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ON REPRESENTATIONS AND SIMULATION OF CONDITIONED RANDOM WALKS ON INTEGER LATTICES

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ABSTRACT. We consider a random walk on a multidimensional integer lattice with random bounds on local times. We introduce a family of auxiliary "accompanying" processes that have regenerative structures and play key roles in our analysis. We obtain a number of representations for the distribution of the random walk in terms of similar distributions of the "accompanying" processes. Based on that, we obtain representations for the conditional distribution of the random walk, conditioned on the event that it hits a high level before its death. Under more restrictive assumptions a representation of such type has been obtained earlier by the same authors in a recent paper published in the Springer series on Progress in Probability, 77 (2021), where a certain "limiting" process was used in place of "accompanying" processes of the present paper.

Keywords: conditioned random walk, bounded local times, regenerative sequence, potential regeneration, separating levels, skip-free distributions, accompanying process.

1. INTRODUCTION

Consider a d-dimensional random walk

$$S_t = (S_t[1], \dots, S_t[d]) = S_0 + \sum_{j=1}^{t} \xi_j, \quad t = 0, 1, 2, \dots, \tag{1}$$

on the integer lattice $\mathbb{Z}^d$, where $\xi_j = (\xi_j[1], \dots, \xi_j[d]) \in \mathbb{Z}^d$, $j = 1, 2, \dots$, are i.i.d. random vectors that do not depend on the initial value $S_0 \in \mathbb{Z}^d$. We assume that,

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for each $x \in \mathbb{Z}^d$ and at any time $t = 0, 1, 2, \ldots$, the number of possible/allowed visits to state x is limited above by a counting number $H_t(x) \geqslant 0$. Let

$$T_* = \inf\{t \geqslant 0 : H_t(S_t) = 0\} \leqslant \infty \tag{2}$$

be the first time when the walk visits a state with zero number of possible/allowed visits to it.

We assume also that, at any time $t < T_*$, the random walk jumps from S_{t-1} to S_t and changes the environment at point S_{t-1} by decreasing the number of remaining allowed visits by 1, so that

$$H_t(x) := H_{t-1}(x) - \mathbf{1}\{S_{t-1} = x\} \quad \text{for all} \quad x \in \mathbb{Z}^d, \quad \text{for} \quad 0 \leqslant t \leqslant T_*.$$

If T_* is finite, we assume that the random walk "freezes" at the time instant T_* (or it "dies", or "is killed" at time T_*). Thus, we consider a multidimensional integer-valued random walk in a changing random environment.

Consider, as a natural example, a model of a random walk on atoms of a "harmonic crystal" (see, e.g., [2] and [3]). An electron jumps from one atom to another, taking from a visited atom for the next jump a fixed unit of energy, that cannot be recovered. Thus, if S_t is a position of the electron at time t, then it takes a unit of energy to make the next jump to position $S_{t+1} = S_t + \xi_{t+1}$, which may be in any direction from S_t since the ξ's are signed random variables. We interpret the first coordinate $S_t[1]$ of S_t as its *height* and assume further that the height cannot increase by more than one unit:

$$\xi_t[1] \leqslant 1 \quad a.s., \qquad t = 1, 2, \ldots. \tag{3}$$

When the electron arrives at an atom with insufficient energy level, it "freezes" there.

We may formulate two natural problems. Firstly, to find exact or approximate formulas for the probability of the event B_n that our random walk reaches the level n before it "freezes", i.e.

$$B_n := \{\alpha(n) < T_*\} \tag{4}$$

with $\alpha(n)$ being the hitting time of the level n:

$$\alpha(n) := \inf\{t \geqslant 0 : S_t[1] \geqslant n\} = \inf\{t \geqslant 0 : S_t[1] = n\}, \tag{5}$$

where the latter equality follows from the skip-free property (3). Secondly, given that the random walk is still active by the time of hitting level n, a question of interest is to find exact or approximate formulas for the conditional distribution of its sample path.

To clarify the presentation, we will use the low-case "star" in the probability $\mathbf{P}_*(\cdot)$ in order to underline the influence of the random environment. We omit the "star" in $\mathbf{P}(\cdot)$ if the environment is not involved.

In the previous paper [1] on random walks with local constraints, we show that, under a number of technical assumptions,

$$\mathbf{P}_*(B_n) \sim c_0 q_\infty^n \quad \text{as} \quad n \to \infty, \quad \text{where} \quad 0 < q_\infty \leqslant 1, \quad 0 < c_0 < \infty. \tag{6}$$

Based on that, we prove convergence of the conditional distributions:

$$\mathbf{P}_*((S_0, \ldots, S_K) \in A \mid B_n) \to \mathbf{P}((\overline{S}_0, \ldots, \overline{S}_K) \in A), \tag{7}$$

for any $k = 0, 1, 2, \dots$ and all $A \subset \mathbb{Z}^{(K+1)\times d}$, where $\mathbb{Z}^{(K+1)\times d}$ denotes the space of vectors $\vec{x} = (x_0, x_1, \dots, \vec{x}_K)$ having d-dimensional vectors as their components. Further, we find that the limiting sequence $\{\overline{S}_k\}$ in (7) has a regenerative structure with an infinite sequence of random regenerative levels $\{\overline{\nu}_i\}$ (see Definition 1 below for details) and increases to infinity with a linear speed, i.e.

$$\overline{S}_n[1]/n \to a_1 \in [0,1] \quad \text{a.s.} \qquad \text{as} \quad n \to \infty. \tag{8}$$

Our proofs of results (6) – (8) in [1] are based on establishing a number of representations for the distribution of the random walk $\{S_t\}$ in random environment $\{H_t(x)\}$, that is linked to the distribution of the limiting sequence $\{\overline{S}_t\}$. For example, it is shown in [1] that, under some technical assumptions,

$$\mathbf{P}_*(B_n) = \psi_0 q_\infty^n \mathbf{P}(\overline{B}_n), \quad \text{where} \quad \overline{B}_n := \cup_{m=0}^n \{\overline{\nu}_m = n\}, \tag{9}$$

for a well-defined positive constant ψ_0 and for q_∞ as in (6); and that

$$\mathbf{P}_*((S_0, \dots, S_K) \in A \mid B_n) = \mathbf{P}((\overline{S}_0, \dots, \overline{S}_K) \in A \mid \overline{B}_n), \tag{10}$$

for any $n \geqslant K = 0, 1, 2, \dots$ and all $A \in \mathbb{Z}^{(K+1)\times d}$. Here event $\overline{B}_n$ occurs if (and only if) n is one of the regenerative levels of the limiting random walk.

In the present paper, we find that the limiting process is not the only one for which such representations do exist. We show that there exist random sequences $\{\overline{S}_t\}$, that are called "accompanying" sequences, or processes, and are such that the representation (10) takes place (for any fixed n), together with the formula:

$$\mathbf{P}_*(B_n) = \psi_n(q) q^n \mathbf{P}(\overline{B}_n), \quad \text{for all} \quad q \geqslant q_n > 0, \tag{11}$$

where $0 < q_n \leqslant 1$ and $0 < \psi_n(q) < \infty$ are well-defined constants. We have to underline that in Theorem 1 below and in its corollaries we obtain several representations that are more general than (10) and (11).

There are several advantages in studying accompanying and limiting processes (rather than the initial sequence $\{S_t\}$ and corresponding constraints). We show that the structure of $\{\overline{S}_t\}$:
(a) does not involve any counting constraints,
(b) does not involve an environment,
(c) is regenerative;
(d) limiting and accompanying processes with optimal $q = q_n$ operate with proper distributions only (see (19) for the definition).

