S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 19, стр. ???–??? (2022) УДК 510.67
DOI 10.33048/semi.2019.16.xxx MSC 03C07, 03C10, 03C68

SPECIAL RELATIONS FOR FORMULAE, THEIR EQUIVALENCE RELATIONS AND THEORIES

S.V. SUDOPLATOV

Abstract. We study properties and characteristics for special relations of formulae in given languages producing equivalence relations both in general case and for elementary theories. Characteristics for chains of special relations and equivalence relations are described.

Keywords: formula, special relation, equivalence relation, theory.

Studying valuable links between theories structural relations were introduced both for theories in given languages and for theories in distinct languages including their similarity [1], definability [2], interpretability [3], etc. We introduced and studied properties and ranks for families of theories in a given language [4, 5, 6, 8, 7, 9, 10, 11].

In the present paper we introduce and study both special relations for formulae and equivalence ones both in general case and for elementary theories. Characteristics for chains of special relations and equivalence relations are described. These considerations are based both on transformations of the rank RS [4] as well as on a general approach for a hierarchy of links between definable relations [12].

The paper is arranged as follows. Preliminary notions, notations and assertions are represented in Section 1. In Section 2, we consider special and equivalence relations on sets of formulae, their forms and properties. Elementary special and equivalence relations are studies in Section 3, including categories related to these

Sudoplatov, S.V., Special relations for formulae, their equivalence relations and theories.

The work was carried out in the framework of the State Contract of the Sobolev Institute of Mathematics, Project No. FWNF-2022-0012, of the Committee of Science in Education and the Science Ministry of the Republic of Kazakhstan, Grant No. AP08855497 (Sections 2 and 3), and of Russian Scientific Foundation, Project No. 22-21-00044 (Sections 4 and 5).

Received ???, ??, 2022, published ???, ??, 2022.

relations. Complete and precomplete special and equivalence relations, and their completions, possibilities for ranks are described in Section 4. In Section 5 chains of special and equivalence relations are studied. Spectra for finite lengths of chains are characterized and spectra for the lengths with respect to countable languages are described.

1. Preliminaries

Following [4] we define the *rank* $\mathrm{RS}(\cdot)$ for families $\mathcal{T}$ of first-order theories in a language $\Sigma = \Sigma(\mathcal{T})$, similar to Morley rank for a fixed theory, and a hierarchy with respect to these ranks in the following way.

By $F(\Sigma)$ we denote the set of all formulae in the language Σ and by $\mathrm{Sent}(\Sigma)$ the set of all sentences in $F(\Sigma)$.

For a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ we denote by $\mathcal{T}_\varphi$ the set of all theories $T \in \mathcal{T}$ with $\varphi \in T$.

Any set $\mathcal{T}_\varphi$ is called the *φ-neighbourhood*, or simply a *neighbourhood*, for $\mathcal{T}$, or the (φ-)*definable* subset of $\mathcal{T}$. The set $\mathcal{T}_\varphi$ is also called (*formula-* or *sentence-*)*definable* (by the sentence φ) with respect to $\mathcal{T}$, or (*sentence-*)*$\mathcal{T}$-definable*, or simply *s-definable*.

Definition [6]. If $\mathcal{T}$ is a family of theories and Φ is a set of sentences, then we put $\mathcal{T}_\Phi = \bigcap\limits_{\varphi\in\Phi} \mathcal{T}_\varphi$ and the set $\mathcal{T}_\Phi$ is called (*type-* or *diagram-*)*definable* (by the set Φ) with respect to $\mathcal{T}$, or (*diagram-*)*$\mathcal{T}$-definable*, or simply *d-definable*.

Clearly, finite unions of d-definable sets are again d-definable. Considering infinite unions $\mathcal{T}'$ of d-definable sets $\mathcal{T}_{\Phi_i}$, $i \in I$, one can represent them by sets of sentences with infinite disjunctions $\bigvee\limits_{i\in I} \varphi_i$, $\varphi_i \in \Phi_i$. We call these unions $\mathcal{T}'$ are called *d_∞-definable* sets.

Definition [4]. For the empty family $\mathcal{T}$ we put the rank $\mathrm{RS}(\mathcal{T}) = -1$, and for nonempty families $\mathcal{T}$ we put $\mathrm{RS}(\mathcal{T}) \geq 0$.

For a family $\mathcal{T}$ and an ordinal $\alpha = \beta + 1$ we put $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if there are pairwise $\mathcal{T}$-inconsistent $\Sigma(\mathcal{T})$-sentences φ_n, $n \in \omega$, such that $\mathrm{RS}(\mathcal{T}_{\varphi_n}) \geq \beta$, $n \in \omega$.

If α is a limit ordinal then $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \beta$ for any $\beta < \alpha$.

We set $\mathrm{RS}(\mathcal{T}) = \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \alpha$ and $\mathrm{RS}(\mathcal{T}) \not\geq \alpha + 1$.

If $\mathrm{RS}(\mathcal{T}) \geq \alpha$ for any α, we put $\mathrm{RS}(\mathcal{T}) = \infty$.

A family $\mathcal{T}$ is called *e-totally transcendental*, or *totally transcendental*, if $\mathrm{RS}(\mathcal{T})$ is an ordinal.

If $\mathcal{T}$ is e-totally transcendental, with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, we define the *degree* $\mathrm{ds}(\mathcal{T})$ of $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences φ_i such that $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

As noticed in [13, 14] these notions are valid both for families of complete theories and for arbitrary families of theories including incomplete ones.

Below in this section we consider results on families of complete theories in a given language.

The following theorem characterizes the property of e-total transcendency for countable languages.

Theorem 1.1 [4, 15]. *For any family $\mathcal{T}$ with $|\Sigma(\mathcal{T})| \leq \omega$ the following conditions are equivalent:*

(1) $|\mathrm{Cl}_E(\mathcal{T})| = 2^\omega$;

(2) *e*-$\mathrm{Sp}(\mathcal{T}) = 2^\omega$*;*
(3) $\mathrm{RS}(\mathcal{T}) = \infty$*;*
(4) *there exists a* 2*-tree of sentences* φ *for s-definable sets* $\mathcal{T}_\varphi$*.*

Let Σ be a language. If Σ is relational we denote by $\mathcal{T}_\Sigma$ the family of all theories of the language Σ. If Σ contains functional symbols f then $\mathcal{T}_\Sigma$ is the family of all theories of the language Σ', which is obtained by replacements of all n-ary symbols f with $(n+1)$-ary predicate symbols R_f which interpreted by $R_f = \{(\bar{a}, b) \mid f(\bar{a}) = b\}$.

Theorem 1.2 [5]. *For any language* Σ *either* $\mathrm{RS}(\mathcal{T}_\Sigma)$ *is finite, if* Σ *consists of finitely many* 0*-ary and unary predicates, and finitely many constant symbols, or* $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$*, otherwise.*

For a language Σ we denote by $\mathcal{T}_{\Sigma,n}$ the family of all theories in $\mathcal{T}_\Sigma$ having n-element models, $n \in \omega$, as well as by $\mathcal{T}_{\Sigma,\infty}$ the family of all theories in $\mathcal{T}_\Sigma$ having infinite models.

