there exists an enumerable family F of m-reducing functions. It is clear that $x = y \pmod{\eta} \Leftrightarrow \exists f \in F[f(x) \in \beta \wedge f(y) \in \beta]$, i.e. η is positive. Additionally $x \neq y \pmod{\eta} \Leftrightarrow \exists f \in F[f(x) \in \beta \wedge f(y) \notin \beta]$, which allows us to state that η is negative. □

Corollary 2. *If a non-solvable equivalence relation has at least one decidable class then no group is representable over it.*

Let η be an equivalence. The set $\alpha \subseteq \omega$ is said to be η-closed if $x \in \alpha \wedge x = y \pmod{\eta} \Rightarrow y \in \alpha$.

Definition 3. *An equivalence η is called computably separable (separable) if for every pair of natural numbers x, y that are different modulo η there exists such an η-closed computable (computably enumerable) set α such that $x \in \alpha$ and $y \notin \alpha$ $((x \in \alpha \wedge y \notin \alpha) \vee (x \notin \alpha \wedge y \in \alpha))$.*

Consider the following examples of equivalences (see [13], pages 58, 174-175):
(1) $\eta^\alpha = \{\langle 2x, 2x+1\rangle, \langle 2x+1, 2x\rangle | x \in \alpha\} \cup id\,\omega, \alpha \subseteq \omega$;
(2) $\eta_\alpha = \{\langle x, y\rangle | x, y \in \alpha\} \cup id\,\omega, \alpha \subseteq \omega$;
(3) $\eta^*_\alpha = \{\langle x, y\rangle | \gamma_x \triangle \gamma_y \subseteq \alpha\}, \alpha \subseteq \omega$ (γ is a canonical numbering of finite sets).

Proposition 4. *For every $\alpha \subseteq \omega$ it is true that:*
1. For any η^α is an m-equivalence. Moreover, this equivalence is uniform iff α is decidable.
2. η_α is an m-equivalence iff α is decidable.

Proof. 1. By proposition 2 for any $\alpha \subseteq \omega$ the equivalence η^α is an m-equivalence and by proposition 3 its uniformity implies decidability.

2. It is clear that for η_α we require the decidability of the class α to be able to map it (by a computable map) to any other class since all η_α classes (except for maybe α itself) consist of only one element. □

Corollary 3. *There exists a group that is presentable over η^α (as well as over η_α) iff α is decidable.*

Proposition 5. *Any pair of equivalence classes of the kernel of an undecidable positive numbering of any group is computably isomorphic (i.e. for any pair of classes of the kernel there exists a computable permutation of ω that maps one of the classes onto the other one and also induces a permutation of ω/η). Additionally there exists a uniform procedure that given x and y outputs a characteristic index of a computable isomorphism on ω which is a permutation on ω/η and it takes $\{x\}/\eta$ to $\{y\}/\eta$.*

Proof. Let η be a kernel of an undecidable positive numbering of a group. By theorem 1, η is a uniform m-equivalence and by proposition 3 all classes of η are non-computable. For given x, y we want to find a permutation $f_0 \in F$ that performs an m-reduction $\{x\}/\eta$ to $\{y\}/\eta$ and construct an injective function g_0 that is equal to f_0 modulo η. We will do so as follows:

Step 0. $g_0(0) = f_0(0)$;

Step $s+1$. If $f_0(s+1) \notin \{g_0(0), g_0(1), \ldots, g_0(s)\}$, then define $g_0(s+1) = f_0(s+1)$. Otherwise, when enumerating the class $\{f_0(s+1)\}/\eta$ look for the first z that is not equal to any $g_0(i)$ $(0 \leqslant i \leqslant s)$ and is η-equivalent to $f_0(s+1)$ (such a z exists since the class $\{f_0(s+1)\}/\eta$ is infinite) and set $g_0(s+1) = z$, thus finishing the step $s+1$.

Hence the class $\{x\}/\eta$ is 1-reducible to $\{y\}/\eta$ via the function g_0. Similarly find $f_1 \in F$ that m-reduces $\{y\}/\eta$ to $\{x\}/\eta$ and construct the corresponding injective computable function g_1 that is equal to f_1 modulo η. Then g_1 1-reduces $\{y\}/\eta$ to $\{x\}/\eta$ and so we obtain that classes $\{x\}/\eta$ and $\{y\}/\eta$ lie in the same 1-degree.

Now we will construct finite sets of correspondences of the type

$$\{\langle u_0, v_0\rangle, \ldots, \langle u_s, v_s\rangle\}$$

such that $u_i = u_j \pmod{\eta} \Leftrightarrow v_i = v_j \pmod{\eta}$ and $\forall n \in \omega \exists k, l(n = u_k \wedge n = v_l)$. We will base our approach on the back-and-forth method used in the proof of Myhill theorem on computable isomorphism of 1-equivalent sets. Positive infiniteness of all η-classes greatly simplifies this method in our case. On even steps we will use the function g_0 and we will use g_1 on odd steps. We shall omit further details.