There are further advantages in using representations based on accompanying processes with comparison to that based on the limiting process:
(e) these representations take place under less restrictive assumptions,
(f) it is easier to determine the values of parameters in these representations (to determine q_n or $\psi_n(q)$, we need to know only either n or $n+1$ values of probabilities, whereas for determining q_∞ and ψ_0 we need to use two infinite sequences of probabilities(see Subsection 3.1 where these sequences are introduced),
(g) they are more stable in calculation.

The latter means that, for example, if we have a problem with evaluation of q_n, then we may take instead any value $q > q_n$ (preferably, as close to q_n as possible).

There is a number of papers dealing with random walks having constraints on local times. We have already mentioned papers [2] and [3]. In [3], a simple symmetric one-dimensional random walk on the integers ("one-dimensional atoms") has been considered under the assumptions that

$$\mathbf{P}(\xi_1 = 1) = 1/2 = \mathbf{P}(\xi_1 = -1), \quad S_0 = 0 \quad \text{and} \quad H_0(x) = L_0 = const \geqslant 2$$

for all $x \in \mathbb{Z}$. The latter means that initially each atom has a fixed (the same for all) amount of energy L_0. Paper [3] has motivated us to study the model, and, prior to [1], we have made a number of preliminary observations in [4] where we consider a reasonable one-dimensional generalisation of the discrete-time model in [3] with non-random boundary constraints.

In papers [5] and [6], the authors consider a random walk on the line (see also [7] for a generalisation onto a class of Markov processes), assuming that the initial energy level $H_0(x)$ of a point $x > 0$ is a deterministic function of x that increases to infinity with x, and analyse various recurrence/transience properties of the random walk that depend on the shape of the function $H_0(x)$. A generalisation of the model onto random trees may be found in [8]. See also [1] for further references on related topics.

We have to add that a number of known results for conditioned random walks that do not have local-time constraints (see, e.g. [9] and [10]) may be represented, in some particular cases, as corollaries of our results.

To the best of our knowledge, representations of type (9) – (11) do appear only in our past and present papers.

The paper is organized as follows. In Section 2, we introduce the main assumptions on the model and the notions of separating and regenerative levels. In Section 3, we first describe accompanying random sequences and their structure, and, finally, formulate the Representation Theorem and limiting results as its Corollaries. Then Section 4 is devoted to the proofs.

Note that, in (2), (5) and throughout the paper, we follow the standard conventions that

$$\inf \emptyset = \infty, \quad \sup \emptyset = -\infty \quad \text{and} \quad \sum_{k \in \emptyset} a_k = 0. \tag{12}$$

For $n \in \mathbb{Z}$, introduce the following subsets of $\mathbb{Z}^d$:

$$\mathbb{Z}^d_n := \{x = (x[1], x[2], \ldots, x[d]) \in \mathbb{Z}^d : x[1] = n\}, \quad \mathbb{Z}^d_{n+} := \{x \in \mathbb{Z}^d : x[1] \geqslant n\}.$$

In particular, $\mathbb{Z}^d_{n+}$ is a half-space in $\mathbb{Z}^d$.

2. Main Assumptions and Definitions

2.1. **Basic Assumptions.** The following assumptions (**A1**) – (**A3**) are supposed to hold throughout the paper.

(**A1**). The increments $\{\xi_t : t \geqslant 1\}$ of the random walk $\{S_t\}$ from (1) are i.i.d. random vectors taking values in $\mathbb{Z}^d$, and their first components have a skip-free distribution:

$$\sum_{k=-\infty}^{1} \mathbf{P}(\xi_1[1] = k) = 1 \quad \text{and} \quad \mathbf{P}(\xi_1[1] = 1) > 0.$$

(**A2**). The random constraints $\{H_t(x),\, x \in \mathbb{Z}^d\}$ are non-negative integer-valued random variables which may take the infinite value: for any $x \in \mathbb{Z}^d$ and each

$t = 0, 1, 2, \dots,$

$$\sum_{l=0}^{\infty} \mathbf{P}(H_t(x) = l) + \mathbf{P}(H_t(x) = \infty) = 1.$$

Moreover, the next three families of random variables

$$\{S_0;\ H_0(x),\, x \notin \mathbb{Z}^d_{0+}\}, \quad \{\xi_i,\ i \geqslant 1\} \quad \text{and} \quad \{H_0(x),\ x \in \mathbb{Z}^d_{0+}\}$$

are mutually independent, $S_0[1] \leqslant 0$ a.s. and $\mathbf{P}_*(B_0) > 0$.

(**A3**). The family $\{H_0(x) : x \in \mathbb{Z}^d_{0+}\}$ consists of i.i.d. random variables with

$$\mathbf{P}(0 \leqslant H_0(0) \leqslant \infty) = 1 \quad \text{and} \quad \mathbf{P}(H_0(0) = 0) < 1. \tag{13}$$

We say that *Assumption* (**A**) *holds if Assumptions* **(A1)**, **(A2)** *and* **(A3)** *take place.*

Remark 2.1. The first added value of the present paper in comparison with [1] is that we do not use any additional assumptions in the rest of the paper, including our main results. For example, we do not suppose that $\mathbf{E}H_0(0) < \infty$ in the case where our random walk (1) is recurrent.

Remark 2.2. The second added value is that our Assumption **(A3)** is weaker than a similar assumption in [1]. Namely, we allow the event $\{H_0(0) = 0\}$ to have a positive probability in (13).

2.2. Further Notation and Comments.

We have a certain flexibility in the initial value S_0 and in the random environment $\{H_0(x)\}$ outside the set $\mathbb{Z}^d_{0+}$. Recall that we use notation $\mathbf{P}_*(\cdot)$ for probabilities of events when the environment is involved. We will also use a special notation, $\mathbf{P}_0$, instead of $\mathbf{P}_*$ for a particular environments where $S_0 = 0$ and $H_0(0) > 0$. For any event B, let

$$\mathbf{P}_0(B) := \mathbf{P}_*(B \,|\, S_0 = 0,\ H_0(0) > 0,\ H_0(y) = 0\ \ \forall y \notin \mathbb{Z}^d_{0+}). \tag{14}$$

In (14), *it is prohibited for the random walk to visit any states* $y \notin \mathbb{Z}^d_{0+}$.

Definition (14) differs from an analogous definition in [1] since we use Assumption **(A3)** that is weaker than a similar assumption in [1].

Remark 2.3. We may interpret Assumption **(A3)** as follows: at time $t = 0$, the environment in $\mathbb{Z}^d_{0+}$ is stochastically homogeneous (in other words, "virgin"). Further, condition $\mathbf{P}_*(B_0) > 0$ in Assumption **(A2)** may be read as "the random walk S_t arrives at the virgin domain of the random environment with a positive probability."

Remark 2.4. Assumptions **(A1)**–**(A3)** yield that, for any $n \geqslant 0$,

$$\begin{aligned}\mathbf{P}_*\big(B_n\big) &\geqslant \mathbf{P}_*\big(\alpha(0) < T_*,\, \xi_{\alpha(0)+j}[1] = 1,\, H_0(S_{\alpha(0)+j}) > 0, \quad j = 1, \dots, n\big)\\ &= \mathbf{P}_*\big(B_0\big)\mathbf{P}^n\big(\xi_1[1] = 1\big)\mathbf{P}^n\big(H_0(0)\big) > 0,\end{aligned}$$

where the events B_n were introduced in (4). In particular,

$$\mathbf{P}_0(B_0) = 1 \quad \text{and} \quad \mathbf{P}_0(B_n) \geqslant \mathbf{P}^n\big(\xi_1[1] = 1\big)\mathbf{P}^n\big(H_0(0)\big) > 0 \ \ \forall n \geqslant 0.$$

Thus, for any $n \geqslant 0$, the event B_n occurs with a positive probability. Hence, as we can see later, all conditional probabilities in all our main assertions are well-defined.

