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ON CENTERS OF SOLUBLE GRAPHS

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Abstract. Let G be a finite group and $V = \pi(G)$ be a set of all prime divisors of its order. A soluble graph $\Gamma_{sol}(G)$ is a graph with a set of vertices V, where two vertices p and q in V are adjacent if there exists a soluble subgroup H of G whose order is divisible by pq. We study centers of soluble graphs of finite sporadic and exceptional simple groups of Lie types.

Keywords: finite group, π-subgroup, exceptional simple group of Lie type, sporadic simple group, soluble graph.

1. Introduction

A soluble graph $\Gamma_{sol}(G)$ of a finite group G is an ordinary graph whose vertices are prime divisors $\pi(G)$ of its order. Moreover, two vertices $p, q \in \pi(G)$ are adjacent if and only if there exists a soluble subgroup in G, whose order is divisible by pq.

The notion of a soluble graph was introduced by S. Abe and N. Iiyori [3] as a generalization of the notion of a Grünberg–Kegel graph. It is easy to see that the Grünberg–Kegel graph $\Gamma(G)$ of the group G is a subgraph of the graph $\Gamma_{sol}(G)$ with the same set of vertices, however, the soluble graph of the finite simple group G is always connected (Theorem 1 in [3]).

A centre of a soluble graph $Z(\Gamma_{sol}(G))$ of the group G is a set of vertices, each of which is adjacent to all vertices of $\Gamma_{sol}(G)$. If the group G is soluble, then the graph $\Gamma_{sol}(G)$ is a clique (a complete graph). The graph center of a sporadic simple group is often empty. On the contrary, the number of elements of the soluble graph center of the alternating group of degree n for larger n given a special choice of n may be great. Note that the notion of a center of a soluble graph is not in general use in the graph theory, but its use is intuitive in this case.

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The participants of the XIII conference school on the group theory, dedicated to 85th aniversary of V.A. Belonogov, held on August 3-7, 2020 in Ekaterinburg, presented a number of unsolved problems. These problems were published in [2]. In particular, L.S. Kazarin in problem 7 posed the following question: describe finite groups G of Lie type, whose graph $\Gamma_{sol}(G)$ has non-empty center.

The main goal of this work is to describe the centers of soluble graphs of sporadic simple groups and exceptional simple groups of Lie types. A complete description of centers of soluble graphs of exceptional simple groups was obtained for all exceptional simple groups of Lie type except for $E_7(q)$ and $E_8(q)$. For the latter ones, non-triviality of their centers is established.

2. Main results

The notation we use is mostly the standard one. When dealing with sporadic simple groups, we follow notation [4].

Theorem 1. *Let G be a finite simple exceptional group of Lie type. Then:*
1. $G \cong Sz(q)$ and $Z(\Gamma_{sol}(G)) = \{2\}$.
2. $G \cong {}^2G_2(q)$ and $Z(\Gamma_{sol}(G)) = \{2, 3\}$.
3. $G \cong {}^3D_4(q)$ and $Z(\Gamma_{sol}(G)) = \{2\}$.
4. $G \cong {}^2F_4(q)'$, where $q \geq 2$ and $Z(\Gamma_{sol}(G)) = \{2, 3\}$.
5. $G \cong G_2(q)$ and $\{2, 3\} = Z(\Gamma_{sol}(G))$.
6. $G \cong F_4(q)$ and $\{2\} = Z(\Gamma_{sol}(G))$.
7. $G \cong E_6(q)$. If $q = 1 + 3t$ and $(t, 3) = 1$, then $Z(\Gamma_{sol}(G)) = \emptyset$. Otherwise, $Z(\Gamma_{sol}(G)) = \{3\}$.
8. $G \cong {}^2E_6(q)$ and if $q = -1 + 3t$ when $(t, 3) = 1$, then $Z(\Gamma_{sol}(G) = \emptyset$. Otherwise, $Z(\Gamma_{sol}(G)) = \{3\}$.
9. $G \cong E_7(q)$ or $G \cong E_8(q)$, then $2 \in Z(\Gamma_{sol}(G))$.

Theorem 2. *Let G be a sporadic simple group. Then:*
1. The center of the graph $\Gamma_{sol}(G)$ is empty for the following groups: $G \cong M_{11}, M_{12}.M_{22}, M_{23}$,
$M_{24}, J_3, HS, McL, O'N, HN, Co_3, Co_2, BM, M$;
2. the center of the graph $\Gamma_{sol}(G)$ consists of the vertex 2 for the following groups: $G \cong J_1, J_4, Ly, He, Co_1, Suz, Ru, Fi_{22}, Fi_{23}, Fi'_{24}$.
3. the center of the graphs $G \cong J_2$ and $G \cong Th$ consists of vertices 2 and 3.

The center of the graph $\Gamma_{sol}(G)$ in the case when $|\pi(G)| = 3$ is always non-empty. This immediately follows from connectedness of the mentioned graph. The list of simple nonabelian 4-primary groups consists of 35 simple groups and three series of such groups (see [6]). Among 4-primary simple sporadic groups, only the soluble graph of the group J_2 has a non-empty center and $|Z(\Gamma_{sol}(J_2))| = 2$. Hence, it seems useful to list all finite simple 4-primary groups that have a soluble graph center consisting of two vertices, that is, the analogues of the group J_2.

Theorem 3. *Let G be a finite 4-primary simple group whose soluble graph center consists of two vertices. Then G is isomorphic to one of the following groups: J_2; $S_6(2)$; A_8; A_9; A_{10}; $O_8^+(2)$; $G_2(3)$; ${}^2F_4(2)'$, $L_2(31)$, $L_2(127)$.*

3. Proof of Theorem 1.

Consider all cases for simple exceptional groups of Lie type.

1. $G \cong Sz(q)$, where $q = 2^{2n+1}$ and $n \geq 1$.

In this case, $|G| = q^2(q^2+1)(q-1) = q^2(q-\sqrt{2q}+1)(q+\sqrt{2q}+1)(q-1)$. From [8, Theorem 4.1] if follows that the group G contains soluble subgroups $(q-1):2$, $(q-\sqrt{2q}+1):4$, and $(q+\sqrt{2q}+1):4$. From this, we easily obtain that $Z(\Gamma_{sol}(G))$ contains vertex 2. Taking into account that vertex 2 is adjacent to vertices from $\pi(q-1)$, and the latter ones are not adjacent to vertices from $\pi(q \pm \sqrt{2q}+1)$, we conclude that $Z(\Gamma_{sol}(G)) = \{2\}$.