Theorem 1.3 [5]. *For any language* Σ *either* $\mathrm{RS}(\mathcal{T}_{\Sigma,n}) = 0$*, if* Σ *is finite or* $n = 1$ *and* Σ *has finitely many predicate symbols, or* $\mathrm{RS}(\mathcal{T}_{\Sigma,n}) = \infty$*, otherwise.*

Theorem 1.4 [5]. *For any language* Σ *either* $\mathrm{RS}(\mathcal{T}_{\Sigma,\infty})$ *is finite, if* Σ *is finite and without predicate symbols of arities* $m \geq 2$ *as well as without functional symbols of arities* $n \geq 1$*, or* $\mathrm{RS}(\mathcal{T}_{\Sigma,\infty}) = \infty$*, otherwise.*

By the definition the families $\mathcal{T}_\Sigma$, $\mathcal{T}_{\Sigma,n}$, $\mathcal{T}_{\Sigma,\infty}$ are E-closed. Thus, combining Theorem 1.1 with Theorems 1.2–1.4 we obtain the following possibilities of cardinalities for the families $\mathcal{T}_\Sigma$, $\mathcal{T}_{\Sigma,n}$, $\mathcal{T}_{\Sigma,\infty}$ depending on Σ and $n \in \omega$:

Proposition 1.5. *For any language* Σ *either either* $\mathcal{T}_\Sigma$ *is countable, if if* Σ *consists of finitely many* 0*-ary and unary predicates, and finitely many constant symbols, or* $|\mathcal{T}_\Sigma| \geq 2^\omega$*, otherwise.*

Proposition 1.6. *For any language* Σ *either* $\mathcal{T}_{\Sigma,n}$ *is finite, if* Σ *is finite or* $n = 1$ *and* Σ *has finitely many predicate symbols, or* $|\mathcal{T}_{\Sigma,n}| \geq 2^\omega$*, otherwise.*

Proposition 1.7. *For any language* Σ *either* $\mathcal{T}_{\Sigma,\infty}$ *is at most countable, if* Σ *is finite and without predicate symbols of arities* $m \geq 2$ *as well as without functional symbols of arities* $n \geq 1$*, or* $|\mathcal{T}_{\Sigma,\infty}| \geq 2^\omega$*, otherwise.*

Definition [6]. Let $\mathcal{T}$ be a family of theories, Φ be a set of sentences, α be an ordinal $\leq \mathrm{RS}(\mathcal{T})$ or -1. The set Φ is called *α-ranking* for $\mathcal{T}$ if $\mathrm{RS}(\mathcal{T}_\Phi) = \alpha$. A sentence φ is called *α-ranking* for $\mathcal{T}$ if $\{\varphi\}$ is *α-ranking* for $\mathcal{T}$.

The set Φ (the sentence φ) is called *ranking* for $\mathcal{T}$ if it is α-ranking for $\mathcal{T}$ with some α.

Proposition 1.8 [6]. *For any ordinals* $\alpha \leq \beta$*, if* $\mathrm{RS}(\mathcal{T}) = \beta$ *then* $\mathrm{RS}(\mathcal{T}_\varphi) = \alpha$ *for some* (α*-ranking*) *sentence* φ*. Moreover, there are* $\mathrm{ds}(\mathcal{T})$ *pairwise* $\mathcal{T}$*-inconsistent* β*-ranking sentences for* $\mathcal{T}$*, and if* $\alpha < \beta$ *then there are infinitely many pairwise* $\mathcal{T}$*-inconsistent* α*-ranking sentences for* $\mathcal{T}$*.*

Theorem 1.9 [6]. *Let* $\mathcal{T}$ *be a family of a countable language* Σ *and with* $\mathrm{RS}(\mathcal{T}) = \infty$*,* α *be a countable ordinal,* $n \in \omega \setminus \{0\}$*. Then there is a* d_∞*-definable subfamily* $\mathcal{T}^* \subset \mathcal{T}$ *such that* $\mathrm{RS}(\mathcal{T}^*) = \alpha$ *and* $\mathrm{ds}(\mathcal{T}^*) = n$*.*

2. Special and equivalence relations on sets of formulae

Definition. Let $F(\Sigma)$ be the set of all formulae in a language Σ, V be an infinite set of variables, V^* be the set of all tuples $\overline{x} \in V^n$, $n \in \omega$. A ternary relation $E \subseteq F(\Sigma) \times F(\Sigma) \times V^*$ is called *special* (for $F(\Sigma)$) if for each $\overline{x} \in V^*$,

$$E_{\overline{x}} = \{(\varphi, \psi) \mid (\varphi, \psi, \overline{x}) \in E\}$$

is an equivalence relation on the set $X \subset F(\Sigma)$ consisting of all formulae φ whose each free variable belongs to $\overline{x}$.

Since each formula $\varphi = \varphi(\overline{x}) \in F(\Sigma)$ can be considered as a formula $\varphi(\overline{y})$ for any $\overline{y} \supseteq \overline{x}$, then for any special relation E for $F(\Sigma)$, φ belongs to some $E_{\overline{y}}$-class for any $\overline{y} \supseteq \overline{x}$. Therefore for any $\varphi \in F(\Sigma)$ there are infinitely many equivalence classes for E, within distinct $\overline{y} \in V^*$, containing φ.

Definition. The special relation $E \in \mathrm{SR}(\Sigma)$ is called *coordinated* if the relation

$$E^* = \{(\varphi, \psi) \mid (\varphi, \psi, \overline{x}) \in E \text{ for some } \overline{x}\}$$

is an equivalence relation on $F(\Sigma)$.

We denote by $\mathrm{id}_F(\Sigma)$ the special relation E for $F(\Sigma)$ whose all $E_{\overline{x}}$-classes are singletons.

By the definition $\mathrm{id}_F(\Sigma)$ is the least special relation for $F(\Sigma)$.

Clearly, there are both coordinated special relations (for instance, $\mathrm{id}_F(\Sigma)$) and non-coordinated ones since equivalence classes for $E_{\overline{x}}$ can be arbitrarily varied extending $\overline{x}$. Notice also that by the definition the restriction of E^*, for coordinated E, to the set of all sentences in $F(\Sigma)$ is an equivalence relation, which both can be equal to $E_\emptyset$ or strictly contain $E_\emptyset$.

Definition. A special relation E is called *upward directed* if $E_{\overline{x}} \cup E_{\overline{y}} \subseteq E_{\overline{x}\hat{\ }\overline{y}}$ for any $\overline{x}, \overline{y} \in V^*$.

By the definition each upward directed special relation is coordinated.

For each formula $\varphi = \varphi(\overline{x}) \in F(\Sigma)$ and a special relation E we consider the set $\varphi_E = \{\forall \overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})) \mid (\varphi, \psi, \overline{x}) \in E\}$. We denote by ∇_E the set $\bigcup\limits_{\varphi \in F(\Sigma)} \varphi_E$.

Example 2.1. Let $\varphi(\overline{x})$ be a formula in a language Σ, R be a new relational symbol of arity $l(\overline{x})$, Σ' be an expansion of Σ containing R. If E is the equivalence relation on $F(\Sigma')$ with one two-element class $\{\varphi(\overline{x}), R(\overline{x})\}$ and singletons for all other formulae we obtain ∇_E generated by the φ-*naming*, or the naming for $\varphi(\overline{x})$: $\forall \overline{x}(\varphi(\overline{x}) \leftrightarrow R(\overline{x}))$.