2.3. **Regenerative and n-separating Levels.** Recall that we have introduced d-dimensional random vectors

$$\overline{S}_t = (\overline{S}_t[1], \dots, \overline{S}_t[d]) \in \mathbb{Z}^d, \qquad t = 0, 1, 2, \dots,$$

on the integer lattice $\mathbb{Z}^d$; and let $\overline{\alpha}(n)$ be the hitting time of level n:

$$\overline{\alpha}(n) := \inf\{t \geqslant 0 : \overline{S}_t[1] \geqslant n\}, \qquad n = 0, 1, 2, \dots.$$

In what follows, a "block" is any collection of random variables that may contain a random number of these variables.

Definition 1. *A random sequence $\overline{S} = (\overline{S}_0, \overline{S}_1, \dots)$ is regenerative with regenerative levels $\overline{\nu}_0 < \overline{\nu}_1 < \dots < \overline{\nu}_n < \dots$, if $\{\overline{\nu}_i\}$ is an infinite sequence of integer-valued random variables such that, firstly, the following "blocks" of random variables*

$$\{\overline{\nu}_i - \overline{\nu}_{i-1},\, \overline{\alpha}(\overline{\nu}_i) - \overline{\alpha}(\overline{\nu}_{i-1}),\, \big(\overline{S}_{\overline{\alpha}(\overline{\nu}_{i-1})+t} - \overline{S}_{\overline{\alpha}(\overline{\nu}_{i-1})},\, t = 1, 2, \dots, \overline{\alpha}(\overline{\nu}_i) - \overline{\alpha}(\overline{\nu}_{i-1})\big)\},$$

where $i = 1, 2, \dots,$ are i.i.d. and do not depend on the following initial "block" $\{\overline{\nu}_0,\, \overline{\alpha}(\overline{\nu}_0),\, \big(\overline{S}_t; t \leqslant \overline{\alpha}(\overline{\nu}_0)\big)\}$; and, secondly,

$$\inf_{t \geqslant \overline{\alpha}(\overline{\nu}_i)} \overline{S}_t[1] = \overline{S}_{\overline{\alpha}(\overline{\nu}_i)}[1] = \overline{\nu}_i > \sup_{0 \leqslant t < \overline{\alpha}(\overline{\nu}_i)} \overline{S}_t[1], \quad i = 0, 1, 2, \dots.$$

We then say that $\overline{\alpha}(\overline{\nu}_i)$ is the *regenerative time* that corresponds to regenerative level ν_i.

Note that analogous constructions of "blocks" do appear (however, in "non-conditional" settings) in a variety of stochastic processes, see e.g. [11] or [12] or [13].

Definition 2. *A number $k \in \{0, 1, \dots, n\}$ is an "n-separating level" of the sequence $\{S_0, S_1, S_2, \dots\}$ if*

$$S_{\alpha(k)}[1] \leqslant \inf_{\alpha(k) < t < \alpha(n)} S_t[1] \qquad \textit{and} \quad n \geqslant 0.$$

It is useful to recall that $\sup_{0 \leqslant t < \alpha(k)} S_t[1] < k = S_{\alpha(k)}[1]$, for all $k \geqslant 0$.

For any $n \geqslant 0$ such that $\alpha(n) < T_*(n)$, let $\eta_*(n)$ count the number of n-separating levels, excluding n itself. And we let $\eta_*(n) = -1$ when $T_*(n) \leqslant \alpha(n)$. Since n is the last n-separating level, the variable $\eta_*(n) + 1$ counts the number of n-separating levels in the case when the event $B_n = \{\eta_*(n) \geqslant 0\} = \{\alpha(n) < T_*(n)\}$ occurs.

Remark 2.5. One can see that if k is an n-separating level, then it may not be an N-separating level for some $N > n$. For example, $k = n$ is always the last n-separating level if $\alpha(n)$ is finite, but it is not an $(n+1)$-separating level if $S_{\alpha(n)+1}[1] < 0$.

These levels play an important role in our analysis. One can view n-separating levels as "potential candidates" for regenerative levels and talk about "potential regeneration".

2.4. **Random Blocks and Vector Notation.** For $n > 0$ with $\eta_*(n) \geqslant 0$, let

$$0 \leqslant \nu_0(n) < \dots < \nu_{\eta_*(n)}(n) = n$$

be the sequence of all n-separating levels (where $\nu_0(n) = \nu_{\eta_*(n)}(n) = n$ if $\eta_*(n) = 0$). For $n > 0$ with $\eta_*(n) \geqslant i \geqslant 1$, we let

$$\lambda_i(n) := \nu_i(n) - \nu_{i-1}(n), \quad T_i(n) := \alpha(\nu_i(n)), \quad \tau_i(n) := T_i(n) - T_{i-1}(n).$$

We need more notation. Introduce the random vectors

$$\vec{S}_K = (S_0, \dots, S_K), \quad \vec{S}_{K,N} = (S_{K,K+1}, \dots, S_{K,N}), \quad N > K \geqslant 0, \tag{15}$$

where

$$S_{K,K+t} := S_{K+t} - S_K = \sum_{j=1}^{t} \xi_{K+j}, \quad t = 0, 1, \ldots. \tag{16}$$

On the event $B_n = \{\eta_*(n) \geqslant 0\}$, introduce a random block $(\nu_0(n), T_0(n), \vec{S}_{T_0(n)})$. This is the *initial block* of our random walk. Further, if $\eta_*(n) \geqslant 1$, then we may introduce consecutive blocks of random variables:

$$(\nu_0(n), T_0(n), \vec{S}_{T_0(n)}), \qquad (\lambda_i(n), \tau_i(n), \vec{S}_{T_{i-1}(n),T_i(n)}), \quad i = 1, 2, \ldots, \eta_*(n), \tag{17}$$

where $\lambda_i(n)$ is the *height* of the i-th block and $\tau_i(n)$ its *duration*. Property 1 below shows that there is a certain conditional independence of each block in (17) from the previous blocks. This property is the base for the proof of our main Theorem 1 below.

3. Main Results

3.1. **Technical Conditions.** We say that *Assumption* (**B**) *holds* if Assumption **(A)** takes place and, in addition, we are given a real number q and an integer M such that

$$0 < M, q < \infty \qquad \text{and} \qquad P_M(q) := \sum_{l=1}^{M} \mathbf{P}_0(\eta_*(l) = 1)/q^l \leqslant 1. \tag{18}$$

In this case

$$0 < \psi_M(q) := \sum_{k=0}^{M} \mathbf{P}_*(\eta_*(k) = 0)/q^k < \infty.$$

It is easy to see that (18) may be rewritten in the following form:

$$0 < M < \infty \quad \text{and} \quad q \geqslant q_M > 0, \quad \text{where} \quad P_M(q_M) = 1. \tag{19}$$

It is clear, that such a q_M exists and is unique, for each finite M.