2. $G \cong {}^2G_2(q)$, where $q = 3^{2n+1}$ and $n \geq 1$.

In this case, $|G| = q^3(q^3+1)(q-1) = q^3(q-\sqrt{3q}+1)(q+\sqrt{3q}+1)(q-1)(q+1)$. From [8, Theorem 4.2] it follows that the group G contains subgroups $[q^3]:(q-1)$, $(q-\sqrt{3q}+1):6$, $(q+\sqrt{3q}+1):6$, $2\times L_2(q)$, and $(2^2\times D_{(q+1)/2}):3$. From this, we easily obtain that $Z(\Gamma_{sol}(G))$ contains vertices $\{2\}$ and $\{3\}$. Note that the subgroups of orders $(q\pm\sqrt{3q}+1):6$ are Frobenius groups with abelian kernels, and q^2-1 is coprime with $(q\pm\sqrt{3q}+1)$. Moreover, prime divisors of these numbers do not divide the order of the parabolic subgroup of the group G. Hence, the only vertices, except for the ones from $\pi(q\pm\sqrt{3q}+1)$, which can be incident to them in the soluble graph of the group G, are vertices $\{2\}$ and $\{3\}$. Therefore, $Z(\Gamma sol(G)) = \{2,3\}$.

3. $G \cong {}^3D_4(q)$.

In this case, $|G| = q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$. The following equalities hold.

$q^8+q^4+1 = (q^4-q^2+1)(q^4+q^2+1) = (q^4-q^2+1)(q^2-q+1)(q^2+q+1)$,

$q^6-1 = (q^3-1)(q^3+1) = (q^2+q+1)(q^2-q+1)(q-1)(q+1)$, and

$q^2-1 = (q-1)(q+1)$.

From the representation of $|G|$, item (13) [8, Theorem 4.3], and simple arithmetical calculations, it follows that the primitive prime divisor of the number q^4-q^2+1 can divide only the order of the group (13) in the list [8, Theorem 4.3]. From this, it follows that only number 2 can belong to $Z(\Gamma_{sol}(G))$. Now, from items (2), (11), (12) [8, Theorem 4.3] it follows that $Z(\Gamma_{sol}(G)) = \{2\}$.

4. $G \cong {}^2F_4(q)$, where $q = 2^{2n+1} \geq 8$.

In this case, $|G| = q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$. The following equalities hold.

$q^3+1 = (q+1)(q^2-q+1)$,

$q^4-1 = (q^2-1)(q^2+1) = (q-1)(q+1)(q\pm\sqrt{2q}+1)$,

$q^6+1 = (q^2+1)(q^4-q^2+1) = (q\pm\sqrt{2q}+1)(q^4-q^2+1) = (q\pm\sqrt{2q}+1)$ $(q^2+q+1\pm\sqrt{2q}(q+1))$.

From here, it follows that $|G| = (q-1)^2(q+1)^2(q\pm\sqrt{2q}+1)^2(q^2-q+1)$ $(q^2+q+1\pm\sqrt{2q}(q+1))$. From the representation of $|G|$ and items (8), (9), (10) [8, Theorem 4.5], it follows that only numbers 2 and 3 can belong to $Z(\Gamma_{sol}(G))$. The same items [8, Theorem 4.5] yield that numbers 2 and 3 are adjacent to vertices from the sets $\pi(q+1)$, $\pi(q\pm\sqrt{2q}+1)$, and $\pi(q^2+q+1\pm\sqrt{2q}(q+1))$.

From item (1) [8, Theorem 4.5], it follows that vertex 2 is adjacent to the vertices from the set $\pi(q-1)$. Since 3 divides $q+1$, then from item (3) [7, Theorem 4.5] we conclude that vertex 3 is adjacent to vertices from the set $\pi(q-1)$.

From item (3) [8, Theorem 4.5], it follows that the group G contains a maximal subgroup $SU_3(q):2$. However, the group $SU_3(q)$ contains the subgroup $(q^2-q+1):3$, invariant under the authomorphism of the group $SU_3(q)$ of order 2. Therefore, vertices 2 and 3 are adjacent to vertices from the set $\pi(q^2-q+1)$.

Hence, $Z(\Gamma_{sol}(G)) = \{2,3\}$ and $|Z(\Gamma_{sol}(G))| = 2$.

5. $G \cong {}^2F_4(2)'$.

From [4, p. 74], it is easy to conclude that $Z(\Gamma_{sol}(G)) = \{2,3\}$ and

$$|Z(\Gamma_{sol}(G))| = 2.$$

6. $G \cong G_2(q)$, where $q = p^n > 2$.

In this case, $|G| = q^6(q^2-1)(q^6-1) = q^6(q-1)^2(q+1)^2(q^2-q+1)(q^2+q+1)$. The group G contains maximal tori of orders $(q \pm 1)^2$, $(q^2 \pm q + 1)$, $q^2 - 1$. Note that for every $r \in \pi(G) \setminus \{p\}$, number r divides the order of some of the listed tori. From the description of the maximal subgroups of the group $G_2(q)$ in [8], it follows that vertex 2 of the graph $\Gamma_{sol}(G)$ is adjacent to all vertices from $\pi(G) \setminus \{p\}$. Since the order of a Borel subgroup of the group G equals $q^6(q-1)^2$, then it is clear that for a field of any characteristic, the vertex $\{2\}$ is also adjacent to the vertex $p \neq 2$. Hence, $2 \in Z(\Gamma_{sol}(G))$.

Since for $r \in \pi(q^2 \pm q + 1)$ the vertex r only belongs to the maximal subgroups of $G_2(q)$, isomorphic to $SL_3(q)$ or $SU_3(q)$, using the structure of the groups $SL_3(q)$ and $SU_3(q)$, we conclude that in all cases, vertex $\{3\}$ also belongs to $Z(\Gamma_{sol}(G))$. Therefore, $Z(\Gamma_{sol}(G)) = \{2,3\}$.

7. $G \cong F_4(q)$. From [1, Table 8, p. 49], it follows that for every maximal torus T the index $|N_G(T) : T|$ is an even number. It easily follows from here that $2 \in Z(\Gamma_{sol}(G))$. On the other hand, there exist two maximal tori, B_4 of order q^4+1 and F_4 of order $q^4 - q^2 + 1$. Moreover, the index of the torus B_4 in its normalizer equals 8, and the index of the torus F_4 in its normalizer equals 12 (see [1, Table 8], p. 49). Since a subgroup, isomorphic to the maximal torus F_4, is strongly isolated in G, then taking into account the existence of the maximal torus B_4, we conclude that none of the vertices from $\pi(q^4 - q^2 + 1)$ is adjacent to vertices from $\pi(q^4+1)$. Therefore, $Z(\Gamma_{sol}(G)) = \{2\}$.