If ∇_E is generated by φ-namings for each $\varphi \in F(\Sigma)$ such that distinct formulae $\varphi(\overline{x})$ are E-equivalent to distinct predicates $R_\varphi(\overline{x})$, then ∇_E produces a Morleyzation for $F(\Sigma)$. □

Remark 2.2. Let $F'(\Sigma)$ be the subset of $F(\Sigma)$ consisting of all formulae $\varphi(x_1, \ldots, x_n)$ equivalent to $\varphi(x_1, \ldots, x_n) \wedge \bigcap\limits_{i<j} \neg x_i \approx x_j$. As shown in [12], each formula $\varphi \in F(\Sigma)$ is equivalent to a Boolean combination of formulae in $F'(\Sigma)$. Therefore equivalence relations on $F(\Sigma)$ are reduced to equivalence relations on $F'(\Sigma)$. Thus, notions and results below, for $F(\Sigma)$, can be naturally transformed for $F'(\Sigma)$, and vice versa.

We denote by $\mathrm{SR}(\Sigma)$ the family of all special relations for $F(\Sigma)$, by $\mathrm{SR}_f(\Sigma)$ the subfamily of $\mathrm{SR}(\Sigma)$ consisting of all special relations E with finite equivalence $E_{\overline{x}}$-classes only, and by $\mathrm{SR}_{\mathrm{ff}}(\Sigma)$ the subfamily of $\mathrm{SR}_f(\Sigma)$ consisting of special relations E with only finitely many, including 0, nontrivial $E_{\overline{x}}$-classes, i.e., $E_{\overline{x}}$-classes which are not singletons.

Proposition 2.3. *A special relation $E \in \mathrm{SR}_{\mathrm{ff}}(\Sigma)$ for $F(\Sigma)$ is upward directed if and only if $E = \mathrm{id}_F(\Sigma)$.*

Proof. Clearly, $\mathrm{id}_F(\Sigma) \in \mathrm{SR}_{\mathrm{ff}}(\Sigma)$ since all equivalence classes for $\mathrm{id}_F(\Sigma)$ are singletons.

Conversely, if $E \neq \mathrm{id}_F(\Sigma)$ then there is $\overline{x} \in V^*$ such that some $E_{\overline{x}}$-class contains at least two formulae. Assuming that E is upward directed we obtain, for each $\overline{y} \in V^*$, $E_{\overline{x}\hat{\ }\overline{y}}$-classes which are not singletons. Thus, $E \notin \mathrm{SR}_{\mathrm{ff}}(\Sigma)$. □

Remark 2.4. Any special relation $E \in \mathrm{SR}_{\mathrm{ff}}(\Sigma)$ can be naturally extended till a upward directed relation $E' \in \mathrm{SR}_f(\Sigma)$ just extending $E_{\overline{x}\hat{\ }\overline{y}}$-classes by (finitely many) formulae in appropriate $E_{\overline{x}}$-classes and $E_{\overline{y}}$-classes.

Definition. Let E be a special relation for $F(\Sigma)$.

The relation E is called *inessential* if ∇_E consists of identically true formulae. The relation E is called *essential* if it is not inessential.

For a formula $\varphi = \varphi(\overline{x}) \in F(\Sigma)$ the equivalence class $E_{\overline{x}}(\varphi)$ is called *regular* if the set φ_E is consistent.

The relation E is called *weakly regular* if for each formula $\varphi = \varphi(\overline{x}) \in F(\Sigma)$ the equivalence class $E_{\overline{x}}(\varphi)$ is regular.

The relation E is called *locally regular* if each finite part of ∇_E is consistent.

The relation E is called *strongly regular* if ∇_E is consistent.

The following properties hold:

1. (Monotony) If $E, E' \in \mathrm{SR}(\Sigma)$, $E \subseteq E'$, and E' is inessential (respectively, weakly regular, strongly regular) then E inessential (weakly regular, strongly regular), too.

2. Any inessential relation is strongly regular, but not vice versa, as Example 2.1 shows.

3. Any strongly regular relation is locally regular, and, by compactness, any locally regular relation is strongly regular. Besides, any strongly regular relation is weakly regular, but not vice versa.

Indeed, we can consider a naming

$$\forall \overline{x}(\psi(\overline{x}) \leftrightarrow R(\overline{x})) \tag{1}$$

for a consistent formula $\psi(\overline{x})$ generating the set ψ_E, and the formula

$$\forall \overline{x}(\neg\psi(\overline{x}) \leftrightarrow (R(\overline{x}) \wedge R(\overline{x}))) \tag{2}$$

producing the set $(\neg\psi)_E$ such that all others $E_{\overline{y}}$-classes are singletons. Clearly, each set φ_E is consistent. i.e., E is weakly regular. At the same time, the formulae (1) and (2) imply $\forall \overline{x}(\psi(\overline{x}) \leftrightarrow \neg\psi(\overline{x}))$ producing that ∇_E is inconsistent, i.e., E is not strongly regular.

4. Any singleton $E_{\overline{x}}(\varphi)$ is regular, moreover, any special relation $E \in \mathrm{SR}(\Sigma)$ with consistent set $\bigcup\{\varphi_E \mid |E(\varphi)| > 1\}$ is strongly regular. As a corollary we obtain that

any disjoint nonempty sets $\Phi_i \subset F(\Sigma)$, $i \in I$, of formulae with consistent set $\{\forall\overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})) \mid \varphi, \psi \in \Phi_i$ for some $i \in I\}$ has a strongly regular relation E such that $\{\Phi_i \mid i \in I\} \subseteq \{E(\varphi) \mid \varphi \in F(\Sigma)\}$, extending $\{\Phi_i \mid i \in I\}$ by singletons.

5. Any Morleyzation is produced by a strongly regular relation, as in Example 2.1.

By Property 3 we unite the notions of locally regular and strongly regular special relations obtaining the notion of *regular* special relation. We denote by $\mathrm{RSR}(\Sigma)$ the subfamily of $\mathrm{SR}(\Sigma)$ consisting of all regular special relations.

Proposition 2.5. *For any language Σ there is the largest inessential relation E_0 for $F(\Sigma)$. The relation E_0^* equals the relation $\equiv$ for $F(\Sigma)$.*

Proof. We denote the relation $\equiv$ by E_0^*. It naturally produces a special relation E_0 which is inessential by the definition, since $\varphi \equiv \psi$ if and only if $\vdash (\varphi \leftrightarrow \psi)$ for any $\varphi, \psi \in F(\Sigma)$, and thus ∇_{E_0} consists of identically true formulae.

Now we note that any proper extension $E \in \mathrm{SR}(\Sigma)$ of E_0 is essential. Indeed, by the definition there is $(\varphi, \psi) \in E^*$ such that $\nvdash (\varphi \leftrightarrow \psi)$. Therefore, by Gödel completeness theorem, $(\varphi \leftrightarrow \psi)$ is not identically true. Thus, E is essential. □

Property 1 and Proposition 2.5 imply:

Corollary 2.6. *A special relation $E \in \mathrm{SR}(\Sigma)$ is inessential if and only if $E^* \subseteq E_0^*$. In particular, the least special relation $\mathrm{id}_{F(\Sigma)}$ is inessential.*

Proposition 2.7. (1) *A relation $E \in \mathrm{SR}(\Sigma)$ is inessential if and only if each its subrelation $E' \in \mathrm{SR}_f(\Sigma)$ is inessential, and if and only if each its subrelation $E'' \in \mathrm{SR}_{\mathrm{ff}}(\Sigma)$ is inessential.*

(2) *A relation $E \in \mathrm{SR}(\Sigma)$ is weakly regular if and only if each its subrelation $E' \in \mathrm{SR}_f(\Sigma)$ is weakly regular, and if and only if each its subrelation $E'' \in \mathrm{SR}_f(\Sigma)$ is weakly regular.*

(3) *A relation $E \in \mathrm{SR}(\Sigma)$ is regular if and only if each its subrelation $E' \in \mathrm{SR}_f(\Sigma)$ is regular, and if and only if each its subrelation $E'' \in \mathrm{SR}_{\mathrm{ff}}(\Sigma)$ is regular.*

Proof. (1) follows by Proposition 2.5 and Corollary 2.6.