For $N = \infty$, we say that *Assumption* ($\mathbf{B}_\infty$) *takes place if Assumption* **(A)** *holds and we are given a real number* $q > 0$ *such that*

(20)

$$P_\infty(q) := \sum_{l=1}^{\infty} \mathbf{P}_0(\eta_*(l) = 1)/q^l \leqslant 1 \quad \text{and} \quad \psi_\infty(q) := \sum_{k=0}^{\infty} \mathbf{P}_*(\eta_*(k) = 0)/q^k < \infty.$$

Then (20) may be rewritten in the following form:

$$M = \infty \quad \text{and} \quad q \geqslant q_\infty > 0, \quad \text{where} \quad P_\infty(q_\infty) \leqslant 1 \leqslant P_\infty(q_\infty + 0).$$

Such a q_∞ exists, but, in general, it may appear that $P_\infty(q_\infty) < 1$.

Remark 3.1. It is proved in [1] that, under more restrictive assumptions than **(A)**, there exists a unique number $q_\infty \in (0, 1]$ such that

$$P_\infty(q_\infty) = 1, \qquad 0 < \psi_0 := \psi_\infty(q_\infty) < \infty, \tag{21}$$

and that

$$0 < \mu := \sum_{n=1}^{\infty} n\mathbf{P}_0(\eta_*(n) = 1)/q^n < \infty. \tag{22}$$

We conjecture that this result may not hold under Assumption **(A)** on its own.

3.2. **Transformed Distributions of Random Blocks.** Suppose that either Assumption **(B)** or Assumption $(\mathbf{B}_\infty)$ takes place. We will introduce an infinite sequence

$$(\overline{\nu}_0, \overline{T}_0, \widetilde{S}_{\overline{T}_0}) \quad \text{and} \quad (\overline{\lambda}_i, \overline{\tau}_i, \widetilde{Y}_{i,\overline{\tau}_i}), \quad i = 1, 2, \ldots, \tag{23}$$

of mutually independent random blocks with special distributions, where

$$\widetilde{S}_{\overline{T}_0} = (\overline{S}_0, \ldots, \overline{S}_{\overline{T}_0}) \quad \text{and} \quad \widetilde{Y}_{i,\overline{\tau}_i} = (\overline{Y}_{i,1}, \ldots, \overline{Y}_{i,\overline{\tau}_i}) \tag{24}$$

are random vectors of random lengths. We determine their distributions step by step. First, we let

$$\mathbf{P}(\overline{\nu}_0 = k) = \mathbf{P}_*(\eta_*(k) = 0)/(\psi_N(q)q^k), \quad k = 0, 1, \ldots, M; \tag{25}$$

$$\mathbf{P}(\overline{\lambda}_i = l) = \mathbf{P}_0(\eta_*(l) = 1)/q^l, \quad l = 1, \ldots, M. \tag{26}$$

Thus, we have determined the distributions of random vectors $\overline{\nu}_0$ and $\overline{\lambda}_i$ as the Cramér-type transforms of the characteristics of the initial random walk $\{S_t\}$. It follows from (25) that the random vector $\overline{\nu}_0$ has a proper distribution, whereas the distribution of $\overline{\lambda}_i$ may be improper and

$$\mathbf{P}(\overline{\lambda}_1 = 1) = \mathbf{P}_0(\eta_*(1) = 1)/q \geqslant \mathbf{P}(\xi_1[1] = 1)/q > 0.$$

We determine next the distributions of other components of the vectors in (23). We let

$$\mathbf{P}(\overline{T}_0 = K, \widetilde{S}_K = \vec{y}_K | \overline{\nu}_0 = k) := \mathbf{P}_*(\alpha(k) = K < T_*, \vec{S}_K = \vec{y}_K | \eta_*(k) = 0), \tag{27}$$

for any $K \geqslant k + 1 \geqslant 1$ and $\vec{y}_K := (y_0, \ldots, y_K) \in \mathbb{Z}^{(K+1)\times d}$; and then

$$\mathbf{P}(\overline{\tau}_i = L, \widetilde{Y}_{i,L} = \vec{x}_L | \overline{\lambda}_i = l) := \mathbf{P}_0(\alpha(l) = L < T_*, \vec{S}_{0,L} = \vec{x}_L | \eta_*(l) = 1), \tag{28}$$

for any $L \geqslant l \geqslant 1$ and $\vec{x}_L := (x_1, \ldots, x_L) \in \mathbb{Z}^{L\times d}$.

Remark 3.2. We have introduced all joint distributions of random elements from (23). The distribution of the initial block $(\overline{\nu}_0, \overline{T}_0, \widetilde{S}_{\overline{T}_0})$ is proper, since it is determined by proper distributions from (25) and (27). By the construction, with probability 1,

$$\overline{\nu}_0 \geqslant 0, \quad \overline{T}_0 \geqslant 0, \text{ and } \overline{\tau}_i \geqslant \overline{\lambda}_i = \widetilde{Y}_{i,\overline{\tau}_i}[1] \geqslant 1, \quad \text{for all } i \geqslant 1.$$

Moreover, the random vectors $\{(\overline{\lambda}_i, \overline{\tau}_i, \widetilde{Y}_{i,\overline{\tau}_i}), \ i = 1, 2, \ldots\}$ are i.i.d., but they may have improper distributions when $P_M(q) < 1$.

3.3. **Sample-path construction of accompanying random sequences.** Using mutually independent random blocks introduced in (23), we may define random variables

$$\overline{\nu}_m = \overline{\nu}_0 + \sum_{i=1}^{m} \overline{\lambda}_i > \overline{\nu}_{m-1}, \quad \overline{T}_m = \overline{T}_0 + \sum_{i=1}^{m} \overline{\tau}_i > \overline{T}_{m-1}, \quad m = 1, 2, \ldots.$$

Now we introduce random vectors $\overline{S}_j$ for all $j \geqslant 0$ using the induction argument. For $j \leqslant \overline{T}_0$, the vectors are given in (24). Suppose we have defined $\overline{S}_j$ for all $j \leqslant \overline{T}_{i-1}$. Then we let

$$\overline{S}_{\overline{T}_{i-1}+j} := \overline{S}_{\overline{T}_{i-1}} + \overline{Y}_{i,j}, \quad j = 1, \ldots, \overline{\tau}_i = \overline{T}_i - \overline{T}_{i-1}. \tag{29}$$

Thus, we have defined $\overline{S}_j$ for all $j \leqslant \overline{T}_i$. Repeating this procedure for all $i = 1, 2, \ldots$ we define random vectors $\overline{S}_j$ for all $j \geqslant 0$.

Similarly to (15), we introduce vectors with multivariate components:

$$\widetilde{S}_N = (\overline{S}_0, \dots, \overline{S}_N), \quad \widetilde{S}_{K,N} = (\overline{S}_{K+1} - \overline{S}_K, \dots, \overline{S}_N - \overline{S}_K), \quad N > K \geqslant 0.$$

Consider now the random blocks

$$(\overline{\nu}_0, \overline{T}_0, \widetilde{S}_{\overline{T}_0}) \quad \text{and} \quad (\overline{\lambda}_i, \overline{\tau}_i, \widetilde{S}_{\overline{T}_{i-1}, \overline{T}_i}), \quad i = 1, 2, \dots, \tag{30}$$

and note that, by (29), the i-th block in (30) coincides with the i-th block in (23). Thus, all blocks in (30) are mutually independent and all of them are i.i.d. (apart from the initial block).