8. $G \cong E_6(q)$, where $q = p^n$. Then

$$|G| = (1/d)q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)$$

for $d = (q-1, 3)$. Consider the following subcases.

(a) $q = 3^n \geq 3$. In the maximal parabolic subgroups $q^{16} : (\Omega_{10}^+(q) \times (q-1))$ and $q^{1+16} : (SL_6(q) \times (q-1))$, the Levi complements have the factors $\Omega_{10}^+(q)$ and $SL_6(q)$. Hence, vertex 3 is adjacent to the vertices $\pi((q^8-1)(q^6-1)(q^4-1)(q^2-1)(q^5-1))$ and $\pi((q^2-1)(q^3-1)(q^4-1)(q^5-1)(q^6-1)$. From the decomposition of $|G|$ it follows that the remaining prime divisors of $|G|$ belong to $\pi(q^6+q^3+1)$ and $\pi(q^4-q^2+1)$. To the graph $\Gamma = E_6(a_1)$ from [1] corresponds the maximal torus $T = E_6(a_1)$ of order $(1/d)(q^6+q^3+1)$. From [1, Table 9, p. 50], we conclude that vertex 3 is adjacent to the vertices $\pi(q^6+q^3+1)$. To the graph $\Gamma = E_6$ from [1] corresponds the maximal torus T_1 of order $(1/d)(q^4-q^2+1)(q^2+q+1)$. From [1, Table 9, p. 50], we conclude that vertex 3 is adjacent to the vertices $\pi((q^4-q^2+1)(q^2+q+1)$. Therefore, in the considered case, $3 \in Z(\Gamma_{sol}(G))$.

(b) $q+1 \equiv 0 \pmod 3$. Obviously, $d = 1$. We will show that $3 \in Z(\Gamma_{sol}(G))$.

The group G contains the parabolic maximal subgroup

$$q^{5+20} : (SL_2(q) \times SL_5(q) \times (q-1)).$$

Therefore, the vertex 3 is adjacent to the vertices

$$\pi((q^2-1)(q^3-1)(q^4-1)(q^5-1).$$

The following equalities hold: $(q^8-1)=(q^4-1)(q^4+1)$, $q^9-1=(q^3-1)(q^6+q^3+1)$, $q^{12}-1=(q^6-1)(q^6+1)$. Consider the following divisors of order of the group G: q^4+1, q^6+q^3+1, q^3+1, q^4-q^2+1.

To the graph $\Gamma = D_5$ in [1, Table 9, p. 50] corresponds the maximal torus T_1 of order $(q^4+1)(q^2-1)$ and $|T_1|$ is divisible by 3. Hence, vertex 3 is adjacent to the vertices $\pi(q^4+1)$. To the graph $\Gamma = E_6(a_1)$ from [1] corresponds the maximal torus T of order q^6+q^3+1. From [1, Table 9, p. 50], we conclude that vertex 3 is adjacent to the vertices $\pi(T_2)$. To the graph $\Gamma = D_4$ from [1] corresponds the maximal torus T_3 of order $(q^3+1)(q+1)(q-1)^2$ and 3 divides $|T|$. Therefore, vertex 3 is adjacent to the vertices $\pi(q^3+1)$. To the graph $\Gamma = E_6$ corresponds the maximal torus T_4 of order $(q^4-q^2+1)(q^2+q+1)$. From [1, Table 9, p. 50], it follows that vertex 3 is adjacent to the vertices $\pi(q^4-q^2+1)$. From the representation of $|G|$, we conclude that $3 \in Z(\Gamma_{sol}(G))$.

(c) $q-1 \equiv 0 \pmod 3$. First, let $q-1 \equiv 0 \pmod 9$. We will show that $3 \in Z(\Gamma_{sol}(G))$. The group G contains parabolic maximal subgroups $1/3q^{16} : (\Omega_{10}^{+}(q) \times (q-1))$ and $1/3q^{20}(SL_6(q) \times (q-1))$.

Since $q-1 \equiv 0 \pmod 9$, vertex 3 is adjacent to the vertices $\pi((q^2-1)(q^3-1)(q^4-1)(q^5-1)(q^6-1)$ and $\pi(q^8-1)(q^6-1)(q^4-1)(q^2-1)$. It suffices to show that vertex 3 is adjacent to the vertices $\pi(q^6+q^3+1)$ and $\pi(q^4-q^2+1)$. This is done similarly as in the case (b). Therefore, $3 \in Z(\Gamma_{sol}(G))$.

(d) $q-1 = 3t$, where $(3,t)=1$. We will show that in this case $Z(\Gamma_{sol}(G)) = \emptyset$.

Consider case (d) in more detail. The group G contains a parabolic maximal subgroup $(1/3)q^{5+20} : (SL_2(q) \times SL_5(q) \times (q-1))$. Hence, vertex 3 is adjacent to the vertices $\pi((q^2-1)(q^3-1)(q^4-1)(q^5-1)$. Similarly to the case (b), it is easy to conclude that the remaining prime divisors of $|G|$ are located in $\pi((q^6+q^3+1)/3)$, $\pi(q^3+1)$, $\pi(q^4-q^2+1)$, $\pi(q^4+1)$.

To the graph $\Gamma = E_6(a_1)$ from [1] corresponds the maximal torus T of order $(q^6+q^3+1)/3$. From [1, Table 9, p. 50], we conclude that vertex 3 is adjacent to the vertices $\pi((q^6+q^3+1)/3)$. To the graph $\Gamma = D_4$ from [1] corresponds the maximal torus T_2 of order $(q^3+1)(q+1)(q-1)^2/3$ and 3 divides $|T_2|$. Therefore, vertex 3 is adjacent to the vertices $\pi(q^3+1)$. To the graph $\Gamma = E_6$ corresponds the maximal torus T_3 of order $(q^4-q^2+1)(q^2+q+1)/3$. From [1, Table 9, p. 50], it follows that $|N_G(T_3)/T_3| = 12$. It easily follows from here that vertex 3 is adjacent to the vertices $\pi(|T_3|)$. The order of the torus T_4, corresponding to the graph D_5 from [1], equals $(q^4+1)(q^2-1)/3$ and this number is not divisible by 3. From [1, Table 9, p. 50], it follows that $|N_G(T_4)/T_4| = 8$. Note that there exists a Hall cyclic subgroup of the group G whose order is odd and divides q^4+1.