It is clear that all constructions presented in this proof uniformly depend on x, y. □

At the end of this section we note that if there exists a group that is representable over the equivalence η then there are many ways to present a group over it. In particular, any equivalence class can be interpreted as a unit element of such a group.

Proposition 6. *If there exists a group representable over η then for any class of η there exists such an η-representation of some group, for which this class is a unit element.*

Proof. Let $\mathfrak{G} = \langle G; \cdot, ^{-1} \rangle$ be a group with the unit element e. Define the translation $\varphi_d = \lambda x.[x \cdot d]$, where d is some fixed element that is not equal to e. It is obvious that φ_d is bijective on G. Define a structure that consists of one binary operation $*$ and one unary operation $[]^{-1}$ on the set G as follows: $a * b = ad^{-1}b$; $[a]^{-1*} = da^{-1}d$. Then $\mathfrak{G}^* = \langle G; *, []^{-1*} \rangle$ is a group with the unit element d that is isomorphic to $\mathfrak{G}$ with φ_d acting as an isomorpism. Indeed, by definition $a * b = ad^{-1}b$ but $\varphi_d^{-1}(a) = ad^{-1}, \varphi_d^{-1}(b) = bd^{-1}$ and $\varphi_d((ad^{-1})(bd^{-1})) = ad^{-1}bd^{-1}d = ad^{-1}b = a * b$. Similarly since $\varphi_d^{-1}(a) = ad^{-1}$ we have that $\varphi_d((ad^{-1})^{-1}) = \varphi_d(da^{-1}) = da^{-1}d = [a]^{-1*}$.

Consider a group $\langle G; \cdot \rangle$ that is representable over η and has a unit element e. Fix any element $d \in G$ and some of the ν-numbers of this element and of its inverse, for example $\nu(n_0) = d, \nu(n_1) = d^{-1}$. Now define a computable function $f_d = \lambda x.[x \cdot n_0]$, and for the group $\langle G; \cdot \rangle$ consider such a numbering ν_d of G that $\nu_d = \nu f_d$. Also define computable operations $* : \nu_d(x) * \nu_d(y) = \nu_d(x \cdot n_1 \cdot y)$ and $[\nu_d(x)]^{-1*} = \nu(n_0 \cdot x^{-1} \cdot n_0)$. It is clear that d is the unit element of the group $\langle G; * \rangle$, that is given by a numbering ν_d with the kernel η. □

2. Computably stitched sets

In this section we will consider uniform m-equivalences with pairwise computably isomorphic classes (i.e. there exists a computable permutation on the set ω that maps one of the classes to another).

Definition 4. *A set is called computably stitched if there exists a computable permutation without cycles of finite length and with infinite number of orbits with each of the orbits containing exactly one element of the set.*

Note that by definition a computably stitched set is both infninite and coinfinte

We say that a computable permutation f stitches a set α (or equivalently α is stitched by f) if α contains exactly one element in every orbit of the cycleless permutation f that has infinite number of orbits. For a set α that is stitched by a computable permutation f by $\eta_{\alpha,f}$ we denote an equivalence relation

$$x = y \pmod{\eta_{\alpha,f}} \Leftrightarrow \exists n \in Z(f^n(x) \in \alpha \wedge f^n(y) \in \alpha),$$

that is obviously a uniform m-equivalence with pairwise computably isomorphic classes.

Proposition 7. *Every computable cycleless permutation with infinite number of orbits stitches a continnuum of sets.*

Proof. The proof is similar to the proof of proposition 1 □

Proposition 8. *Every computably stitched set is an equivalence class of some uniform m-equivalence that has pairwise computably isomorphic classes.*

Proof. Assume that α is stitched by a computable permutation f. Then a family of sets $\{f^n\alpha | n \in Z\}$ consists of classes of a uniquely determined uniform m-equivalence $\eta_{\alpha,f}$. □

By using a non-effective diagonalisation we can easily show the existence of sets that are not computably stitched.

Proposition 9. *If a computably enumerable set α is stitched by a computable function f then the equivalence relation $\eta_{\alpha,f}$ is decidable.*

Proof. Indeed, $x = y \pmod{\eta_{\alpha,f}} \Leftrightarrow \exists n \in Z(f^n(x) \in \alpha \wedge f^n(y) \in \alpha)$. □

Corollary 4. *A computably enumerable set is computably stitched iff it is infinite, coninfinite and decidable.*

Proof. It is clear that the an infinite, coninfinite and solvable set is computably stitched. The converse follows from proposition 8 □

We will call an equivalence relation η computably stitched if it has a class α such that there exists a computable permutation f such that $\eta_{\alpha,f} = \eta$.

Corollary 5. *If (G, ν) is an unsolvable positive group then the kernel $ker(\nu)$ is not computably stitched.*

Proof. By corollary 4 if the kernel of the numbering is computably stitched then it is decidable. □

By the characteristic transversal of the equivalence relation η we mean the set of all natural numbers that are minimal in the η-classes that contain them (formally $tr(\eta) = \{x | \forall y(x = y \pmod{\eta} \Rightarrow x \leqslant y)\}$).