(2) and (3) hold by compactness since if some φ_E (respectively ∇_E) is inconsistent then some finite part of φ_E (∇_E) is inconsistent producing a special relation $E'' \subseteq E$ in $\mathrm{SR}_{\mathrm{ff}}(\Sigma)$, and in $\mathrm{SR}_f(\Sigma)$ by $\mathrm{SR}_{\mathrm{ff}}(\Sigma) \subset \mathrm{SR}_f(\Sigma)$, which is not weakly regular (regular). □

3. Elementary special and equivalence relations

Definition. A special relation $E \in \mathrm{SR}(\Sigma)$ is called *elementary* if there exists a consistent theory T of given language Σ such that $E = \{(\varphi, \psi, \overline{x}) \mid T \vdash \forall\overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))\}$.

Here the relation E (respectively, E^*) is called the *special* (*equivalence*) *relation for* T.

The set of all elementary special relations $E \in \mathrm{SR}(\Sigma)$ is denoted by $\mathrm{ESR}(\Sigma)$.

By the definition any elementary special relation $E \in \mathrm{ESR}(\Sigma)$ for T is upward directed, so it is coordinated, and the relation E^* is the equivalence relation $\equiv_T$ modulo T, where for any $\overline{x} \in V^*$, $E_{\overline{x}}$ is the restriction of $\equiv_T$ to the set of formulae $\varphi(\overline{x})$.

Clearly, each relation $E \in \mathrm{ESR}(\Sigma)$ is regular, and that relation E is inessential if and only if $E^* = \equiv_{T_\Sigma}$ for the least, by inclusion, theory T_Σ of the language Σ.

Besides, each regular special relation E generates both the unique consistent theory T with $\nabla_E \vdash T$, which is axiomatized by ∇_E, and the unique elementary special relation E' for T, containing E. Thus the set $\mathrm{RSR}(\Sigma)$ of all regular special relations for $F(\Sigma)$ has a closure operator Cl acting by the rule $\mathrm{Cl}(E) = E'$, similar to [17]. This closure operator satisfies the following obvious properties for any $E, E' \in \mathrm{RSR}(\Sigma)$:

i) $E \subseteq \mathrm{Cl}(E)$ (Extensiveness),

ii) if $E \subseteq E'$ then $\mathrm{Cl}(E) \subseteq \mathrm{Cl}(E')$ (Monotony),

iii) $\mathrm{Cl}(\mathrm{Cl}(E)) = \mathrm{Cl}(E)$ (Idempotency).

The structure $(\mathrm{RSR}(\Sigma), \mathrm{Cl})$ is extensible till the structure $(\mathrm{SR}(\Sigma), \mathrm{Cl}^*)$ with the closure operator Cl^* producing the special relation $E' = \mathrm{Cl}^*(E)$, for $E \in \mathrm{SR}(\Sigma) \setminus \mathrm{RSR}(\Sigma)$, corresponding the equivalence relation $(E')^* = \equiv_{T_0}$ for the contradictory theory T_0.

Proposition 3.1. *For any relation $E \in \mathrm{ESR}(\Sigma)$ there is unique theory T such that $E^* = \equiv_T$.*

Proof. The required theory T exists by the definition of elementary special relation E. Now taking the elementary special relation E we reconstruct T as the set T' of sentences φ such that $(\varphi, \forall x(x \approx x), \emptyset) \in E$. Indeed, if $\varphi \in T$ then $\varphi \equiv_T \forall x(x \approx x)$ producing $\varphi \in T'$. Conversely, if the sentence $\varphi \in F(\Sigma)$ does not belong to T then φ can not belong to T' since otherwise $\varphi \equiv_T \forall x(x \approx x)$ implies $\varphi \in T$. □

The arguments for Proposition 3.1 show that any relation $E \in \mathrm{ESR}(\Sigma)$ uniquely defines a theory $T(E) = T$, and conversely any consistent theory T uniquely defines an elementary special relation E with $E^* = \equiv_T$. Moreover, both $T(E)$ and $\equiv_T$ are uniquely defined by the E^*-class $E^*(1)$ containing identically true formulae. Therefore there are one-to-one correspondences between the class EE of elementary equivalence relations E^*, the class $\mathcal{T}$ of consistent theories $T = T(E)$, and the class EE(1) consisting of the E^*-classes $E^*(1)$. These correspondences are denoted by $\mathrm{EE} \to \mathcal{T}$, $\mathrm{EE} \to \mathrm{EE}(1)$. Additionally we obtain correspondences $\mathcal{T} \to \mathrm{EE}$, $\mathrm{EE}(1) \to \mathrm{EE}$, $\mathcal{T} \to \mathrm{EE}(1)$, $\mathrm{EE}(1) \to \mathcal{T}$.

Any elementary special relation E, for a theory $T = T(E)$, produces an appropriate Lindenbaum–Tarski algebra [18] defined by the partial order $\leq$ in the following way: for equivalence classes $\equiv_T(\varphi)$ and $\equiv_T(\psi)$, where $\varphi, \psi \in F(\Sigma)$,

$$\equiv_T(\varphi) \leq \equiv_T(\psi) \Leftrightarrow (\varphi \wedge \psi, \varphi) \in E^*.$$

We denote this algebra by $\mathrm{LT}(E)$ and by $\mathrm{LT}(T)$, and the class of these algebras by $\mathcal{LT}$.

The algebras in $\mathcal{LT}$ are connected with theories in $\mathcal{T}$ by the one-to-one correspondences $\mathcal{LT} \to \mathcal{T}$ and $\mathcal{T} \to \mathcal{LT}$ with the rules $\mathrm{LT}(T) \mapsto T$ and $T \mapsto \mathrm{LT}(T)$.

Clearly, homomorphisms between algebras in $\mathcal{LT}$ induce by these correspondences homomorphisms between elements in $\mathcal{T}$, in EE, and in EE(1). Therefore we obtain four categories $\mathcal{LT}$, $\mathcal{T}$, $\mathcal{EE}$, $\mathcal{EE}(1)$, whose morphisms are homomorphisms. These categories are connected by functors $\alpha \to \beta$ above, where $\alpha, \beta \in \{\mathcal{LT}, \mathcal{T}, \mathcal{EE}, \mathcal{EE}(1)\}$.

Thus we have the following:

Theorem 3.2. *The categories $\mathcal{LT}$, $\mathcal{T}$, $\mathcal{EE}$, $\mathcal{EE}(1)$ are connected by the functors $\alpha \to \beta$, where $\alpha, \beta \in \{\mathcal{LT}, \mathcal{T}, \mathcal{EE}, \mathcal{EE}(1)\}$.*

Definition [16]. A theory T is said to be Δ-*based*, where Δ is some set of formulae without parameters, if any formula of T is equivalent in T to a Boolean combination of formulae of Δ.

For Δ-based theories T, it is also said that T has *quantifier elimination* or *quantifier reduction* modulo Δ.

Clearly, if Δ consists of quantifier free formulae then the Δ-based theory T has quantifier elimination.

The following proposition gives an obvious characterization for the Δ-basedness of a theory.