Since in our case $q = 1+3t$ when t is mutually coprime to number 3, then the order of the Sylow 3-subgroup G is at most 3^{10}.

The complete description of maximal subgroups of the group G can be found in work by Craven [12, Tables 7.3 and 1.2]. There only exist two kinds of maximal subroups whose orders are divisible by prime odd divisors of the number q^4+1. In particular, the order of the group $P\Omega_{10}^{+}(q)$ contains a co-factor q^8-1 and $|PSp_8(q)|$ is divisible by q^8-1.

In both cases, a Sylow 3-subgroup of the mentioned groups does not have elementary abelian sections which are groups of order more than 3^6. If G has a soluble $\{3,r\}$-subgroup M for an odd prime r, divisible by q^4+1, then $O_r(M) = 1$.

Therefore, $O_3(M) \neq 1$ and there exists an elementary abelian subgroup of order 3^k, normalized by a cyclic subgroup of order r.

If every vertex r that belongs to the set of primitive odd prime divisors of the number (q^4+1) is adjacent to vertex 3 in the soluble graph of the group G, then the cyclic subgroup of order r will normalize the subgroup of order 3^k for some $1 < k \leq 6$ and act regularly on the elementary abelian subgroup of order 3^k. Hence, $r|(3^k-1)$. A direct check of divisors of the numbers 3^k-1 for $1 < k \leq 6$ shows that that is impossible. Therefore, vertex 3 does not belong to $Z(\Gamma_{sol}(G))$.

In the previously considered cases (a), (b), and (c), when $q \equiv 1 \pmod 9$, there exists a torus T of order $(q^6+q^3+1)/(3,q-1)$, coprime to the orders of parabolic maximal subgroups. Hence, the prime divisors of $|T|$ cannot divide the orders of p-local subgroups of the group G. On the other hand, the order T is coprime to the order of the Weyl group W of the group G. From the description in [12, Tables 7.3 and 1.2], it follows that in the group G there are no 2-primary $\{r,s\}$ – subgroups, where $r \in \pi(q^j-1)$ and $r \neq 3$ for $j \in \{2,3,5,6,8\}$, and $s \in \pi(T)$. Hence, in all cases ((a),(b), and (CЃ)), mentioned above, we have $Z(\Gamma_{sol}(G)) = \{3\}$. To exclude groups from the list in Table 1.2 from [12], it suffices to notice that the order of the torus T is neither divisible by 11 nor by 13. In the remaining cases, new pairs of adjacent vertices belonging to the soluble graph of the group G are not added. The order of the group J_3 equals $2^7 \cdot 3^5 \cdot 5 \cdot 7 \cdot 19$, the order of ${}^2F_4(2)'$ equals $2 \cdot 3 \cdot 5 \cdot 13$, and the order of the group M_{12} equals $2^6 \cdot 3^3 \cdot 5^2 \cdot 11$. In the first case, in the soluble graph of the group G there is an edge $(19,3)$, which is already known, and in the other cases, edges with vertices from $\pi(T)$ do not emerge.

Now it is easy to see that in the case (d) 3 does not belong to the center of the graph $\Gamma_{sol}(G)$. Therefore, in the case $q = 3t+1$ with $(t,3) = 1$ we have that the center of the soluble graph of the group G is empty.

9. $G \cong {}^2E_6(q)$. Then $|G| = (1/d)q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)$, where $d = (q+1,3)$. For every maximal torus T of the group G, the number $d|T|$ equals one of the following numbers [10]:

$(q+1)^k \cdot (q-1)^{6-k}, 2 \leq k \leq 6$; $(q^k-(-1))^k) \cdot (q^{6-k}-(-1)^{6-k}),\ 1 \leq k \leq 5$; $(q^k-(-1)^k)\cdot(q+1)^{6-k}$, $3 \leq k \leq 6$; $(q^3+1)(q^2-1)(q\pm 1)$; $(q^5+1)\cdot(q-1)$; $(q^3-1)(q^2\pm 1)(q+1)$; $(q^4+1)(q^2-1)$; $(q^2+1)^2(q+1)^2$; $(q^2-q+1)^3$; $(q^2-q+1)^2(q^2-1)$; $(q^4-1)(q-1)^2$; $q^3-1)(q^2-q+1)(q-1)$; $(q^4-q^2+1)(q^2-q+1)$; $q^6-q^3+1)$; $(q^2-q+1)^2(q^2+q+1)$.

Moreover, for every number n of the mentioned above numbers, there exists a torus T such that $d|T| = n$.

From [5, Table 5.1], it follows that among the maximal subgroups of the group $G \cong {}^2E_6(q)$ there exists a subgroup $U_3(q^3)\cdot 3$. Therefore, there exists a maximal torus T_0 of order $(q^6-q^3+1)/d$, where $d = (3,q+1)$, belonging to the maximal torus of order $q^{12}+q^6+1$, corresponding to the graph $E_6(a_1)$ from [1] of the group $E_6(q^2)$. Due to the fact that the mentioned torus if strongly isolated in $E_6(q^2)$, we obtain that T_0 is strongly isolated in G. Moreover, the torus T_0 is a Hall cyclic subgroup of the group G.

Assume that there exists a vertex $s \in (\pi(G) \setminus \pi(N_G(T_0))$, incident to every vertex from $\pi(N_G(T_0))$. That is, there exists a soluble $\{r,s\}$-subgroup M, where $r \in \pi(T_0)$. Obviously, $O_r(M) = 1$, otherwise (because T_0 is strongly isolated) it is easy to obtain a contradiction. Therefore, $O_s(M) \neq 1$, and from [12, Table 7.4], it follows that the group G does not have such soluble $\{r,s\}$-subgroups. Similarly as it was in the case of the group $E_6(q)$, the order of the maximal torus T_0 is neither divisible by 11, nor by 13. Therefore, the obtained conclusion on the center of a soluble graph based on [12, Table 7.4] remains true when analyzing Table 1.3 from [12]. The groups Fi_{22}, ${}^2F_4(2)'$ due to the mentioned above do not change the conclusion about the center of the soluble graph G, since they do not touch the edges of the form (r, s), where $r \in \pi(N_G(T_0))$ and $s \in \pi(G) \setminus \pi(N_G(T_0))$.