Proposition 10. *Every infinite, coinfinite, and coenumerable set is computably stitched.*

Proof. Let α be an infinite coenumerable set with an infinite complement. Now consider the following positive equivalence η with infinite classes, with α as its charasteric transversal (see [13], page 296 for information about exsistence of such equivalence relations and general methods of constructing them). By using the enumeration algorithm for η it is possible to construct finite approximations of a computable permutation f of ω that is trivial modulo η and has each class of η as its orbit. We will present a moderately detailed sketch of the algorithm of computing the graph G_f of f.

By η^s denote a part of the equivalence relation η that is constructed after s steps of a certain fixed effective enumeration of η.

Step 0. $G_f^0 = \emptyset$, $\eta^0 = \emptyset$.

Step $s+1$. By $C_0, \ldots, C_m$ denote all nontrivial equivalence classes of η^s, each of which is a chain $a_0, f(a_0), \ldots, f^n(a_0)$ of η^s-equivalent numbers (a_0 has no f-preimage and $f^n(a_0)$ has no f-image at the step s). We will call a_0 the initial element of the chain and we will call $f^n(a_0)$ the end element of the chain. We order the chains with respect to their initial elements. Expand the classes $C_0, \ldots, C_m$

modulo equivalence η^{s+1}. If the chains were different at the step s and intersected at the step $s+1$ (i.e. some elements became η^{s+1}-equivalent), then for these chains expand the graph G_f^s in such a way that the end elements of smaller chains are mapped by f onto the initial elements of the greater chains if the number of the step is even and the end elements of greater chains are mapped onto the initial elements of the smaller chains if the number of the step is odd, thus constructing G_f^{s+1}. This concludes step $s+1$.

We set

$$G_f = \bigcup_{s\in\omega} G_f^s.$$

It is now clear that f is a computable permutation that stitches the set α. $\square$

Now we will show that "being a uniform m-equivalence relation" is not a sufficient condition for existence of a group representable over said equivalence relation.

Before stating the next theorem we introduce some notions. In correspondence with the definition 3 for a given equivalence relation η construct a topological space $[\eta]_{comp}$, with the family of all η-closed computably enumerable sets serving as its base of open sets. We will call this topology separable (T_1-separable, Hausdorff, regular, etc) if the space $[\eta]_{comp}$ satisfies the similarly named property. It can be seen that if a universal algebra $\mathfrak{A}$ is representavle over η then all its operations, supported by suitable computable functions with respect to the natural projective numbering $\nu(x) = \{x\}/\eta$ are continuous in that topology. This fact is obvious for unary operations, since the preimage of a computably enumerable set is always computably enumerable. Another obvious fact is that a morphism between two numbered systems is a continuous map. Continuity in the most general case is shown in [17].

If a computably stitched equivalence relation is positive then by proposition 9 it is decidable and there exists a group that is representable over it. We now show that there exists a computably stitched equivalence of low algorithmic complexity, over which no group is represented.

Theorem 2. *There exists a computable permutation that is m-reduicng for such a uniform m-equivalence over which no group is representable.*

Proof. In [18] one can find an example of set α computably stitched by a suitable computable permutation f of the set ω such that the topological space $[\eta_{\alpha,f}]_{comp}$ is T_1-separable but not Hausdorff. Moreover, there are no two non-empty disjoint computable enumerable $\eta_{\alpha,f}$-closed sets. It is well-known that if a topological group is T_1-separable, then it is T_2-separable, i.e. Hausdorff. Therefore, if there existed a group that would be representable over $\eta_{\alpha,f}$ then the space $[\eta_{\alpha,f}]_{comp}$ would be Hausdorff (note that the continuity of group operations with respect to the recently intoduced topology is of key importance here). Thus there is no group that is representable over an unsolvable uniform m-equivalence $\eta_{\alpha,f}$. $\square$

Note that the algorithmic complexity of the equivalence relation $\eta_{\alpha,f}$ from the theorem 2 is in the class Π_2^0, more precisely, it is effectively T_1-separable, i.e. there exists such an effective family S of computably enumerable $\eta_{\alpha,f}$-closed sets, such that for all $x \neq y \pmod{\eta_{\alpha,f}}$ there exist $\sigma_0, \sigma_1 \in S$, such that $\{x\}/\eta_{\alpha,f} \subseteq \sigma_0 \wedge \{y\}/\eta_{\alpha,f} \cap \sigma_0 = \emptyset$.

3. Lattices

Recall (see Section 1) that $\eta^*_\alpha = \{\langle x,y\rangle | \gamma_x \triangle \gamma_y \subseteq \alpha\}, \alpha \subseteq \omega$ (where γ is a canonical numbering of finite sets).