3.4. **Representation Theorem.** We are now ready to present our main results. The following statement summarises the main structural properties of the accompanying random sequences and provides an inverse formula for the distributions of the random walk in terms of the accompanying processes.

Let $\mathbb{Z}_*^d := \cup_{n=1}^{\infty} \mathbb{Z}^{n\times d}$. We consider $\mathbb{Z}_*^d$ as the state space for random sequences of random lengths.

Theorem 1. *Suppose that either Assumption* ($\mathbf{B}$) *or Assumption* ($\mathbf{B}_\infty$) *takes place and that* $n < \infty$*. Then, for any set* $\mathcal{A} \subset \mathbb{Z}_*^d$ *and for each* $M \geqslant n \geqslant m \geqslant 0$*,*

$$\begin{aligned} &\mathbf{P}_*(\alpha(n) < T_*,\, \eta_*(n) = m,\, (S_0, \dots, S_{\alpha(n)}) \in \mathcal{A}) \\ &= \psi_M(q) q^n \mathbf{P}(\overline{\nu}(m) = n,\, (\overline{S}_0, \dots, \overline{S}_{\overline{\alpha}(n)}) \in \mathcal{A}). \end{aligned} \tag{31}$$

Remark 3.3. As it follows from the theorem, when $n \leqslant M$ and $0 < n < \infty$, the distribution of the part $(\overline{S}_0, \dots, \overline{S}_{\overline{\alpha}(n)})$ of the trajectory of each accompanying random sequence has the same support with the distribution of the corresponding part $(\overline{S}_0, \dots, \overline{S}_{\overline{\alpha}(n)})$ of the trajectory of the initial random walk (any finite sample path of this type has positive probabilities to occur simultaneously for the accompanying sequences and for the random walk, however these probabilities may differ). In particular, for all $j = 1, 2, \dots$, the following inequalities hold with probability 1:

$$\overline{\xi}_j[1] \leqslant 1 \quad \text{and} \quad \overline{S}_j[1] \leqslant j, \quad \text{where} \quad \overline{\xi}_j = \overline{S}_j - \overline{S}_{j-1}.$$

Since $B_n = \{\alpha(n) < T_*\} = \{0 \leqslant \eta_*(n) \leqslant n\}$, we have from (31) that, for any set $A \subset \mathbb{Z}^{(k+1)\times d}$,

$$\mathbf{P}_*((S_0, \dots, S_k) \in A,\, B_n) = \sum_{m=0}^{n} \mathbf{P}_*((S_0, \dots, S_k) \in A,\, \eta_*(n) = m) \tag{32}$$

$$= \sum_{m=0}^{n} \psi_M(q) q^n \mathbf{P}((\overline{S}_0, \dots, \overline{S}_k) \in A,\, \overline{\nu}(m) = n) = \psi_M(q) q^n \mathbf{P}((\overline{S}_0, \dots, \overline{S}_k) \in A,\, \overline{B}_n)$$

Now (11) follows from (32) with $A = \mathbb{Z}^{(k+1)\times d}$. Equating the ratio of the left-hand sides of (32) and (9) to the ratio of the right-hand sides leads to (10). Thus, we have proved

Corollary 1. *Suppose that either Assumption* ($\mathbf{B}$) *or Assumption* ($\mathbf{B}_\infty$) *takes place and that* $n < \infty$*. Then relations* (10), (11) *and* (32) *hold for any set* $A \subset \mathbb{Z}^{(k+1)\times d}$ *when* $M \geqslant n \geqslant k \geqslant 0$*.*

3.5. **Limiting Results.** Assume that, for a fixed n, we are going to evaluate one of the probabilities in the left-hand sides of formulas (10), (11), (31) or (32). Then it makes sense to use Assumption (**B**) with $M = n$ and $q = q_M$; and to take some $q > q_M$ if we have problems with determining q_M. On the other hand, if we like to investigate the asymptotic behavior of these probabilities, we have to take $M = \infty$ and use a fixed $q \geqslant q_\infty$. However, we can proceed only in the case where $q = q_\infty$ and conditions (21) and (22) take place, since this is the only case where the random variable $\overline{\lambda}_1$ has a proper distribution with $0 < \mathbf{E}\overline{\lambda}_1 < \infty$. For example, in this case we can see from (32) that

$$\mathbf{P}(\overline{B}(n)\,|\,\overline{\nu}_0 = 0) = \mathbf{P}(\overline{\nu}_m = n \quad \text{for some} \quad m \geqslant 0\,|\,\overline{\nu}_0 = 0)$$
$$= \sum_{m=0}^{n} \mathbf{P}(\overline{\nu}_i = n\,|\,\overline{\nu}_0 = 0) = V_n := \mathbf{I}\{n = 0\} + \sum_{m=1}^{n} \mathbf{P}\left(\sum\nolimits_{i=1}^{m} \overline{\lambda}_i = n\right)$$

is the renewal function of the undelayed renewal process with i.i.d. increments $\{\overline{\lambda}_i,\ i = 1, 2, \dots\}$. Then, by the local renewal theorem, as $n \to \infty$,

$$V_n \to 1/\mu, \quad \mathbf{P}(\overline{B}_n) \to 1/\mu \quad \text{and} \quad \mathbf{P}_*(B_n)/q^n \to c_0 := \psi_0/\mu. \tag{33}$$

Repeating now the elementary arguments used in Subsection 3.5 of [1], we obtain that

$$\mathbf{P}_*((S_0, \dots, S_K) \in A,\ B_n)/q^n \to c_0 \mathbf{P}((\overline{S}_0, \dots, \overline{S}_K) \in A) \tag{34}$$

as $n \to \infty$. Thus, we have proved the following result.

Theorem 2. *Suppose that Assumption* ($\mathbf{B}_\infty$) *and conditions* (21) *and* (22) *take place. Then* $0 < q_\infty \leqslant 1$ *and, for all* $K \geqslant 0$ *and any* $A \subset \mathbb{Z}^{(K+1)\times d}$, *the convergences* (34), (33) *and* (7) *hold as* $n \to \infty$, *together with the asymptotics* (6). *In addition, by the Strong Law of Large Numbers,* (8) *holds with* $a_1 := \mu/\mathbf{E}\overline{\tau}_1 \in [0, 1]$.

Remark 3.4. Formally, the presented Theorem 2 is a more general result than Theorem 3 in [1], which was the main result there. However, that Theorem 3 from [1] has an evident advantage: its conditions are simpler and easier to be verified.

4. Proofs

4.1. **Properties of Trajectories and Additional Notation.** For a finite or infinite sequence $\vec{y} = (y_0, y_1, y_2, \dots)$ of $\mathbb{Z}^d$-valued vectors and for any $n \geqslant 0$, we let

$$\alpha(n|\vec{y}) := \inf\{t \geqslant 0 : y_t[1] \geqslant n\} \leqslant \infty,$$

where $y_t[1]$ is the first coordinate of y_t, for $t = 0, 1, \dots$. Recall that we follow the standard conventions (12).

Definition 3. *A number* $k \in \{0, 1, \dots, n\}$ *is an "n-separating level" of the sequence* $\vec{y}$ *if*

$$\sup_{0 \leqslant t < \alpha(k|\vec{y})} y_t[1] < k = y_{\alpha(k|\vec{y})}[1] \leqslant \inf_{\alpha(k|\vec{y}) < t < \alpha(n|\vec{y})} y_t[1] \quad \textit{and} \quad \alpha(n|\vec{y}) < \infty.$$

For $n \geqslant 0$, let $\eta(n|\vec{y}) + 1$ count the number of n-separating levels; and let $\varkappa(n|\vec{y})$ be the supremum of all $k < n$ such that k is an n-separating level.