From what was said it follows that the center of the graph $\Gamma_{sol}(G)$ either consists of vertex 3, or contains the vertex $r \in \pi(T_0)$, or is empty. According to [5, Table 5.1], the group G has a subgroup $e.(L_2(q) \times U_6(q)).e$, where $e = (q-1, 2)$. Moreover, there exist maximal tori T_1 of order $(q^5+1)(q+1)/d$ and T_2 of order $(q^5+1)(q-1)/d$, where $d = (q-1, 3)$. Assume that there is a soluble subgroup K of order divisible by rs, where s is a primitive prime divisor of the number q^5+1. It is clear that $O_r(K) = 1$. Therefore, there exists an abelian subgroup S, belonging to $O_s(K)$. Due to the fact that the Sylow s-subgroup of the group G is cyclic, we can assume that $|S| = s$. Therefore, r divides $s-1$. It is easy to see that in the group $C_G(S)$ there is a Hall cyclic subgroup of order $(q^5+1)/(q+1)$, normalized by a subgroup of order r. In particular, r divides the greatest common divisor $((q^5+1)/(q+1)-1, |T_0|)$. The mentioned greatest common divisor equals 1, which is a contradiction.

Therefore, the only possibility for the center of the soluble graph not to be empty is that this center consists of vertex 3. The list of parabolic maximal subgroups of the group G can be found in work [7].

According to [7], there exist exactly 4 conjugacy classes of maximal parabolic subgroups of the universal group $(3, q+1).G$: P_1, P_2, P_3, and P_4. Moreover,

$$\pi(P_1) \cup \pi(P_2) \cup \pi(P_3) \cup \pi(P_4) = \pi(G) \setminus \pi(T_0).$$

The structure of the subgroups P_i for $i > 1$ is the following: $P_i = U_i L_i$, where L_i is a Levi factor, U_i is a unipotent p-subgroup (p is a characteristic of the field $GF(q)$). In the case P_4, the subgroup $L_4 \simeq {}^2D_4(q^2) : (q^2-1)/d$, where $d = (3, q+1)$. Recall that there also exists a subgroup $e.(L_2(q) \times U_6(q)).e$, where $e = (q-1, 2)$. In the case $q \equiv \pm 1 \pmod 9$, for every $r \in \pi(G) \setminus \{3\}$ the group G contains a soluble subgroup of order divisible by $3r$. In the case when $q = -1 + 3t$, where $(t, 3) = 1$, there is no soluble subgroup of order divisible by $3r$ in the group G. In this case, the center of the graph $\Gamma_{sol}(G)$ is empty.

If q is a power of 3, then $Z(\Gamma_{sol}(G)) = \{3\}$. Indeed, due to description [7], for every $r \in \pi(G) \setminus \{3\}$ there exists a soluble subgroup whose order is divisible by $3r$.

10. $G \in \{E_7(q), E_8(q)\}$. According to [1, Table 10, pp. 51 – 53] and [1, Table 11, pp. 54 – 58], the normalizer of every maximal torus of the mentioned groups is of an even order. Therefore, vertex 2 belongs to $Z(\Gamma_{sol}(G))$ for every of the considered groups.

Theorem 1 is proved.

4. Proof of Theorem 2.

There are 26 sporadic simple groups. We will find the centers of the soluble graphs for each of these groups.

1. Group $G \cong M_{11}$. According to [4, p. 18], the group G has a maximal soluble subgroup of order 55 and a maximal subgroup $2.S_4$. The only maximal subgroup G, whose order is divisible by 11, is a subgroup of order 55. This and the list of

maximal subgroups of G yield that there are no soluble subgroups of order divisible by 15 in G. Therefore, the center of the soluble graph of this group is empty.

2. Group $G \cong M_{12}$. According to [4, p. 33], the group G has a maximal soluble subgroup of order 55 and a maximal subgroup $A_4 \times S_3$. This and the list of maximal subgroups of G yield that the center of the soluble graph of this group is empty.

3. Group $G \cong M_{22}$. According to [4, p. 39], the group G has a maximal soluble subgroup of order 55 and a maximal subgroup $2^3 : L_3(2)$. This and the list of maximal subgroups of G yield that the center of the soluble graph of this group is empty.

4. Group $G \cong M_{23}$. According to [4, p. 71], the group G has a maximal soluble subgroup of order $23 \cdot 11$ and a maximal subgroup A_8. This and the list of maximal subgroups of G yield that the center of the soluble graph of this group is empty.

5. Group $G \cong M_{24}$. According to [4, p. 96], group G has a maximal soluble subgroup of order $23 : 11$ and a maximal subgroup $2^4 : A_8$. This and the list of maximal subgroups of G yield that the center of the soluble graph of this group is empty.

6. Group $G \cong J_1$. According to [4, p. 36], the group G has maximal subgroups $19 : 6$ and $11 : 10$. The only maximal subgroup G, whose order is divisible by 19, is $19 : 6$. Moreover, there exists a maximal subgroup $2^3 : 7 : 3$. Therefore, the center of the soluble graph of this group is $Z(\Gamma_{sol}(G)) = \{2\}$.

7. Group $G \cong J_2$. According to [4, p. 42], the group G has a maximal subgroup $5^2 : D_{12}$ and a maximal subgroup $L_3(2) : 2$. But there are no soluble subgroups of order divisible by 35. Therefore, the center of the soluble graph of this group is $Z(\Gamma_{sol}(G)) = \{2, 3\}$.

8. Group $G \cong J_3$. According to [4, p. 82], the group G has maximal soluble subgroups of orders $19 \cdot 9$ and $17 \cdot 8$. And there are no soluble subgroups of order divisible by $19 \cdot 17$. Therefore, the center of the soluble graph of this group is empty.

9. Group $G \cong J_4$. According to [4, p. 190], group G has maximal soluble subgroups $37 : 12$, $43 : 14$, $29 : 28$, $(23 : 11) : 2$. From the list of maximal subgroups of G it follows that the center of the soluble graph of this group is $Z(\Gamma_{sol}(G)) = \{2\}$.

10. Group $G \cong HS$. According to [4, p. 80], the group G has a maximal soluble subgroup of order $11 \cdot 5$ and a maximal soluble subgroup of order $4^3 \cdot 3 \cdot 7$. From the list of maximal subgroups of G it follows that the center of the soluble graph of this group is empty.

11. Group $G \cong Suz$. According to [4, p. 131], group G has a soluble subgroup of order $11 \cdot 10$ and a maximal subgroup $J_2 : 2$. Moreover, in G there is a soluble subgroup of order $13 \cdot 2$. From the list of maximal subgroups of G it follows that the center of the soluble graph of this group is $Z(\Gamma_{sol}(G)) = \{2\}$.

12. Group $G \cong McL$. According to [4, p. 100], the group G has a maximal soluble subgroup $11 : 5$ and a maximal subgroup $5^3 : 3 : 8$. The group G also contains the maximal subgroup $2^4 : A_7$. From the list of maximal subgroups of G it follows that the center of the soluble graph of this group is empty.