Theorem 3. *For every $\alpha \subseteq \omega$ the equivalence η^*_α is a computably separable m-equivalence with pairwise isomorphic classes. Moreover, if α is coenumerable, then the equivalence η^*_α is negative and uniform.*

Proof. By $\Gamma_{x,y}$ denote the set of γ-numbers of all finite expansions of the set γ_x outside of $\gamma_y \setminus \gamma_x$, i.e. $\Gamma_{x,y} = \{z | \gamma_x \subseteq \gamma_z \wedge (\gamma_y \setminus \gamma_x) \cap \gamma_z = \emptyset\}$. For instance $\Gamma_{x,0}$, where $\gamma_x \neq \emptyset, \gamma_0 = \emptyset$ is a union of canonical indices of all finite expansions of the set γ_x, while $\Gamma_{0,x}$ is the set of numbers of all finite sets that do not intersect γ_x.

We show that η^*_α is computably separable for every α. Indeed, let $x \neq y \pmod{\eta^*_\alpha}$. Select x_0, y_0 such that $\gamma_x \setminus \alpha = \gamma_{x_0} \setminus \alpha, \gamma_y \setminus \alpha = \gamma_{y_0} \setminus \alpha$ and $\gamma_{x_0}, \gamma_{y_0} \subseteq \omega \setminus \alpha$, i.e. x_0, y_0 are γ-numbers of sets, completely lying in $\omega \setminus \alpha$. It is easy to check that Γ_{x_0,y_0} is an η^*_α-closed computable set that contains x and does not contain y.

We move on to constructing the required permutation $f_{x,y}$. For given $x \neq y$ $(\mathrm{mod}\ \eta^*_\alpha)$ such that $\gamma_x, \gamma_y \subseteq \omega \setminus \alpha$ we construct sets (their characteristic indices) $\Gamma_{x,y}, \Gamma_{y,x}$ and create a computable bijection in the following way. For every $z \in \Gamma_{x,y}$ find such u that $\gamma_u = \gamma_z \setminus \gamma_x$ and define $\gamma_v = \gamma_u \cup \gamma_y$. Then $v \in \Gamma_{y,x}$. We set $f_{x,y}(z) = v$ and $f_{x,y}(v) = z$. On all numbers from $\omega \setminus (\Gamma_{x,y} \cup \Gamma_{y,x})$ we define $f_{x,y}$ to act trivially.

We will show that $f_{x,y}$ is a function that is consistent with η^*_α, and is acting as an infninte set of cycles of length 2 on $\Gamma_{x,y} \cup \Gamma_{y,x}$. Let $z_1 \in \Gamma_{x,y}$ and $f_{x,y}(z_1) = p_1$. Then we can uniquely define such a u_1 that $\gamma_{u_1} = \gamma_{z_1} \setminus \gamma_x$ and $\gamma_{p_1} = \gamma_y \cup \gamma_{u_1}$ with γ_{u_1} and γ_y disjoint. For $z_2 \in \Gamma_{x,y}$ there exist uniquely determined u_2, p_2. Now if $z_1 = z_2 \pmod{\eta^*_\alpha}$ then $\gamma_{z_1} \setminus \alpha = \gamma_{z_2} \setminus \alpha$, i.e. $\gamma_{u_1} \setminus \alpha = \gamma_{u_2} \setminus \alpha$. Then $p_1 = p_2$ $(\mathrm{mod}\ \eta^*_\alpha)$. Injectiveness of $f_{x,y}$ is obvious. Bijectiveness follows from the fact that for every $p \in \Gamma_{y,x}$ there exists such a unique i (more specifically, $\gamma_p = \gamma_y \cup \gamma_u$) such that $f_{x,y}(z) = p$, where $\gamma_z = \gamma_x \cup \gamma_u$. In a similar way one can check that $f_{x,y}$ is correctly defined on the arguments from $\Gamma_{y,x}$. Therefore $f_{x,y}$ is a computable permutation on the set ω that induces a permutation on ω/η^*_α that takes the class $\{x\}/\eta^*_\alpha$ to the class $\{y\}/\eta^*_\alpha$.

Negativeness of η^*_α is obvious if α is coenumerable, since the set of all minimal elements (in the equivalence classes that contain them) is computably enumerable, then we can apply the procedure described above for every pair of numbers $\langle x,y\rangle$, such that $\gamma_x, \gamma_y \subseteq \omega \setminus \alpha$ to obtain a computable family $\{f_{x,y} | \gamma_x, \gamma_y \subseteq \omega \setminus \alpha\}$ of computable permutations with the required property, i.e. η^*_α is a uniform m-equivalence with pairwise isomorphic classes. □

Corollary 6. *For every α there exists a lattice that is representable over η^*_α and is isomorphic to the sublattice of the lattice of all finite subsets of some set.*

Proof. The nontrivial case is the one, where α is coinfnite. For any $x, y \in \omega$ define $x \sqcup y = \gamma^{-1}(\gamma_x \cup \gamma_y), x \sqcap y = \gamma^{-1}(\gamma_x \cap \gamma_y)$. Consistence of these operations with η^*_α, existence of exact upper and lower bounds and the existence of the minimal element are obvious. □

Every torsion-free group that has a computable copy admits an unsolvable negative numbering [19]. Because of this, a quesrion arises: Is a group representable over the equivalence relation η^*_α for some coenumerable undecidable α?