For any vector of a form $\vec{y}_N = (y_0, \dots, y_N)$, we use below notations of the following type:

(35)
$$\vec{y}_K = (y_0, \dots, y_K), \quad \vec{y}_{K,N} = (y_{K+1} - y_K, \dots, y_N - y_K), \quad \text{when} \quad N > K \geqslant 0.$$

Note that all notations and definitions introduced in this subsection are similar to those introduced earlier in (15) for the vector $\vec{S}_t$.

We need a number of additional notation that allow us to clarify the role of the changing random environment used in our probabilities $\mathbf{P}_*(\cdot)$. Note that

(36)
$$\{T_* > N\} = \{h(\vec{S}_N) \geqslant 0\}, \quad \text{where} \quad h(\vec{S}_N) := \min_{0 \leqslant j \leqslant N} H_j(S_j).$$

In what follows, we consider a random walk that starts at time $t \geqslant 0$ from a state x, rather that at time $t = 0$ from the state S_0. The following notation will be helpful:

(37)
$$\alpha_t(l) = \inf\{j \geqslant 0 : S_{t,t+j}[1] = l\}, \quad s(K, N) := \inf_{0 \leqslant j \leqslant N-K} S_{K,K+j}[1],$$

(38)
$$h_t(x, K, N) := \inf_{0 \leqslant j \leqslant N-K} H_{t+j}(x + S_{K,K+j}),$$

for $N > K, t, l \geqslant 0$, where notation $S_{t,j} := S_j - S_t$ for $t \geqslant j$ was introduced earlier in (16). Note that

$$\forall\, l \geqslant 0 \quad \alpha_0(l) = \alpha(l) \qquad \text{iff} \quad S_0 = 0.$$

Later on we will use the following properties of notation from (37) and (38):

(39)
$$\{S_K = x, \, T_* > N\} = \{S_K = x, \, h(\vec{S}_N) \geqslant 0\}$$
$$= \{S_K = x, \, h(\vec{S}_{K-1}) \geqslant 0, \, h_K(x, K, N) \geqslant 0\} \quad \text{if} \quad N > K \geqslant 0.$$

In what follows, we use notation $\mathbf{P}(\cdot)$ instead of $\mathbf{P}_*(\cdot)$ in the cases where we describe the role of the environment by using more explicit characteristics $h(\cdot)$ or $h_\bullet(\cdot, \cdot, \cdot)$ introduced in (36) and (38) instead of T_*.

4.2. **Important Property of n-Separating Levels.** In this subsection we fix integers:

(40)
$$N \geqslant n > k > 0, \quad N > K \geqslant k > 0 \quad \text{and} \quad \vec{y}_N = (y_0, \dots, y_N) \in \mathbb{Z}^{(N+1) \times d},$$

and use notation (35). Our aim is to find a convenient formula for the probability of the following event

(41)
$$\mathcal{D}_{K,N} := \{\alpha(k) = K < \alpha(n) = N < T_*, \, s_{K,N} \geqslant 0, \, \vec{S}_N = \vec{y}_N\}.$$

It is clear that k is an n-separating level if $\mathcal{D}_{K,N}$ occurs.

Property 1. *Under Assumption* $(\mathbf{A})$,

(42)
$$\mathbf{P}_*(\mathcal{D}_{K,N}) = \mathbf{P}_*(\alpha(k) = K < T_*, \, \vec{S}_K = \vec{y}_K)$$
$$\cdot \mathbf{P}_0(\alpha(n-k) = N - K < T_*, \, \vec{S}_{0,N-K} = \vec{y}_{K,N})$$

for all integers that satisfy (40).

The proof of this statement is carried out in several steps. For any $t \geqslant 0$ and $x \in \mathbb{Z}^d$ introduce events:

(43)
$$\mathcal{A}_{k,K} := \{h(\vec{S}_{K-1}) \geqslant 0, \, \alpha(k) = K, \, \vec{S}_K = \vec{y}_K\},$$
$$\mathcal{C}_t(x, k, K, n, N) := \{h_t(x, \vec{S}_{K,N}) \geqslant 0, \alpha_K(n-k) = N - K, s_{K,N} \geqslant 0, \vec{S}_{K,N} = \vec{y}_{K,N}\}.$$

Lemma 1. *Under the assumptions of Property 1,*

$$\mathbf{P}_*(\mathcal{D}_{K,N}) = \mathbf{P}(\mathcal{A}_{k,K}) \cdot \mathbf{P}(\mathcal{C}_0(y_K, k, K, n, N)). \tag{44}$$

In addition,

$$\mathbf{P}_*(\alpha(k) = K < T_*,\ \vec{S}_K = \vec{y}_K) = \mathbf{P}(\mathcal{A}_{k,K}) \cdot \mathbf{P}(H_0(0) > 0). \tag{45}$$

Proof. Using property (39), we obtain from definitions (41) and (43) that

$$\mathcal{D}_{K,N} = \mathcal{A}_{k,K} \cap \mathcal{C}_K(y_K, k, K, n, N). \tag{46}$$

Random variables $\alpha(k)$, $\vec{S}_K$ and $h(\vec{S}_{K-1})$ that define the event $\tilde{A}_{k,K}$ are functions only of random variables from the following two families:

$$\{\xi_j : j \leqslant K\} \quad \text{and} \quad \{H_0(y) : y \notin \mathbb{Z}^d_{k+}\}. \tag{47}$$

On the other hand, all random variables that determine the event $\mathcal{C}_K(y_K, k, K, n, N)$, are functions only of random variables from the following two families:

$$\{\xi_j : j > K\} \quad \text{and} \quad \{H_K(y) : y \in \mathbb{Z}^d_{k+}\}.$$

Note that if the event $\mathcal{A}_{k,K}$ occurs, then the environment in the half-space $\mathbb{Z}^d_{k+}$ remains virgin at time K. Hence,

$$\forall\ y \in \mathbb{Z}^d_{k+} \quad H_K(y) = H_0(y) \quad \text{and} \quad h_K(y_K, \vec{S}_{K,N}) = h_0(y_K, \vec{S}_{K,N}).$$

Here we have used definition (38). Thus, $\mathcal{C}_K(y_K, k, K, n, N) = \mathcal{C}_0(y_K, k, K, n, N)$ by (43) when $\mathcal{A}_{k,K}$ occurs, and we may rewrite (46) in the following way:

$$\mathcal{D}_{K,N} = \mathcal{A}_{k,K} \cap \mathcal{C}_K(y_K, k, K, n, N) = \mathcal{A}_{k,K} \cap \mathcal{C}_0(y_K, k, K, n, N). \tag{48}$$

Now, all random variables that determine the event $\mathcal{C}_0(y_K, k, K, n, N)$, are functions only of variables from the following two families:

$$\{\xi_j : j > K\} \quad \text{and} \quad \{H_0(y) : y \in \mathbb{Z}^d_{k+}\}. \tag{49}$$

Since the families in (49) and (47) do not overlap, they are independent. Hence, events $\tilde{A}_{k,K}$ and $\mathcal{C}_0(y_K, k, K, n, N)$ are independent too. This independence allows us to obtain (44) from (48).