13. Group $G \cong Ru$. According to [4, p. 126], the group G has maximal soluble subgroups of orders $29 \cdot 14$ and $13 \cdot 12$. Moreover, the group G contains the maximal subgroup $2^{11} : L_3(2)$. From the list of maximal subgroups of G it follows that the center of the soluble graph of this group is $Z(\Gamma_{sol}(G)) = \{2\}$.

14. Group $G \cong He$. According to [4, p. 104], the group G has maximal subgroups $5^2 : 4A_4$ and $7^3 : (S_3 \times 3)$. In the group G, there is a maximal subgroup $S_4(4) : 2$,

containing a maximal subgroup of order $17 \cdot 8$ ([4, p. 44]). From the list of maximal subgroups of G it follows that the center of the soluble graph of this group is $Z(\Gamma_{sol}(G)) = \{2\}$.

15. Group $G \cong Ly$. According to [4, p. 174], the group G has maximal subgroups $37 : 18$ and $67 : 22$. Therefore, the center of the soluble graph of the group G cannot contain any other vertices except for vertex 2. In the group G, there is a maximal subgroup $G_2(5)$. From the list of maximal subgroups of G it follows that the center of the soluble graph of this group is $Z(\Gamma_{sol}(G)) = \{2\}$.

16. Group $G \cong O'N$. According to [4, p. 132], the group G has a maximal soluble subgroup of order $31 \cdot 15$ and a maximal subgroup J_1, considered in item 6. It easily follows from here that the center of the soluble graph of the group G is empty.

17. Group $G \cong Co_1$. According to [4, p. 183], the group G has a maximal soluble subgroup of order $7^2 \cdot 3^2 \cdot 2^3$ and a maximal subgroup $(A_5 \times J_2) : 2$. Therefore, there exists a complete subgraph of the graph $\Gamma_{sol}(G)$ on vertices 2, 3, 5, 7. The existence of a maximal subgroup $2^{11} : M_{24}$ shows that vertex 23 is adjacent to vertices 2 and 11. In the group G, there is a maximal subgroup $(A_4 \times G_2(4)) : 2$, containing the subgroup $(A_4 \times 13) : 2$. Hence, vertex 13 is adjacent to vertices 3 and 2. This yields that $Z(\Gamma_{sol}(G)) = \{2\}$.

18. Group $G \cong Co_2$. According to [4, p. 154], the group G has a maximal subgroup M_{23}, considered in item 4. This yields that the center of the soluble graph (if it is not empty) can only consist of vertex 11. Taking into account the existence in the group G of maximal subgroups $HS : 2$ and McL, we conclude that $Z(\Gamma_{sol}(G))$ must contain vertex 2. Contradiction. Therefore, the center of the graph $\Gamma_{sol}(G)$ is empty.

19. Group $G \cong Co_3$. According to [4, p. 134], the group G has a maximal subgroup M_{23}, considered in item 4. This yields that the center of the soluble graph (if it is not empty) can only consist of vertex 11. Taking into account the existence in the group G of maximal subgroups HS and $McL : 2$, we conclude that $Z(\Gamma_{sol}(G))$ must contain vertex 2. Contradiction. Therefore, the center of the graph $\Gamma_{sol}(G)$ is empty.

20. Group $G \cong Fi_{22}$. In this case, $\pi(G) = \{2, 3, 5, 7, 11, 13\}$. According to [4, p. 163], the group G has a maximal subgroup $2^{10} : M_{22}$. It immediately follows from here that the center of the soluble graph of the group G (if it is not empty) can only consist of vertex 2. The group G contains a maximal subgroup ${}^2F_4(2)'$, that has a subgroup $L_3(3) : 2$. Therefore, $Z(\Gamma_{sol}(G)) = \{2\}$.

21. Group $G \cong Fi_{23}$. Then $\pi(G) = \{2, 3, 5, 7, 11, 13, 17, 23\}$. According to [4, p. 177], the group G has a maximal subgroup $2^{11} : M_{23}$. Therefore, we conclude that the center of the soluble graph of the group G (if it is not empty) can only consist of vertex 2. From the existence in the group G of a maximal subgroup $2.Fi_{22}$, we conclude that the center of the soluble graph of the group G is $Z(\Gamma_{sol}(G)) = \{2\}$.

22. Group $G \cong Fi_{24}'$. In this case, $\pi(G) = \{2, 3, 5, 7, 11, 13, 17, 23, 29\}$. According to [4, p. 207], the group G has maximal subgroups $2.Fi_{22}.2$, Fi_{23}, and $29 \cdot 14$. Therefore, the center of the soluble graph $\Gamma_{sol}(G)$ can only consist of vertex 2. From the list of other maximal subgroups of the group G it follows that $Z(\Gamma_{sol}(G)) = \{2\}$.

23. Group $G \cong HN$. Then $\pi(G) = \{2, 3, 5, 7, 11, 19\}$. According to [4, p. 166], the group G has a maximal subgroup $U_3(8) : 3$, whose order is divisible by 19. There are no other maximal subgroups of order divisible by 19 in the group G. Therefore, the center of the soluble graph of the group G (if it is not empty) can only consist

of vertex 3. Since the group G has a maximal subgroup $2 \cdot HS.2$, we conclude that $Z(\Gamma_{sol}(G))$ must contain vertex 2. From the list of other maximal subgroups of the group G it follows that the center of the graph $\Gamma_{sol}(G)$ is empty.

24. Group $G \cong Th$. In this case, $\pi(G) = \{2, 3, 5, 7, 13, 19, 31\}$. According to [4, p. 177], the group G has a maximal subgroup $31 \cdot 15$. Taking into account the existence of a maximal subgroup $L_2(19) : 2$, we conclude that $Z(\Gamma_{sol}(G))$ has to contain vertex 3. However, there is also a subgroup of order $2^5{\cdot}31$. Having considered the list of other maximal subgroups of G, we obtain that $Z(\Gamma_{sol}(G)) = \{2, 3\}$.

25. Group $G \cong BM \cong F_2+$. In this case, $\pi(G) = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 31, 47\}$. In [4, p. 217], a complete description of maximal p-local subgroups of the group G is provided. In particular, the group G contains a maximal subgroup $47 : 23$ and the order of every p-local subgroup for $p \neq 47$ is not divisible by 47. This easily yields that the center of the soluble graph of the group G (if it is not empty) can consist only of vertex 23. On the other hand, for example, all 17-local subgroups of the group G have an order divisible only by prime numbers 17 and 2. Therefore, vertex 23 is not adjacent to vertex 17. Hence, we obtain that the center of the graph $\Gamma_{sol}(G)$ is empty.