Recall ([13], page 288) that a positive equivalence relation η is called perfect is there exist no nontrivial η-closed computable sets.

In the next result a lattice is considered as an algebra (in the signature $\langle \sqcup, \sqcap \rangle$).

Proposition 11. *There exists such a positive equivalence, over which no upper (lower) semi-lattice is representable.*

Proof. Let η be a perfect equivalence with the compressed characteristic transversal (examples of such equivalences can be found in [13]). In [16, 20] it is shown that any n-ary computable function, consistent with η acts on the set ω/η like a constant or like a projection. Assume that the binary computable operation f acts as an operation of taking exact upper (lower) bound on ω/η. Then the positive partial order relation $x \leqslant_\eta y \Leftrightarrow f(x,y) = y \pmod{\eta}$ that is induced by f cannot be linear [21] and therefore there exist two incomparable η-classes with respect to $\leqslant_\eta$, say $\{k\}/\eta$ and $\{l\}/\eta$. Consider a translation $t = \lambda x.[f(x,k)]$. Then, by the idempotency we have $t(k) = k \pmod{\eta}$, but $t(l) \neq k \pmod{\eta} \wedge t(l) \neq l \pmod{\eta}$ and the unary computable function t is neither a constant nor a projection (trivial), thus leading us to a contraditction. $\square$

This proposition prompts us to pose an important question: For a given negative equivalence relation is there a lattice that is representable over this relation as an algebra?

Definition 5. *The graph $\langle \Gamma; P \rangle$ is said to be positively (negatively) presented over the equivalence η if there exists such a numbering ν of this graph such that $ker(\nu) = \eta$ and the set $\{\langle x,y \rangle | \langle \nu x, \nu y \rangle \in P\}$ is computably enumerable (coenumerable).*

Note that this definition does not place any restrictions on the complexity of η Now we will view lattices as a posets (partially ordered sets).

Proposition 12. *(a) For any negative equivalence there exists a linear order that is negatively presented over said equivalence.*

(b) For any positive equivalence there exists a lattice that is positively presented over said equivalence, however there exists a positive equivalence, over which no linear order is representable.

Proof. (a) In [21] it is shown that for any negative equivalence with an infinite number of classes there exists a dense linear order that is presented over it. This obviously implies the existence of the lattice (as a poset) representable over this equivalence.

(b) By η denote a positive equivalence with at least two equivalence classes. We will fix two of them and denote by them α, β. Then the order $\alpha \times \omega \cup \omega \times \beta \cup \eta$ is a positive lattice with a zero (α), a unit (β) and a set of other elements that are pairwise incomparable modulo η. The existence of a positive equivalence over which no linear order is positively representable is proved in [21]. $\square$

4. Translationally precomplete algebras, rings and semigroups

In this section we will consider the properties of equivalence relations that are kernels of numberings of rings, fields and universal algebras with some conditions like finiteness conditions for lattices and their congruences (simple and subdirectly irreducible) and the possibility of representations of wider classes of systems, for example, semigroups.

Since the notion of a ring contains inside of it a notion of an additive abelian group, then the kernel of the numbering of any ring is a uniform m-equivalence. This is even more true for the case of division rings and fields. Thus all general results that were stated for groups in previous sections are easily transferred to the case of rings.

It is known [22] that for any negative equivalence there exist both finitely-generated and congruence-simple universal algebras. Representability of a congruence-free algebra over a positive equivalence is equivalent to its solvability. However, if the equivalence (not necessarily positive) has, for example, a hyperimmune characteristic transversal then every algebra of finite signature that is representable over it is locally finite [16]. If a positive equivalence is not computably separable then the lattice of congruences of any universal algebra that is representable over it has the cardinality of the continuum [16]. Then from the point of view of representability of rich classes over equivalence relations we can prioritise negative equivalence relations over the positive ones. The following important classes of universal algebras support that statement.

Any unary termal operation with fixed parameters from the algebra is called a translation.

Definition 6. *A universal algebra is said to be translationally complete if any pair of its distinct elements can be mapped to any other pair of distinct elements by a suitable translation.*

It is clear that any translationally complete algebra is congruence-simple. The converse, however, is false. Any division ring is a classical example of a translationally complete algebra [23].

Definition 7. *A universal algebra is called translationally precomplete if there exists a pair of distinct elements of this algebra such that any other pair of distinct elements can be mapped into this pair by a sutiable translation.*

From definition it follows that any translationally precomplete algebra is subdirectly irreducible. The converse, however, is yet again false.

Definition 7 is much wider than definition 6. The simplest example of a non-trivial translationally precomplete and not simple algebra is the so-called precedence algebra $P = \langle \omega; p \rangle, p(0) = 0, p(n+1) = n$.