Proposition 3.3. *Let $\Delta \subseteq F(\Sigma)$. A theory T is Δ-based if and only if for each formula $\varphi(\overline{x}) \in F(\Sigma)$ there is a $(\equiv_T)_{\overline{x}}$-class containing both φ and a formula in Δ.*

Proposition 3.3 immediately implies:

Corollary 3.4. *A theory T of a language Σ has quantifier elimination if and only if for each formula $\varphi(\overline{x}) \in F(\Sigma)$ there is a $(\equiv_T)_{\overline{x}}$-class containing both φ and a quantifier free formula.*

Clearly, each E^*-class, for an elementary special relation E, is infinite. Moreover, since there are $\max\{\omega, |\Sigma|\}$ identically true sentences φ_1 of the language Σ and each E^*-class contains $\psi \wedge \varphi_1$ together with a formula ψ, each E^*-class contains $\max\{\omega, |\Sigma|\}$ formulae.

The following natural questions arise:

Question 1. What can be cardinalities of quotients $F(\Sigma)/E^*$ and $F(\Sigma)_{\overline{x}}/E_{\overline{x}}$, for restrictions $F(\Sigma)_{\overline{x}}$ of $F(\Sigma)$ to the sets of formulae $\varphi(\overline{x})$, where $\overline{x} \in V^*$?

Question 2. What special relations for $F(\Sigma)$ are elementary, depending on languages Σ?

Question 3. What are hierarchies for elementary special relations for $F(\Sigma)$, depending on languages Σ?

A partial answer to the Question 3 is represented in the following section.

4. Complete and precomplete special and equivalence relations, their completions

Extending the class $E^*(1)$, for an elementary special relation $E \in \mathrm{ESR}(\Sigma)$, by maximum many sentences preserving the consistency we obtain a completion T' of $T = T(E)$ correspondent to an elementary special relation $E' \supseteq E$. The relation E' for the completion T' is called *complete*, or the *completion* of E. The set of all complete relations $E \in \mathrm{ESR}(\Sigma)$ is denoted by $\mathrm{CSR}(\Sigma)$.

Definition. A relation $E' \in \mathrm{ESR}(\Sigma)$ is called a *a prime extension* of a relation $E \in \mathrm{ESR}(\Sigma)$ if there is a pair $(\varphi, \psi) \notin E$ of formulae such that $\nabla_E \cup \{\varphi \leftrightarrow \psi\}$ forces E'. If E' is the prime extension of E via the pair (φ, ψ) we say that (φ, ψ) *witnesses* that prime extension, and we denote the pair (E, E') by $E \prec_{\varphi,\psi} E'$, or simply $E \prec E'$ if an appropriate pair (φ, ψ) is fixed.

A relation $E \in \mathrm{ESR}(\Sigma) \setminus \mathrm{CSR}(\Sigma)$ is called *precomplete*, denoted by $E \in \mathrm{PCSR}(\Sigma)$, if $E \prec_{\varphi,\psi} E'$ for some pair (φ, ψ) and a complete special relation $E' \in \mathrm{CSR}(\Sigma)$.

If $E \in \mathrm{CSR}(\Sigma)$ (respectively, $E \in \mathrm{PCSR}(\Sigma)$) then the equivalence relation E^* is called (*pre*)*complete*, too.

Example 4.1. If the language Σ consists of unique unary predicate R then the inessential special relation E is precomplete obtaining a complete extension $E' \supseteq E$ just by adding a pair $(\varphi, \forall x \approx x)$ of sentences such that φ asserts a finite cardinality for R and a finite cardinality for its complement.

Similarly, any extension of a complete theory T by new unary predicate R produce a precomplete relation E generated by $\equiv_T$ and tautologies for $\Sigma(T) \cup \{R\}$. The relation E is extended to a complete one by the pair $(\varphi, \forall x \approx x)$, where, for instance, φ asserts that R is either empty or has the empty complement.

By the definition the number of elementary special relations extending E is connected with the number of completions and precomplete extensions of $T(E)$. In particular, by Propositions 1.5–1.7, for certain languages, E can have at most countably many or at least continuum many completions.

For $E \in \mathrm{ESR}(\Sigma)$, we denote by $\mathrm{Comp}(E)$ the set $\{E' \in \mathrm{CSR}(\Sigma) \mid E' \supseteq E\}$. Similarly, for a consistent theory T in the language Σ we denote by $\mathrm{Comp}(T)$ the set of all complete theories T' in the language Σ which contain T.

By the definition the set $\mathrm{Comp}(T)$ is d-definable, and so closed, in the family $\mathcal{T}_\Sigma$ of all theories of the language $\Sigma = \Sigma(T)$: $\mathrm{Comp}(T) = (\mathcal{T}_\Sigma)_T$.

By Theorem 3.2 there is a natural bijection between $\mathrm{Comp}(T(E))$ and $\mathrm{Comp}(E)$ acting by the *canonical* map $f^{\mathrm{Comp}(T(E))}$: $T' \mapsto \equiv_{T'}$ for $T' \supseteq T(E)$.

By the definition for any $E \in \mathrm{ESR}(\Sigma)$, $|\mathrm{Comp}(E)| \geq 1$. Moreover, $\mathrm{Comp}(E) = \{E\}$ and so $|\mathrm{Comp}(E)| = 1$ for complete E. Besides, if $E \in \mathrm{ESR}(\Sigma) \setminus \mathrm{CSR}(\Sigma)$, in particular, if E is precomplete, then $|\mathrm{Comp}(E)| \geq 2$.

Indeed, if E is not complete then there is a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ such that $T(E) \cup \{\varphi\}$ and $T(E) \cup \{\neg\varphi\}$ are consistent. Therefore $T_1 = \mathrm{Th}(T(E) \cup \{\varphi\})$ and $T_2 = \mathrm{Th}(T(E) \cup \{\neg\varphi\})$ are proper extensions of $T(E)$, hence special relations $\equiv_{T_1}$ and $\equiv_{T_2}$ are proper extensions of E, implying $|\mathrm{Comp}(E)| \geq 2$.

Thus we have the following:

Proposition 4.2. *If* $E \in \mathrm{ESR}(\Sigma)$ *then* $E \in \mathrm{CSR}(\Sigma)$ *if and only if* $\mathrm{Comp}(E)$ *is a singleton* (*consisting of* $\{E\}$).

Proposition 4.3. *For any cardinality* $\lambda \geq 2$ *there is a special relation* $E \in \mathrm{PCSR}(\Sigma)$ *such that* $|\mathrm{Comp}(E)| = \lambda$.

Proof. It suffices to take a complete theory T_0 of a language $\{c_i \mid i < \lambda\}$ with λ distinct constants expanded by a unary predicate R with the sentence $\exists^{=1} x R(x)$. Here we assume that $(x \approx x) \leftrightarrow \bigvee\limits_{i<\lambda} (x \approx c_i)$ if λ is finite. The resulted theory T has a precomplete special relation $E = \equiv_T$ with λ completions by sentences $R(c_i)$. If λ is infinite there is a completion by the set of sentences

$$\exists x \left(R(x) \wedge \bigwedge_{i_j \in \{i_1, \ldots, i_k\} \subset \lambda} \neg(x \approx c_{i_j}) \right).$$

In any case we have $|\mathrm{Comp}(E)| = \lambda$. □

The proof above is based on possibilities of interpretation of new predicate R by definable sets $\varphi(\mathcal{M})$ in initial models $\mathcal{M} \models T$ via the sentence $\forall \overline{x}(R(\overline{x}) \leftrightarrow \varphi(\overline{x}))$ extending the special relation E by the pair $(R(\overline{x}), \varphi(\overline{x}))$. In general case, precomplete special relation becomes complete via an identification of some two definable sets $\varphi(\mathcal{M})$ and $\psi(\mathcal{M})$.