26. Group $G \cong M \cong F_1$. In this case, $\pi(G) = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}$. In [4, p. 234], a complete description of maximal p-local subgroups of the group G is provided. The group G contains a maximal subgroup $59 : 29$ and the order of every p-local subgroup of the group G for $p \neq 59$ is not divisible by 59. From here it easily follows that the center of the soluble graph of the group G (if it is not empty) can consist only of vertex 29. On the other hand, for example, all 41-local subgroups of the group G have an order divisible only by prime numbers 41, 5, 2. Therefore, vertex 29 is not adjacent to vertex 41. From here we obtain that the center of the graph $\Gamma_{sol}(G)$ is empty.

Theorem 2 is proved.

5. Proof of Theorem 3.

According to [6, Lemma 1.3], the list of 4-primary simple groups consists of the following groups: (1) $A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, L_2(16), L_2(25), L_2(49), L_2(81), L_3(4), L_3(5), L_3(7), L_3(8), L_3(17), L_4(3), S_4(4), S_4(5), S_4(7), S_4(9), S_6(2), O_8^+(2), G_2(3), U_3(4), U_3(5), U_3(7), U_3(8), U_3(9), U_4(3), U_5(2), Sz(8), Sz(32), {}^3D_4(2), {}^2F_4(2)'$.

(2) $L_2(r)$, where r is prime, $17 \neq r \geq 11, r^2 - 1 = 2^a 3^b s^c, s > 3$ is prime, $a, b \in \mathbb{N}$ and c equals either 1, or 2 for $r \in \{97, 577\}$.

(3) $L_2(2^m)$, where $m, 2^m - 1$ and $(2^m + 1)/3$ are prime numbers exceeding 3.

(4) $L_2(3^m)$, where m and $(3^m - 1)/2$ are odd prime numbers, and $(3^m + 1)/4$ either equals a prime number or 11^2 (for $m = 5$).

Lemma 1. *Let G be one of the groups $A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2$. If the center of the graph $\Gamma_{sol}(G)$ contains two vertices, then G is one of the following groups: A_8, A_9, A_{10}, or J_2.*

Proof. From [4] it easily follows that $Z(\Gamma_{sol}(A_7)) = \emptyset$ and for the groups A_8, A_9, A_{10} the center of their soluble graph coincides with the set $\{2, 3\}$. For the groups M_{11}, M_{12}, J_2, the proof follows from Theorem 2. $\square$

Lemma 2. *Let G be a finite simple 4-primary group isomorphic to $PSL_2(q)$, where $q = p^m$. If $|Z(\Gamma_{sol}(G))| = 2$, then $p = q > 7$ is a Mersenne prime and $(p-1)/2$ has two distinct prime divisors. In the latter case, $p \in \{31, 127\}$.*

Proof. The information on subgroup structure of the group $L_2(q)$ can be found in [9, Theorem II.8.27]. From [6, Lemma 1.3], it follows that it suffices to consider the following cases.

(1) $q = 2^m$. Then G has two dihedral subgroups of orders $2(q \pm 1)$ and a Sylow 2-subgroup of order q. Therefore, $2 \in Z(\Gamma_{sol}(G))$. On the other hand, the vertices from $\pi(q+1)$ are not incident to the vertices from $\pi(q-1)$. Therefore, $|Z(\Gamma_{sol}(G))| = 1$.

(2) $q = 3^m$. Then in G there is a subgroup S_4 and, hence, vertex 3 is adjacent to vertex 2. Since G contains dihedral subgroups of orders $3^m \pm 1$, then $2 \in Z(\Gamma_{sol}(G))$. Moreover, the vertices from $\pi(q+1) \setminus \{2\}$ are not adjacent to the vertices from $\pi(q(q-1)/2)$. Therefore, we conclude that $|Z(\Gamma_{sol}(G))| = 1$.

(3) $q = p$, where p is a prime, $17 \neq p \geq 11, p^2 - 1 = 2^a 3^b s^c, s > 3$ is a prime, $a, b \in \mathbb{N}$ and c either equals 1, or 2 for $p \in \{97, 577\}$. First, suppose that $p \equiv 1 \pmod 4$, then $2 \in Z(\Gamma_{sol}(G))$. Moreover, the vertices from $\pi(q+1) \setminus \{2\}$ are not adjacent to the vertices from $\pi(q(q-1)/2)$. From here it follows that $|Z(\Gamma_{sol}(G))| = 1$. Let $p \equiv -1 \pmod 4$. Then the vertices from $\pi(p+1) \setminus \{2\}$ are not adjacent to the vertices from $\pi(p(p-1)/2)$. If $p+1$ is not a power of 2, then the center of the graph $\Gamma_{sol}(G)$ is empty. Therefore, $p = 2^r - 1$ is a Mersenne prime. All prime divisors of the number $(p-1)/2$ belong to the center of $\Gamma_{sol}(G)$. Note that for the numbers $p = 2^5 - 1 = 31$ and $p = 2^7 - 1 = 127$ the equality $|Z(\Gamma_{sol}(G))| = 2$ holds. We will show that when $p > 127$, the number of prime divisors of $(p-1)/2$ is not less than 3. Indeed, $p = 2^r - 1$ for some prime $r > 7$. Since the order of the Borel subgroup of the group $PSL_2(q)$ equals $p(p-1)/2 = p \cdot (2^{r-1} - 1)$, then there exist at least two distinct prime divisors of the numbers $2^{r-1} - 1$ and $2^{(r-1)/2} - 1$. Indeed, there exists a primitive prime divisor (Zsigmondy's divisor) s of the number $2^{r-1} - 1$, moreover, $s \geq (r-1) + 1 = r$ and a primitive prime divisor (Zsigmondy's divisor) t of the number $2^{(r-1)/2} - 1$, distinct from s, such that $t \geq (r-1)/2 + 1 > (7-1)/2 + 1 = 4$. Moreover, it is obvious that number 3 divides $2^{r-1} - 1 = (2^{(r-1)/2} - 1)(2^{(r-1)/2} + 1)$. Therefore, $2^{r-1} - 1$ for $r > 7$ is divisible at least by three distinct prime numbers.

(4) The group G is isomorphic to one of the groups $L_2(25)$, $L_2(49)$, or $L_2(81)$. Using the reasoning similar to the previous one it is easy to show that in all cases $Z(\Gamma_{sol}(G)) = \{2\}$.