For the set $\alpha \subseteq \omega$ and the equivalence relation η by $[\alpha]_\eta$ denote an η-closure of the set α (i.e. a smalles η-closed extension of α). We say that a number x is η-rejected by the set α is $x \notin [\alpha]_\eta$. It can be seen that for any negative equivalence relation η the the relation «x is η-rejected by a finite set δ» is computably enumerable and uniformly dependent on x and δ (we can assume that δ is given by its canonical index or presented explicitly). If the choice of the equivalence relation η is clear from context we will say that x is rejected by δ.

Theorem 4. *For any negative equivalence relation there exists a translationally complete universal algebra that is representable over that equivalence relation.*

Proof. Let η denote a negative equivalence. If the characteristic transversal $tr(\eta)$ is finite then η is decidable and there is nothing to prove. Note that $tr(\eta)$ since $x \in tr(\eta) \Leftrightarrow \forall z < x(z \neq x \pmod{\eta})$. We will now show that for any pair $x \neq y \pmod{\eta}$ there exists a solvable η-closed set, that separates x and y and, moreover, that this set can be uniformly constructed from given x and y, Step 0. $A_x^0 = \{x\}, A_y^0 = \{y\}$.

Step $s+1$. Let z be a smallest natural number that is not in $A_x^s \cup A_y^s$. Check whether z is rejected by at least one of the sets A_x^s, A_y^s. If z is rejected by A_x^s then we set $A_x^{s+1} = A_x^s, A_y^{s+1} = A_y^s \cup \{z\}$; if z is rejected by A_y^s then $A_x^{s+1} = A_x^s \cup \{z\}, A_y^{s+1} = A_y^s$. If z is rejected by both sets then we put it into A_x^{s+1}. This concludes the step $s+1$.

Define

$$A_x = \bigcup_{s \in \omega} A_x^s, A_y = \bigcup_{s \in \omega} A_y^s.$$

By induction over the construction steps it can be easily shown that every step terminates with inclusion of the natural number in question to one of two sets and it also can be shown that $[A_x^s]_\eta \cap [A_y^s]_\eta = \emptyset$ for every step. Therefore computably enumerable sets A_x, A_y do not intersect, are η-closed and their union is all of ω. The uniform dependence of indices of characteristic functions of A_x, A_y on x and y is obvious. Now for every pair $\langle x, y \rangle$ of numbers different modulo η construct a computable set of translations $T_{x,y} = \{f_{x,y,m,n} | m, n \in \omega\}$ as follows:

$$z \in A_x \Rightarrow f_{x,y,m,n}(z) = t_m; z \in A_y \Rightarrow f_{x,y,m,n}(z) = t_n,$$

where $t_m, t_n \in tr(\eta), m \neq n$. Now define a computable family of translations

$$T = \bigcup_{x \neq y \pmod{\eta}} T_{x,y}.$$

It is clear that every translation $f \in T$ is consistent with η, i.e. $x = y \pmod{\eta} \Rightarrow f(x) = f(y) \pmod{\eta}$ and therefore we can correctly define a quotient algebra $\langle \omega/\eta; T \rangle$ of a computable algebra $\langle \omega; T \rangle$ by the congruence relation η. Since for any pair of numbers x, y that are different modulo η and any different $t_m, t_n \in tr(\eta)$ there exists a translation from T that takes x to t_m and y to t_n we conclude that the algebra $\langle \omega/\eta; T \rangle$ is translationally complete. □

Corollary 7. *For any negative equivalence there exists a translationally precomplete universal algebra that is representable over that equivalence.*

Remark 2. Negative numberings are in some sense just as natural as the positive ones. Foe example an additive group of integers $\langle Z; + \rangle$ has undecidable negative numberings, however it is stable with respect to positive representations; any finite field that has a computable copy (as mentioned before) also possesses an undecidable negative numbering (see [19]), while any positive representation of any field is decidable.

Proposition 13. *Any Hausdorff numbering of a translationally precomplete universal algebra is negative.*

Proof. See [18]. □

Corollary 8. *No translationally precomplete universal algebra is representable over any non-negarive equivalence such that all of its classes are computably enumerable.*

Proof. Otherwise by proposition 13 from obvious T_2 separableness (even discreteness) of the corresponding space we obtain the negativity of the kernel of the representation. □

In particular, no translationally precomplete universal algebra is representable over any undecidable positive equivalence.

Now we will consider wider families of classical systems.