Arguments for the proof of Proposition 4.3 imply the following:

Problem. *Describe possibilities for the cardinalities of* Comp(E) *depending on the languages* Σ *and possibilities of identifications.*

For a consistent theory T we denote by $\mathrm{RS}_{\mathrm{Comp}}(T)$ and by $\mathrm{ds}_{\mathrm{Comp}}(T)$ the values $\mathrm{RS}(\mathrm{Comp}(T))$ and $\mathrm{ds}(\mathrm{Comp}(T))$, respectively, for the family $\mathrm{Comp}(T)$.

These rank and degree values form complexity measured for theories T via correspondent measures for sets of completions. These values have upper bounds by the values for $\mathcal{T}_\Sigma$ [5].

The following three properties are obvious:

1. If T_1 and T_2 are theories in a language Σ with $T_1 \subseteq T_2$ then $\mathrm{Comp}(T_1) \supseteq \mathrm{Comp}(T_1)$, $\mathrm{RS}_{\mathrm{Comp}}(T_1) \geq \mathrm{RS}_{\mathrm{Comp}}(T_2)$, and $\mathrm{ds}_{\mathrm{Comp}}(T_1) \geq \mathrm{ds}_{\mathrm{Comp}}(T_2)$ if

$$\mathrm{RS}_{\mathrm{Comp}}(T_1) = \mathrm{RS}_{\mathrm{Comp}}(T_2).$$

2. If a theory T consists of tautologies in a language Σ then $\mathrm{Comp}(T) = \mathcal{T}_\Sigma$ with the maximal rank $\mathrm{RS}(\mathrm{Comp}(T)) = \mathrm{RS}(\mathcal{T}_\Sigma)$.

3. If T is a complete theory then $\mathrm{Comp}(T) = \{T\}$, $\mathrm{RS}(\mathrm{Comp}(T)) = 0$,

$$\mathrm{ds}(\mathrm{Comp}(T)) = 1.$$

4. For any theory T in a language Σ and a set Φ of Σ-sentences the set $\mathrm{Comp}(T)_\Phi$ equals the set $\mathrm{Comp}(T \cup \Phi)$ of completions for $T \cup \Phi$, and equals the set $\mathrm{Comp}(T')$ for the theory $T' = \mathrm{Th}(T \cup \Phi)$ axiomatized by $T \cup \Phi$:

$$\mathrm{Comp}(T)_\Phi = \mathrm{Comp}(T \cup \Phi) = \mathrm{Comp}(\mathrm{Th}(T \cup \Phi)).$$

Indeed, if $T' \in \mathrm{Comp}(T)_\Phi$ then T' contains $T \cup \Phi$ implying $T' \in \mathrm{Comp}(T \cup \Phi)$ and $T' \in \mathrm{Comp}(\mathrm{Th}(T \cup \Phi))$. If $T' \in \mathrm{Comp}(T \cup \Phi)$ then T' is a complete theory containing both T and Φ, therefore $T' \in \mathrm{Comp}(T)$ with $\Phi \in T'$, i.e., $T' \in \mathrm{Comp}(T)_\Phi$. Finally, by the definition of the operators Comp and Th we have $\mathrm{Comp}(T \cup \Phi) = \mathrm{Comp}(\mathrm{Th}(T \cup \Phi))$.

By Property 4 for any set Φ of Σ-sentences the subfamily $\mathrm{Comp}(T \cup \Phi)$ of $\mathrm{Comp}(T)$ is d-definable. In particular, if Φ is finite then $\mathrm{Comp}(T \cup \Phi)$ is s-definable in $\mathrm{Comp}(T)$. Therefore, $\mathrm{Comp}(T \cup \Phi)$ is E-closed in $\mathrm{Comp}(T)$ by [6, Theorem 4.3].

In view of the properties above, Proposition 1.8 and Theorem 1.9 we have the following:

Proposition 4.4. *For any ordinals* $\alpha \leq \beta$*, if* $\mathrm{RS}(\mathrm{Comp}(T)) = \beta$ *then*

$$\mathrm{RS}(\mathrm{Comp}(\mathrm{Th}(T \cup \{\varphi\}))) = \alpha$$

for some (α*-ranking*) *sentence* φ*. Moreover, there are* $\mathrm{ds}(\mathrm{Comp}(T))$ *pairwise* $\mathrm{Comp}(T)$*-inconsistent* β*-ranking sentences for* $\mathrm{Comp}(T)$*, and if* $\alpha < \beta$ *then there are infinitely many pairwise* $\mathrm{Comp}(T)$*-inconsistent* α*-ranking sentences for* $\mathrm{Comp}(T)$*.*

Theorem 4.5. *Let* $\mathrm{Comp}(T)$ *be a family of a countable language* Σ *and with* $\mathrm{RS}(\mathrm{Comp}(T)) = \infty$*,* α *be a countable ordinal,* $n \in \omega \setminus \{0\}$*. Then there is a* d_∞*-definable subfamily* $\mathcal{T}^* \subset \mathrm{Comp}(T)$ *such that* $\mathrm{RS}(\mathcal{T}^*) = \alpha$ *and* $\mathrm{ds}(\mathcal{T}^*) = n$*.*

The properties above, Proposition 4.1 and Theorem 1.9 can be naturally transformed for relations $E \in \mathrm{ESR}(\Sigma)$ and E^* by the canonical map $f^{\mathrm{Comp}(T(E))}$.

5. Chains of elementary special and equivalence relations

Now we consider both chains of elementary special relations E and equivalence relations E^*, and chains of theories $T(E)$ in a language Σ.

Clearly, these chains define each others, therefore principally we deal with chains $C = \{E_i \mid i < \lambda\}$ of strictly increasing elementary special relations. We will assume that the chains are *prime*, i.e., $E_i \prec E_{i+1}$ for unlimit steps and $E_i = \bigcup\limits_{j<i} E_j$ for limit steps i.

By the definition the cardinalities of these chains have the upper bounds $|F(\Sigma)|$, i.e., $\max\{|\Sigma|, \omega\}$. Taking into consideration these bounds we introduce the following:

Definition. Let $E, E' \in \mathrm{ESR}(\Sigma)$ with $E \subset E'$. If for a cardinality λ there is a prime chain $C = \{E_i \mid i < \lambda + 1\}$ such that $E_0 = E$ and $E_\lambda = E'$, we say that the pair (E, E') has a *prime λ-chain*.

The set $\{\lambda \mid (E, E')$ has a prime λ-chain$\}$ is called the *prime spectrum* for (E, E') and denoted by $\mathrm{pSp}(E, E')$.

The cardinality $\mathrm{pl}(E, E') = \sup(\mathrm{pSp}(E, E'))$ is called the *longness of prime (E, E')-chains*.

A relation $E \in \mathrm{ESR}(\Sigma)$ is called *prime λ-completeable*, for a cardinality λ, if there is a prime λ-chain $C = \{E_i \mid i < \lambda + 1\}$ such that $E_0 = E$ and $E_\lambda \in \mathrm{CSR}(\Sigma)$, i.e., $\lambda \in \mathrm{pSp}(E, E')$ for some $E' \in \mathrm{CSR}(\Sigma)$.