Next, it is not difficult to see from (39) and (43) that

$$\mathbf{P}_*(\alpha(k) = K < T_*,\ \vec{S}_K = \vec{y}_K) = \mathbf{P}(\mathcal{A}_{k,K},\ H_K(y_K) > 0).$$

Here random variable $H_K(y_K) = H_0(y_K)$ belongs to the second family in (49). Hence, events $\mathcal{A}_{k,K}$ and $\{H_0(y_K) = 0\}$ are independent too. Thus, (45) follows since $\mathbf{P}(H_0(y_K) > 0) = \mathbf{P}(H_0(0) > 0)$. $\square$

Lemma 2. *Under the assumptions of Property 1,*

$$\mathbf{P}(\mathcal{C}_0(y_K, k, K, n, N)) = \mathbf{P}(\mathcal{C}_0(0, 0, 0, n-k, N-K)). \tag{50}$$

Proof. Note that, by (16), families of random variables

$$\{S_{K,K+t} = \sum_{j=1}^{t} \xi_{K+j},\ t = 1, \ldots, N-K\} \quad \text{and} \quad \{S_{0,t} = \sum_{j=1}^{t} \xi_j,\ t = 1, \ldots, N-K\}$$

are identically distributed. Hence, random vectors $\vec{S}_{K,N}$ and $\vec{S}_{0,N-K}$ have the same distributions. For the fixed environment, the indicator of the event $\mathcal{C}_0(y_K, k, K, n, N)$

is a function of $\vec{S}_{K,N}$. Thus it is identically distributed with the indicator of the event $\mathcal{C}_0(y_K, 0, 0, n-k, N-K)$, and we have:

$$\mathbf{P}(\mathcal{C}_0(y_K, k, K, n, N)) = \mathbf{P}(\mathcal{C}_0(y_K, 0, 0, n-k, N-K)). \tag{51}$$

Now note that the probability in the right-hand side of (51) does not depend on the choise of the point y_K in the virgin environment. For example, we may choose the "standard" point 0 instead of y_K to obtain

$$\mathbf{P}(\mathcal{C}_0(y_K, 0, 0, n-k, N-K)) = \mathbf{P}(\mathcal{C}_0(0, 0, 0, n-k, N-K)).$$

The latter fact and (51) imply (50). □

Proof of Property 1. Substituting (45) and (50) into (44), we obtain that

$$\mathbf{P}_*(\mathcal{D}_{K,N}) = \mathbf{P}_*(\alpha(k) = K < T_*,\ \vec{S}_K = \vec{y}_K) \cdot \frac{\mathbf{P}(\mathcal{C}_0(0, 0, 0, n-k, N-K))}{\mathbf{P}(H_0(0) > 0)}. \tag{52}$$

Note that the first component of the vector $\vec{S}_{K,N}$ equals to zero by (16). Hence

$$\begin{aligned}\mathbf{P}(\mathcal{C}_0(0, 0, 0, n-k, N-K)) &= \mathbf{P}(\mathcal{C}_0(0, 0, 0, n-k, N-K)|S_0 = 0)\\ &= \mathbf{P}(H_0(0) > 0) \cdot \mathbf{P}_0(\alpha_0(n-k) = N-K < T_*,\ \vec{S}_{0,N-K} = \vec{y}_{K,N}).\end{aligned}$$

Then, the latter fact and equality (52) imply the assertion (42) of Property 1 because $\alpha_0(n-k) = \alpha(n-k)$ under probability $\mathbf{P}_0(\cdot)$. □

4.3. **Properties of Transformed Distributions.** In the rest of the paper we suppose that either Assumption ($\mathbf{B}$) or Assumption ($\mathbf{B}_\infty$) takes place.

Lemma 3. *For arbitrary numbers*

$$N \geqslant n \geqslant 0 \quad \textit{and} \quad M \geqslant n, \tag{53}$$

suppose that we are given a vector $\vec{y}_N = (y_0, \ldots, y_N) \in \mathbb{Z}^{(N+1)\times d}$ *such that*

$$\alpha(n|\vec{y}_N) = N \geqslant 0 \quad \textit{and} \quad \eta(n|\vec{y}_N) = 0. \tag{54}$$

Then

$$\mathbf{P}_*(\vec{S}_N = \vec{y}_N) = \psi_M(q) q^n \mathbf{P}(\overline{\nu}(0) = n, \overline{\alpha}(n) = N, \widetilde{S}_N = \vec{y}_N). \tag{55}$$

Proof. Using conditions (53) and (54), it is easy to see that

$$\begin{aligned}\mathbf{P}_*(\vec{S}_N = \vec{y}_N) &= \mathbf{P}_*(\alpha(n) = N < T_*, \eta_*(n) = 0, \vec{S}_N = \vec{y}_N)\\ &= \mathbf{P}_*(\eta_*(n) = 0) \cdot \mathbf{P}_*(\alpha(n) = N < T_*,\ \vec{S}_N = \vec{y}_N|\eta_*(n) = 0).\end{aligned} \tag{56}$$

Substituting now (25) and (27) with $k = n$ and $K = N$ into (56), we obtain (55). □

Lemma 4. *For arbitrary numbers*

$$L \geqslant l \geqslant 1 \quad \textit{and} \quad M \geqslant l, \tag{57}$$

suppose that we are given a vector $\vec{x}_L = (x_1, \ldots, x_L) \in \mathbb{Z}^{(L+1)\times d}$ *such that*

$$\alpha\big(l|(0, \vec{x}_L)\big) = L \quad \textit{and} \quad \varkappa\big(l|(0, \vec{x}_L)\big) = 0, \tag{58}$$

where we use vector $(0, \vec{x}_L) := (0, x_1, \ldots, x_L)$. *Then*

$$\mathbf{P}_0(\vec{S}_{0,L} = \vec{x}_L) = q^l \mathbf{P}(\overline{\lambda}_1 = l, \overline{\tau}_1 = L, \widetilde{Y}_{1,L} = \vec{x}_L) \tag{59}$$

$$= q^l \mathbf{P}(\overline{\lambda}_m \equiv \overline{\nu}_m - \overline{\nu}_{m-1} = l, \overline{\tau}_m \equiv \overline{T}_m - \overline{T}_{m-1} = L, \widetilde{S}_{\overline{T}_{m-1},\overline{T}_m} = \vec{x}_L), \tag{60}$$

for all $m = 1, 2, \ldots$.

Proof. Note that condition $\varkappa\big(l|(0,\vec{x}_L)\big) = 0$ from (58) means that there are no n-separating levels between 0 and l. Hence $\eta\big(l|(0,\vec{x}_L)\big) = 1$, and it follows from (57) and (58) that

$$\begin{aligned}(61)\qquad \mathbf{P}_0(\vec{S}_L = \vec{x}_L) &= \mathbf{P}_0(\alpha(l) = L < T_*, \eta_*(l) = 1, \vec{S}_L = \vec{x}_L)\\ &= \mathbf{P}_0(\eta_*(l) = 1)\cdot\mathbf{P}_0(\alpha(l) = L < T_*,\ \vec{S}_L = \vec{x}_L|\eta_*(l) = 1).\end{aligned}$$

Substituting now (26) and (28) for $i = 0$ into (61), we obtain (59).

Recall that, by the construction (see Subsection 3.3), all independent random blocks $(\overline{\lambda}_m, \overline{\tau}_m, \widetilde{S}_{\overline{T}_{m-1},\overline{T}_m})$ from (30) are identically distributed with the block $(\overline{\lambda}_1, \overline{\tau}_1, \widetilde{Y}_{1,\overline{\tau}_1})$ from (23). Therefore, (60) follows from (59). $\square$

4.4. **Main Lemma.** In the proof of the following lemma we use the description of the accompanying random sequences that have been introduced in Subsection 3.3.