We say that a function $f : \omega^n \longrightarrow \omega$ is consistent with the equivalence relation η if η is a congruence relation of the algebra $\langle \omega; f\rangle$. As usual, the function C^n_m : $\omega^n \longrightarrow \omega$ with the image $\{m\}(m \in \omega)$ is called a constant function and the function $I^n_m : \omega^n \longrightarrow \omega$, where $1 \leqslant m \leqslant n$ and $\forall\langle x_1, \ldots, x_n\rangle \in \omega^n(I^n_m(x_1, \ldots, x_n) = x_m)$ is called a projective function. Denote

$$U = \bigcup_{m,n\in\omega} \{C^n_m\} \cup \bigcup_{1\leqslant m\leqslant n, n\in\omega} \{I^n_m\}.$$

It is obvious that any function is consistent with two equivalences: the zero equivalence $(id\ \omega)$ and the trivial equivalence $(\{\langle x, y\rangle | x, y \in \omega\})$. It is also clear that any U-function is consistent with any equivalence on the set ω

Proposition 14. *For any equivalence there exists a commutative semigroup that is representable over said equivalence.*

Proof. Define the multiplicatipn on ω as C^2_m, where m is some fixed natural number. Then C^2_m is computable and obviously consistent with any equivalence η on ω and therefore a quotient algebra $\langle \omega/\eta; C^2_m\rangle$ of a computable commutative semigroup $\langle \omega; C^2_m\rangle$, i.e, a semigroup $\langle \omega; C^2_m\rangle$ is representable over η. □

Note that we can take I^2_1 as a semigroup multiplication, but the semigroup will not be commutative.

Proposition 15. *For every $\alpha \subseteq \omega$ there exists a commutative semigroup with a unit element that is representable over the equivalence relation η^*_α.*

Proof. Define the multiplication $*$: $x * y = \gamma^{-1}(\gamma_x \cup \gamma_y)$. It is easy to check that $*$ is consistent with η^*_α for any $\alpha \subseteq \omega$. A unit element of any commutative semigroup is the η^*_α-class that is generated by all α-extensions of an empty set, i.e, $\varepsilon = \{x | \gamma_x \subseteq \alpha\}$. □

Proposition 16. *There exists a positive equivalence over which no semigroup with a unit element is representable.*

Proof. Let η be the aforementioned perfect equivalence with a compressed characteristic transversal. Then any computable function, that is consistent with η acts on ω/η like a constant or like a projection. Then if there exists a semigroup with the unit element e and operation $*$ that is representable over η then this operation acts on ω/η as either a constant function or a projection operation. If $*$ acts like a constant then denote by c such an element of a semigroup that $\forall a, b(a * b = c)$. Then $\forall a(e * a = a * e = a = c)$, which is impossible in a nontrivial semigroup.

Then let $*$ act as a projection, i.e. like I^2_1 or like I^2_2. In the first case $\forall a(e = e * a)$ since $*$ acts like I^2_1 but at the same time $\forall a(e * a = a)$ since e is a unit element of the semigroup. Then $\forall a(e = a)$ which is only possible for a trivial monoid. The case I^2_2 can be worked through in a similar way. □

Note that if we consider semigroups only with left (right) unit element, the following result holds:

Proposition 17. *For any equivalence relation there exists a semigroup with the left (right) unit element that is representable over this relation.*

Proof. Indeed, if we take I_2^2 as the group operation then any element of a semigroup is a left unit. Similary for I_1^2 every element will be a right unit. $\square$

5. Degrees of representability

In the final section we will say a few words regarding the possibilities of applying the aforementioned results in the theory of degrees of algorithmic representability of systems, which is of intereset in the theoretical informatics' approach to the problem of clarifying the notion of algorithmic realisation of the data model [16].

Let $\mathbb{G}_p$ be the class of positively-representable infinite groups, Σ be the set of all infinite positive equivalences and $K_G(\eta)$ the class of all groups representable over the equivalence η. On Σ define the following preorder

$$\eta_0 \leqslant_{G_p} \eta_1 \leftrightharpoons \forall G \in \mathbb{G}_p[G \in K_G(\eta_0) \Rightarrow G \in K_G(\eta_1)].$$

The symmetric closure of the preorder $\leqslant_{G_p}$ splits the set Σ into classes of $\equiv_{G_p}$-equivalence and one can generate a preorder on $\Sigma/\equiv_{G_p}$ as follows: $D_{G_p} = \langle \Sigma/\equiv_{G_p}; \leqslant_{G_p}\rangle$, where $\leqslant_{G_p}$ also denotes a partial order that is induced by the action $\leqslant_{G_p}$ on Σ. It is clear that this transition is correctly defined with respect to taking quotients. Informally, $\eta_0 \leqslant_{G_p} \eta_1$ means that any group that is representable over η_0 is also representable over η_1 and the structure of the partial order D_{G_p} defines the algorithmic nature of equivalences from the «classes-of groups that are realised over these equivalences» point of view. Note that any group from $\mathbb{G}_p$ is defined by some $\equiv_{G_p}$ class. We will call the elements of D_{G_p} degrees of positive representability of groups. Note that the degrees of representability of finite systems are isolated points in the structure of the partially ordered set of degrees and, from the view of descriptive computability theory, can be considered «degenerate cases». Thus we will consider all degrees of representability except for those that are generated by finite equivalences.

For example, theorem 1 implies that all positive equivalences that are not m-uniform lie in one $\equiv_{G_p}$-degree that is the smallest with respect to $\leqslant_{G_p}$ and defines an empty class of groups, However it follows from theorem 2 that some uniform m-equivalences are also inside of that degree.