The cardinality $\mathrm{cpSp}(E) = \bigcup\limits_{E \subset E' \in \mathrm{CSR}(\Sigma)} \mathrm{pSp}(E, E')$ is called the *spectrum of prime completions* of E, and the cardinality

$$\mathrm{cpl}(E) = \sup\{\lambda \mid E \text{ is prime } \lambda\text{-completeable}\}$$

is called the *longness* of prime completion of E.

We assume that $\mathrm{cpSp}(E) = \{0\}$ and $\mathrm{cpl}(E) = 0$ for $E \in \mathrm{CSR}(\Sigma)$.

Clearly, $\mathrm{pSp}(E, E')$ is nonempty with $0 \notin \mathrm{pSp}(E, E')$ for any $E \subset E'$, therefore $\mathrm{pl}(E, E')$ is a positive cardinal. Similarly, $\mathrm{cpSp}(E)$ is nonempty with $0 \notin \mathrm{cpSp}(E)$ for any $E \in \mathrm{ESR}(\Sigma) \setminus \mathrm{CSR}(\Sigma)$, with $\mathrm{cpl}(E) > 0$. Thus, $\mathrm{cpl}(E) = 0$ if and only if $E \in \mathrm{CSR}(\Sigma)$.

By the definition we have $1 \in \mathrm{cpSp}(E)$ if and only if E is precomplete.

Lemma 5.1. *For any* $\mathrm{ESR}(\Sigma)$ *the relation* $\prec$ *is transitive.*

Proof. Let $E_0, E_1, E_2 \in \mathrm{ESR}(\Sigma)$ with $E_0 \prec_{\varphi_1,\psi_1} E_1$, $E_1 \prec_{\varphi_1,\psi_1} E_2$. Since E_2 is forced by $\nabla_{E_0} \cup \{\varphi_1 \leftrightarrow \psi_1, \varphi_2 \leftrightarrow \psi_2\}$ it suffices to find a pair $(\varphi^*, \psi^*) \in E_2^*$ such that $\nabla_{E_0} \cup \{\varphi^* \leftrightarrow \psi^*\}$ forces both $\varphi_1 \leftrightarrow \psi_1$ and $\varphi_2 \leftrightarrow \psi_2$. We can take $(\varphi_1 \leftrightarrow \psi_1) \wedge (\varphi_2 \leftrightarrow \psi_2)$ for φ^* and a tautology for ψ^*. Clearly, $\varphi^* \leftrightarrow \psi^*$ forces the required formulae. □

Lemma 5.1 immediately implies:

Corollary 5.2. *For any* $\mathrm{ESR}(\Sigma)$ *the relation* $\prec \cup \mathrm{id}_{\mathrm{ESR}(\Sigma)}$ *is a partial order.*

The following assertion shows that finite prime (E, E')-chains of prime extensions produce initial segments for values $\mathrm{pSp}(E, E')$

Proposition 5.3. *If* $n \in \mathrm{pSp}(E, E')$ *for a natural number* $n \neq 0$ *then* $n \setminus \{0\} \subseteq \mathrm{pSp}(E, E')$.

Proof. Since $\prec$ is transitive by Lemma 5.1, for each finite prime (E, E')-chain $E_0, E_1, E_2, \dots, E_n$ witnessing $n \in \mathrm{pSp}(E, E')$, for $n > 1$, the (E, E')-chain $E_0, E_2, \dots, E_n$ is prime, too, implying $(n-1) \in \mathrm{pSp}(E, E')$. Continuing the process we obtain $n \setminus \{0\} \subseteq \mathrm{pSp}(E, E')$. □

Proposition 5.3 immediately implies:

Corollary 5.4. *If* $E \subset E'$ *and* $\mathrm{pl}(E, E') \in \omega$ *then* $\mathrm{pSp}(E, E') = n \setminus \{0\}$ *for some* $n \in \omega \setminus \{0, 1\}$.

Corollary 5.5. *If* $E \in \mathrm{ESR}(\Sigma) \setminus \mathrm{CSR}(\Sigma)$ *and* $\mathrm{cpSp}(E) \cap \omega \neq \emptyset$ *then* E *is precomplete.*

Definition. Let $E, E' \in \mathrm{ESR}(\Sigma)$ with $E \subseteq E'$. We say that E' is *finitely axiomatizable* over E if E' is forced by $\nabla_E \cup \{\varphi \leftrightarrow \psi\}$ for some $(\varphi, \psi) \in (E')^*$, i.e. $E \prec_{\varphi,\psi} E'$.

Lemma 5.6. *If* $E' \in \mathrm{Comp}(E)$ *then* E' *is finitely axiomatizable over* E *if and only if* E' *is isolated in* $\mathrm{Comp}(E)$ *by some formula* $\varphi \leftrightarrow \psi$ *with* $(\varphi, \psi) \in (E')^*$.

Proof. If E' is forced by $\nabla_E \cup \{\varphi \leftrightarrow \psi\}$ then $\varphi \leftrightarrow \psi$ forces unique element in $\mathrm{Comp}(E)$ implying that E' is isolated. Conversely, if E' is not isolated in $\mathrm{Comp}(E)$ then there are no formulae $\varphi \leftrightarrow \psi$ forcing E' in $\mathrm{Comp}(E)$. Therefore E' is not finitely axiomatizable over E. □

Theorem 5.7. *For any* $E \in \mathrm{ESR}(\Sigma)$ *the following conditions are equivalent:*
1) $\mathrm{cpSp}(E) \subseteq \omega$;
2) $\mathrm{cpSp}(E) = \{0\}$ *or* $\mathrm{cpSp}(E) = n \setminus \{0\}$ *for some* $n \in \omega \setminus \{0, 1\}$;
3) $\mathrm{Comp}(E)$ *is finite.*

Proof. 1) $\Rightarrow$ 3). We denote by Y the set of all $\varphi \in \mathrm{Sent}(\Sigma)$ such that $T(E) \cup \{\varphi\}$ forces a complete theory $T(E')$, i.e., E' is finitely axiomatizable over E that is E' is isolated by φ in view of Lemma 5.6. Assume on contrary that $\mathrm{Comp}(E)$ is infinite. Then by compactness we have consistent $Z = T(E) \cup \{\neg\varphi \mid \varphi \in Y\}$. Therefore there is a completion $T(E'')$ of Z. We have $E'' \in \mathrm{Comp}(E)$ which is not finitely axiomatizable over E that is non-isolated in $\mathrm{Comp}(E)$ by Lemma 5.6. It means that each prime (E, E'')-chain is infinite implying $\mathrm{cpSp}(E) \not\subseteq \omega$.

Having $|\mathrm{Comp}(E)| < \omega$ we obtain $|\mathrm{Comp}(E)| = 1$, if $E \in \mathrm{CSR}(\Sigma)$, and

$$|\mathrm{Comp}(E)| \geq 2,$$

if $E \notin \mathrm{CSR}(\Sigma)$, since in the latter case E has at least two completions.

3) $\Rightarrow$ 2). If $\mathrm{Comp}(E)$ is finite then completions of $\mathrm{Comp}(E)$ are isolated. So they are finitely axiomatizable. By Proposition 4.2 all prime chains starting with E are finite and moreover have a finite upper bound for lengths. If $E \in \mathrm{CSR}(\Sigma)$ then $\mathrm{Comp}(E) = \{E\}$ and $\mathrm{cpSp}(E) = \{0\}$. Otherwise $\mathrm{cpSp}(E) = n \setminus \{0\}$ for some $n \in \omega \setminus \{0, 1\}$ in view of Proposition 5.3.