The lemma is proved. $\square$

Lemma 3. *Let G be one of 4-primary groups distinct from the groups $L_2(q)$ and the groups listed in Lemma 1. If the center of the graph $\Gamma_{sol}(G)$ contains two vertices, then $G \cong S_6(2), G_2(3), {}^2F_4(2)'$, or $O_8^+(2)$.*

Proof. According to [6, Lemma 1.3], it is necessary to consider the following cases.

1. Group $G \cong L_3(4)$. According to [4, p. 23], the group G contains maximal subgroups $2^4 : A_5$ and $L_2(7)$. From the list of maximal subgroups of G it follows that the center of the soluble graph of this groups is empty.

2. Group $G \cong L_3(5)$. According to [4, p. 38], the group G contains a maximal soluble subgroup $31 : 3$ and a maximal subgroup $5^2 : GL_2(5)$. From the list of maximal subgroups of G it follows that $Z(\Gamma_{sol}(G)) = \{3\}$.

3. Group $G \cong L_3(7)$. According to [4, p. 50], the group G contains a maximal soluble subgroup $19:3$ and a maximal subgroup $L_2(7):2$. From the list of maximal subgroups of G it follows that $Z(\Gamma_{sol}(G)) = \{3\}$.

4. Group $G \cong L_3(8)$. According to [4, p. 74], the group G contains maximal soluble subgroups $73:3$ and $7^2:S_3$. From the list of maximal subgroups of G it follows that $Z(\Gamma_{sol}(G)) = \{3\}$.

5. Group $G \cong L_4(3)$. According to [4, p. 69], the group G contains maximal subgroups $3^3:L_3(3)$ and $U_4(2):2$. From the list of maximal subgroups of G it follows that the center of the soluble graph of this groups is empty.

6. Group $G \cong S_4(4)$. According to [4, p. 44], the group G contains maximal subgroups $L_2(16):2$ and $2^6:(3\times A_5)$. From the list of maximal subgroups of G it follows that $Z(\Gamma_{sol}(G)) = \{2\}$.

7. Group $G \cong S_4(5)$. According to [4, p. 61], the group G contains maximal subgroups $L_2(25):2_2$ and $S_3\times S_5$. From the list of maximal subgroups of G it follows that $Z(\Gamma_{sol}(G)) = \{2\}$.

8. Group $G \cong S_4(7)$. From [7, Table 8.12, Table 8.13], it follows that in the symplectic group $Sp_4(7)$ there are maximal subgroups $2_-^{1+4}.S_5$ and $GL_2(7):2$. From the list of maximal subgroups of $Sp_4(7)$ it is easy to conclude that $Z(\Gamma_{sol}(G)) = \{2\}$.

9. Group $G \cong S_4(9)$. From [8, Table 8.12, Table 8.13] it follows that in the symplectic group $Sp_4(9)$ there is a maximal subgroup $Sp_2(81):2$, therefore, the group G contains a maximal soluble subgroup $41:4$. The group $Sp_4(9)$ also contains a maximal subgroup $GL_2(9):2$. From the list of maximal subgroups of $Sp_4(9)$ it is easy to conclude that $Z(\Gamma_{sol}(G)) = \{2\}$.

10. Group $G \cong S_6(2)$. According to [4, p. 46], the group G contains maximal subgroups $2^6:L_3(2)$ and $S_3\times S_6$. From the list of maximal subgroups of G it follows that $Z(\Gamma_{sol}(G)) = \{2,3\}$.

11. Group $G \cong O_8^+(2)$. According to [4, p. 85], the group G contains a maximal subgroup $S_6(2)$. From the list of maximal subgroups of G it follows that $Z(\Gamma_{sol}(G)) = \{2,3\}$.

12. Group $G \cong U_3(5)$. According to [4, p. 34], the group G contains maximal subgroups A_7 Pë $A_6{}^{\cdot}2_3$. From the list of maximal subgroups of G it follows that the center of the soluble graph of this group is empty.

13. Group $G \cong U_3(8)$. According to [4, p. 66], the group G contains maximal soluble subgroups $19:3$ and $2^{3+6}:21$. From the list of maximal subgroups of G it follows that $Z(\Gamma_{sol}(G)) = \{3\}$.

14. Group $G \cong U_3(9)$. According to [4, p. 79], the group G contains maximal soluble subgroups $73:3$ and $10^2:S_3$. From the list of maximal subgroups of G it follows that $Z(\Gamma_{sol}(G)) = \{3\}$.

15. Group $G \cong U_4(3)$. According to [4, p. 52], the group G contains maximal subgroups A_7 and $3^4:A_6$. From the list of maximal subgroups of G it follows that $Z(\Gamma_{sol}(G)) = \{3\}$.

16. Group $G \cong U_5(2)$. According to [4, p. 73], the group G contains maximal subgroups $L_2(11)$ and $2^{4+4}:(3\times A_5)$. From the list of maximal subgroups of G it follows that $Z(\Gamma_{sol}(G)) = \{5\}$.

17. Group $G \in \{G_2(3), Sz(8), Sz(32), {}^3D_4(2), {}^2F_4(2)^{'}$. According to [4, p. 61], the group $G_2(3)$ contains maximal subgroups $L_2(13)$, $2^3:L_3(2)$ and $L_2(8):3$. From the list of its maximal subgroups it follows that $Z(\Gamma_{sol}(G_2(3))) = \{2,3\}$. The

remaining groups have been considered in Theorem 1. The order of the center of the group ${}^2F_4(2)'$ equals 2, the centers of other groups are less than 2.

Theorem 3 directly follows from Lemmas 1, 2, and 3. □

6. Concluding remarks.

Remark 1. As it follows from Lemma 3, the groups $S_6(2), {}^2F_4(2)'$, and $\Omega_8^+(2)$ are counterexamples to the assumption of the first author that the number of vertices in the center of the graph $\Gamma_{sol}(G)$ for a group of Lie type over the field $GF(q)$ of characteristic p is bounded by the number $q-1$. Possibly, there exists a linear boundary for the number of vertices in the center of a soluble graph for a simple group of Lie type in terms of the field over which it is defined. As it is seen from the results provided above, for exceptional simple groups of Lie type, that is the case.

Remark 2. In work by B. Amberg and L. Kazarin [11], it is stated that the independence number $t_s(G)$ of the graph of the group $G = B = F_2$ equals 7. A more attentive study has shown that $t_s(G) = 6$. We will list the vertices belonging to the maximal system of independent vertices of this graph: $\{47, 31, 19, 17, 13, 11\}$.

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