Similarly, when considering the class of negatively representable infinite groups $\mathbb{G}_n$ and the set of all infninte negative equivalence relations Π we obtain a structure of degrees of negative representability of groups $D_{G_n} = \langle \Pi/\equiv_{G_n}; \leqslant_{G_n}\rangle$. We can consider wider classes $\mathbb{SG}_p$ ($\mathbb{SG}_n$) of positively (negatively) representable infinite semigroups and the corresponding sets of degrees of positive (negative) representability for the classes of semigroups. When doing so, we can note that the $\equiv_{G_p}$ degrees of representations of groups will, generally speaking, «split» for representations of semigroups since similar $\equiv_{G_p}$ degrees for groups can be different for a wider class of semigroups (when considering the groups in a signature with one binary operation).

Currently the cases of degrees of positive and negative representability for linear orders are relatively well studied. For instance, infiniteness of a structure D_{L_p} of degrees of positive representability of linear orders has been shown. Additionally this structure was shown to have no maximal element and it was also proved that it contains a degree that defines and empty class of linear orders (for these and other related results see [24]). The results for the structure D_{L_n} of degrees of infinite representability of linear orders were totally different: this structure is also infinite, but it has a maximal element and every degree of this structure defines a rather rich class of linear orders that are representable over every equivalence from this degree (for these and other related results see [21]).

Cases of groups, semigroups and rings are not studied. The simplest case is the structure of degrees of positive representability of fileds D_{F_p}.

Proposition 18. *A poset of degrees of positive representability of infinite fields consits of two elements and is isomorphic to the ordinal 2.*

Proof. Indeed, any undecidable positive equivalence lies in the degree of positive representability that defines an empty class of fields, since this equivalence has no field that can be represented over it. On the other hand, any infinite field that has a computable copy is representable over any decidable infinite equivalence. □

On the structure of D_{G_p} of degrees of positive representability of groups we present an almost trivial

Proposition 19. *A poset of degrees of positive representability of infinite groups has at least three elements with two of them being incomparable.*

Proof. All equivalences that are not m-unifrom (and even some uniform m-equivalences) form the smalles degree d_0 in D_{G_p}. On the other hand, the degree d_1 that contains $\eta_1 = id\ \omega$ defines a simplest group $G_1 = \langle Z; +\rangle$ that is stable with respect to positive numberings . Now if d_2 is a fegree that contains an undecidable equivalence η_2 that is a kernal of the representation of some finitely generated group G_2 then $G_1 \in K_G(\eta_1) \setminus K_G(\eta_2) \wedge G_2 \in K_G(\eta_2) \setminus K_G(\eta_1)$, i.e. $\eta_1 \nleqslant_{G_p} \eta_2$ and $\eta_2 \nleqslant_{G_p} \eta_1$. □

Some other natural questions concerning the structure of degrees of representability of the aforementioned systems are open. For instance, it is yet unknown whether D_{G_p} is infinite.

Recall [12] that if (A, μ), (B, ν) are enumerated algebras then the homomorphism $\varphi : A \longrightarrow B$ is called a morphism if it is effective with respect to numbers, i.e. there exists a computable function f such that $\varphi\mu = \nu f$. If $\mathfrak{B} = \{(B_i, \nu_i) | i \in I\}$ is a family of enumerated algebras then it is said that an enumerated algebra (A, μ) is approximated by $\mathfrak{B}$-algebras if for any pair of distinct elements $a_0, a_1 \in A$ there exists a morphism φ_{a_0,a_1} from (A, μ) to a suitable $\mathfrak{B}$ that sends these elements into distinct elements (i.e. $\varphi_{a_0,a_1}(a_0) \neq \varphi_{a_0,a_1}(a_1)$).

We can also introduce another natural structure of degrees, that contains both positive and negative degrees. Recall that the equivalence η on ω is called separable (effectively separable) if there exists a T_0-separable family of η-closed computably enumerable sets of the space ω/η (computable T_0-separable family of η-closed enumerable sets of the space ω/η). Yu. Ershov in [13] introduced the most natural notion of a separable numbering of the set (i.e. such a numbering, that has a T_0-separable kernel in the enumerable topology) and gave the following characterisation of the kernels of separable numberings of the families of enumberable sets:

Theorem 5. [13, Proposition 8, page 60] *The numbering of a family of computably enumerable sets is separable iff its kernel is effectively separable.*

This theorem, when applied to algebras, almost instantly produces the following result:

Proposition 20. *Enumerated algebra is separable iff it can be approximated by effectively separable algebras*

The proof can be found in [17].

The class of effectively separable algebras lies in Π_2^0 but does not contain Δ_2^0. This class is rather natural since it lies quite low in the arithmetic hierarchy and contains some of the most important lower classes:$\Sigma_1^0 \bigcup \Pi_1^0$. Therefore questions about the structure of degrees of effectively separable groups D_{G_E} can be of interest for the field of theoretical informatics.

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