Lemma 5. *For arbitrary numbers*

$$(62)\qquad N \geqslant n \geqslant m \geqslant 0 \quad \textit{and} \quad M \geqslant n \geqslant 0,$$

suppose that we are given a vector $\vec{y}_N = (y_0, \dots, y_N) \in \mathbb{Z}^{(N+1)\times d}$ *such that*

$$(63)\qquad \alpha(n|\vec{y}_N) = N \geqslant 0 \quad \textit{and} \quad \eta(n|\vec{y}_N) = m \geqslant 0.$$

Then

$$(64)\qquad \mathbf{P}_*(\vec{S}_N = \vec{y}_N) = \mathbf{P}_*(\alpha(n) = N < T_*, \eta_*(n) = m, \vec{S}_N = \vec{y}_N)$$

and

$$(65)\qquad \mathbf{P}_*(\vec{S}_N = \vec{y}_N) = \psi_M(q)q^n\mathbf{P}(\overline{\nu}(m) = n, \overline{\alpha}(n) = N, \widetilde{S}_N = \vec{y}_N).$$

Moreover, all random variables in the right hand-side of (65) *are deterministic functions of random variables from the initial block and from the first* m *blocks in* (30) *only.*

Proof. The first assertion (64) of the lemma follows immediately from the assumption (62) and (63).

We prove now the second assertion (65) by the induction in m. For $m = 0$, (65) follows from Lemma 3. Let m be a strictly positive integer and suppose that (65) holds for each possible $\vec{y}_N$ and all numbers from (62) in the case when $\eta(n|\vec{y}_N) = m-1 \geqslant 0$. Now take numbers and a vector satisfying (62) and (63). Then, for some integers k and K,

$$(66)\qquad M > \varkappa(n|\vec{y}_N) = k \in [0, n-1] \quad \text{and} \quad \alpha(k|\vec{y}_N) = K \in [0, N-1],$$

where the value $\varkappa(\cdot|\cdot)$ was defined after Definition 3. We will use notations introduced in (35). Clearly, $\eta(k|\vec{y}_K) = m-1$ by (66). So, by the induction base, we have that

$$(67)\qquad \mathbf{P}_*(\vec{S}_K = \vec{y}_K) = \psi_M(q)q^k\mathbf{P}(\widetilde{S}_K = \vec{y}_K, \overline{\alpha}(k) = K, \overline{\nu}_{m-1} = k).$$

Ir follows from the definition of $\varkappa(\cdot|\cdot)$ and from (66) that k is an n-separating level. Hence, on the event $\{\vec{S}_N = \vec{y}_N\}$ we have $s_{K,N} \geqslant 0$. Thus,

$$\{\vec{S}_N = \vec{y}_N\} = \{\vec{S}_N = \vec{y}_N,\ s_{K,N} \geqslant 0\} = \mathcal{D}_{K,N},$$

where the event $\mathcal{D}_{K,N}$ was introduced in (41). Hence, by Property 1,

$$(68)\qquad \mathbf{P}_*(\vec{S}_N = \vec{y}_N) = \mathbf{P}_*(\mathcal{D}_{K,N}) = \mathbf{P}_*(\mathcal{A})\cdot\mathbf{P}_0(\mathcal{C}),$$

where

$$\mathcal{A} := \{\alpha(k) = K < T_*,\ \vec{S}_K = \vec{y}_K\} = \{\vec{S}_K = \vec{y}_K\}, \tag{69}$$

$$\mathcal{C} := \{\alpha(n-k) = N-K < T_*,\ \vec{S}_{0,N-K} = \vec{y}_{K,N}\} = \{\vec{S}_{0,N-K} = \vec{y}_{K,N}\}. \tag{70}$$

Next, we are going to apply Lemma 4 with $l = n-k$, $L = N-K$ and $\vec{x}_L = \vec{y}_{N,K}$. Note that the condition $\varkappa(n|\vec{y}_N) = k$ from (66) implies that $\varkappa\big(n-k|(0,\vec{y}_{N,K})\big) = 0$. Hence we have from Lemma 4 that

$$\mathbf{P}_0(\vec{S}_{0,N-K} = \vec{y}_{K,N}) \tag{71}$$
$$= q^{n-k}\mathbf{P}(\overline{\lambda}_m \equiv \overline{\nu}_m - \overline{\nu}_{m-1} = n-k, \overline{\tau}_m \equiv \overline{T}_m - \overline{T}_{m-1} = N-K, \widetilde{S}_{\overline{T}_{m-1},\overline{T}_m} = \vec{y}_{K,N}).$$

Substituting (67) and (71) into (69) and (70), we obtain from (68) that

$$\mathbf{P}_*(\vec{S}_N = \vec{y}_N) = \psi_M(q)q^k\mathbf{P}(\widetilde{S}_K = \vec{y}_K, \overline{\alpha}(k) = K, \overline{\nu}_{m-1} = k) \tag{72}$$
$$\times q^{n-k}\mathbf{P}(\overline{\nu}_m - \overline{\nu}_{m-1} = n-k, \overline{T}_m - \overline{T}_{m-1} = N-K, \widetilde{S}_{\overline{T}_{m-1},\overline{T}_m} = \vec{y}_{K,N}).$$

Notice that the m-th block in (30) is independent of the previous ones. Hence, (72) may be represented as

$$\begin{aligned}
&\mathbf{P}_*(\vec{S}_N = \vec{y}_N)\\
&= \psi_M(q)q^{k+(n-k)}\mathbf{P}(\overline{\nu}_m = n, \overline{T}_m = \overline{\alpha}(\overline{\nu}_m) = N,\\
&\qquad \widetilde{S}_{\overline{T}_{m-1}} = \vec{y}_K, \widetilde{S}_{\overline{T}_{m-1},\overline{T}_m} = \vec{x}_L = \vec{y}_{K,N})\\
&= \psi_M(q)q^n\mathbf{P}(\overline{\nu}(m) = n, \overline{\alpha}(n) = N, \widetilde{S}_N = \vec{y}_N).
\end{aligned}$$

Therefore, we have completed the induction step. This ends the proof of Lemma 5. □

4.5. **Proof of Theorem 1.** Let us notice that any set $\mathcal{A} \in \mathbb{Z}_*^d$ may be represented as

$$\mathcal{A} = \cup_{N=0}^{\infty} A_N, \quad \text{where} \quad A_N \subset \mathbb{Z}^{(N+1)\times d}, \quad N = 0,1,2,\ldots.$$

In addition, all vectors $\vec{y}_N = (y_0,\ldots,y_N)$ from A_N satisfy (64) and (65).

Then summing up the LHS's and RHS's of (64) and (65) over all N and all $\vec{y}_N \in A_N$ leads to (31). This completes the proof of Theorem 1. □

Let us make a concluding remark.

Remark 4.1. Recall that definition (14) of the probability $\mathbf{P}_0(\cdot)$ is essentially different from the analogous definition in [1] since we use Assumption **(A3)** that is weaker than a similar assumption in [1]. Correspondingly, our proof of Property 1 is based on fresh ideas that are not used in [1]. On the other hand, in the proof of our main Lemma 5, we have decided to follow the structure of the proof of Lemma 9 in [1]. However, we provide more details here, in order to show that Theorem 1 take place under weaker assumptions than were used in [1].

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