2) $\Rightarrow$ 1) is obvious. □

Example 5.8. We realize the values $\mathrm{cpSp}(E) = n \setminus \{0\}$, $n \in \omega$, and the cardinalities for $\mathrm{Comp}(E)$ by induction in the following way.

We take a 0-ary relation P_0 with values $P_0 = 0$ and $P_1 = 1$ producing two completions of the theory $T(E_0)$ for the inessential special relation E_0. Thus, $\mathrm{cpSp}(E_0) = \{1\}$ and $|\mathrm{Comp}(E_0)| = 2$.

If the elementary special relation E_n is already constructed in the language $\Sigma_n = \{P_i \mid i \leq n\}$ of 0-ary predicates with $\mathrm{cpSp}(E_n) = \{1, \ldots, n+1\}$ and $|\mathrm{Comp}(E_n)| = n+2$, we define the special relation E_{n+1} in the language $\Sigma_n = \{P_i \mid i \leq n+1\}$ of 0-ary predicates extending $T(E_n)$ by the sentences $P_j \to P_{n+1}$, $j < n+1$.

We have $\mathrm{cpSp}(E_{n+1}) = \{1, \ldots, n+2\}$ and $|\mathrm{Comp}(E_{n+1})| = n+3$ since there are two possibilities $P_{n+1} = 0$ and $P_{n+1} = 1$ for $P_0 = \ldots = P_n = 0$, and unique possibility $P_{n+1} = 1$ if $P_j = 1$ for some $j \leq n$. These possibilities extend both the longest prime chain for $\mathrm{cpSp}(E_n)$ by one step with $P_{n+1} = 0$, and $\mathrm{Comp}(E_{n+1})$ by the value $P_{n+1} = 0$.

Thus the process above produces realizations for $\mathrm{cpSp}(E) = n \setminus \{0\}$.

If we continue the process by countably many steps then we obtain a relation $E \in \mathrm{ESR}(\Sigma)$ with $\mathrm{cpSp}(E) = \omega + 1$.

Now taking inessential relation $E' \in \mathrm{ESR}(\Sigma')$ for Σ' consisting of 0-ary relations Q_k, $k \in \omega$, we obtain 2^ω prime chains for completions of $T(E')$ each of which has the length omega. Thus, $\mathrm{cpSp}(E') = \{\omega\}$.

If we combine E_n and E' by the sentences $P_i \to Q_k$, $i \leq n$, $k \in \omega$, then the resulted relation $E'' \in \mathrm{ESR}(\Sigma'')$, for $\Sigma'' = \{P_i \mid i \leq n\} \cup \{Q_k \mid k \in \omega\}$, satisfies $\mathrm{cpSp}(E') = ((n+1) \setminus \{0\}) \cup \{\omega\}$.

In view of Theorem 5.7 the possibilities for $\mathrm{cpSp}(E)$, where $E \in \mathrm{ESR}(\Sigma)\setminus\mathrm{CSR}(\Sigma)$ and $|\Sigma| \leq \omega$, are exhausted by the values $n\setminus\{0\}$, $(n\setminus\{0\})\cup\{\omega\}$ for some $n \in \omega\setminus\{0\}$, $\{\omega\}$, $\omega + 1$. Therefore having realizations above for these possibilities we obtain:

Corollary 5.9. *If $|\Sigma| \leq \omega$ and $E \in \mathrm{ESR}(\Sigma) \setminus \mathrm{CSR}(\Sigma)$ then $\mathrm{cpSp}(E)$ has the following possibilities: $n \setminus \{0\}$, $(n \setminus \{0\}) \cup \{\omega\}$ for some $n \in \omega \setminus \{0\}$, $\{\omega\}$, $\omega + 1$.*

References

[1] M.G. Peretyatkin, *Finitely axiomatizable theories*, Plenum Publ. Corp., New York, 1997.
[2] Yu.L. Ershov, *Definability and computability*, Plenum Publ. Corp., New York, 1996.
[3] A. Pillay, *Geometric stability theory*, Clarendon Press, Oxford, 1996.
[4] S.V. Sudoplatov, *Ranks for families of theories and their spectra*, Lobachevskii Journal of Mathematics, **42**:12 (2021), 2959–2968.
[5] N.D. Markhabatov, S.V. Sudoplatov, *Ranks for families of all theories of given languages*, Eurasian Mathematical Journal, **12**:2 (2021), 52–58.
[6] N.D. Markhabatov, S.V. Sudoplatov, *Definable subfamilies of theories, related calculi and ranks*, Siberian Electronic Mathematical Reports, **17** (2020), 700–714.
[7] N.D. Markhabatov, S.V. Sudoplatov, *Algebras for definable families of theories*, Siberian Electronic Mathematical Reports, **16** (2019), 600–608.
[8] N.D. Markhabatov, *Ranks for families of permutation theories*, Bulletin of Irkutsk State University. Series Mathematics, **28** (2019), 86–95.
[9] N.D. Markhabatov, *Pseudofiniteness of locally free algebras*, Algebra and Model Theory 12. Collection of papers, Eds. A. G. Pinus, E. N. Poroshenko, S. V. Sudoplatov, NSTU, Novosibirsk, 2019, 41–46.
[10] S.V. Sudoplatov, *Formulas and properties, their links and characteristics*, Mathematics, **9**:12, 2021, 1391, 16 pp.
[11] B.Sh. Kulpeshov, S.V. Sudoplatov, *Properties of ranks for families of strongly minimal theories*, Siberian Electronic Mathematical Reports, **19**:1 (2022), to appear.
[12] A. S. Morozov, D. A. Tussupov, *Minimal predicates for Δ-definability*, Algebra and Logic, **59**:4 (2020), 328–340.

[13] N.D. Markhabatov, S.V. Sudoplatov, *Topologies, ranks, and closures for families of theories.* I, Algebra and Logic, **59**:6 (2021), 437–455.
[14] N.D. Markhabatov, S.V. Sudoplatov, *Topologies, ranks, and closures for families of theories.* II, Algebra and Logic, **60**:1 (2021), 38–52.
[15] In.I. Pavlyuk, S.V. Sudoplatov, *Formulas and properties for families of theories of Abelian groups*, Bulletin of Irkutsk State University. Series Mathematics, **36** (2021), 95–109.
[16] E.A. Palyutin, J. Saffe, S.S. Starchenko, *Models of superstable Horn theories*, Algebra and Logic, **24**:3 (1985), 171–210.
[17] A. Tarski, *Fundamental Concepts of the Methodology of Deductive Sciences = Fundamentale Begriffe der Methodologie der deduktiven Wissenschaften* : [orig. 1930], Logic, Semantics, Metamathematics. Papers from 1923 to 1938 by Alfred Tarski, transl. by J.H. Woodger, Clarendon Press, Oxford, 1956, 60–109.
[18] D. Myers, *Lindenbaum–Tarski Algebras*, Handbook of Boolean Algebras. Vol. 3, eds. J. D. Monk, R. Bonnet, North-Holland, Amsterdam, New York, Oxford, Tokyo, 1989, 1167–1195.

Sergey Vladimirovich Sudoplatov
Sobolev Institute of Mathematics
Academician Koptyug avenue, 4
630090, Novosibirsk, Russia.
E-mail address: sudoplat@math.nsc.ru

Novosibirsk State Technical University
K. Marx avenue, 20
630073, Novosibirsk, Russia.
E-mail address: sudoplatov@corp.nstu.ru