ON FINDING THE EXACT VALUES OF THE CONSTANT IN A $(1, q_2)$-GENERALIZED TRIANGLE INEQUALITY FOR BOX-QUASIMETRICS ON 2-STEP CARNOT GROUPS WITH 1-DIMENSIONAL CENTER

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ABSTRACT. For 2-step Carnot groups with 1-dimensional center, a method for defining the exact values of the constant q_2 in a $(1, q_2)$-generalized triangle inequality for their Box-quasimetrics is developed. The exact values of the constant q_2 are defined for 4-, 5-, and 6-dimensional 2-step Carnot groups with 3-dimensional horisontal subbundle.

Keywords: (q_1, q_2)-quasimetric spase, Carnot group, exact value, Box-quasimetric.

INTRODUCTION

We refer as a (q_1, q_2)-quasimetric space [1]–[9] to a pair (X, d), where X is some set, $d : X \times X \to \mathbb{R}^+ \cup 0$ is some function such that the *identity axiom*

$$d(x, y) = 0 \Leftrightarrow x = y$$

holds for it and (q_1, q_2) is a *generalized triangle inequality*, that is,

$$d(x, y) \leq q_1 d(x, z) + q_2 d(z, y) \quad \forall x, y, z \in X.$$

The expression $d(x, y)$ denotes a (q_1, q_2)-quasi-distance exactly *from the point x to the point y*. If $q_1 = q_2 = 1$, then (X, d) is a quasimetric space [10]. If for a

(q_1, q_2)-quasimetric space (X, d) the following condition holds

$$d(x, y) \leq q_0 d(y, x) \quad \forall x, y \in X,$$

where the constant q_0 does not depend on the choice of the points x, y, then we refer to a (q_1, q_2)-quasimetric space (X, d) as a q_0-*symmetric* one; for the case when $q_0 = 1$, we use the notion of *symmetric* (q_1, q_2)-*quasimetric space*. An important special case of symmetric (q_1, q_2)-quasimetric spaces are the symmetric $(1, q_2)$-quasimetric spaces [1]; these include Carnot groups and more general equiregular Carnot–Carathéodory spaces $(M, \rho_{\mathrm{Box}_M})$, equiped by Box-quasimetrics ρ_{Box_M} [6]–[14]. Moreover, in the general case, the constant q_2 does not equal 1 [18]. Box-quasimetrics were introduced in work [15]. The $(1, q_2)$-generalized triangle inequality plays a crucial role in obtaining the «divergence» estimates of the equiregular Carnot–Carathéodory space $(M, \rho_{\mathrm{Box}_M})$ from its nilpotent tangential cone, see, for example, [16, 17].

For a (q_1, q_2)-quasimetric space (X, d) we denote by $R = R(d)$ the set of points $(q'_1, q'_2) \in \mathbb{R}^2$, such that for ρ, the (q'_1, q'_2)-generalized triangle inequality holds. Directly from the definition of the set R, follows the

Property 0.1 ([1, 2]). 1^0 *The set $R = R(d)$ is convex and closed, and, moreover, $R \subseteq \{(x, y) \in \mathbb{R}^2 \mid x \geq 1, \ y \geq 1\}$; 2^0 the condition $(1, 1) \in R$ is equivalent to the fact that d is a quasimetric; 3^0 if (q_1, q_2)-quasimetric is symmetric, then the set R is symmetric with respect to the bisector of the right upper coordinate angle of the Euclidean plane.*

If $(q'_1, q'_2) \in R$ and $\tilde{q}_i \geq q'_i$, $i = 1, 2$, then $(\tilde{q}_1, \tilde{q}_2) \in R$. Drawing supporting lines through the boundary points of the closed convex set R, we obtain that R has extreme points. (Recall that a point $x_0 \in A$ is called an *extreme point* of a set A, if there are no points $x_1, x_2 \in A$, such that $x_0 \in (x_1, x_2)$, that is, $x_0 = tx_1 + (1 - t)x_2$ for some $0 < t < 1$.) It is easy to see that every extreme point of the set R is a Pareto optimal point of the set R (in the sense of minimisation of components). The point $(q_1^0, q_2^0) \in R$ is called *best*, if for every $(q'_1, q'_2) \in R$ we have that $q'_i \geq q_i^0$, $i = 1, 2$. From the definition of the best point, directly follows the next

Property 0.2 ([1, 2]). *If a best point exists, then it is unique; if such point exists, then $R = R(d) = \{(x, y) \in \mathbb{R}^2 \mid x \geq a, \quad y \geq b\}$ for some $a, b \geq 1$.*

See the examples of (q_1, q_2)-quasimetric spaces with the best points (q_1^0, q_2^0) such that $q_1^0 + q_2^0 > 2$ in [1], [4]–[6].

In work [18], the exact values of the constant q_2 for the $(1, q_2)$-generalized triangle inequality of Box-quasimetrics on the canonical Heisenberg groups $\mathbb{H}_\alpha^n$, $n \in \mathbb{N}$, and the canonical Engel group $\mathbb{E}_{\alpha, \beta}$ were obtained. In this work, we consider a problem of finding the exact values of the constant q_2 for Box-quasimetrics $\rho_{\mathrm{Box}_{\mathbb{D}_n}}$, where $\mathbb{D}_n$ is a canonical 2-step Carnot group with a one-dimensional centre, whose topological dimension equals $n + 1$. Here, the term «exact value» implies such value of the constant q_2 that for every number q'_2, $q'_2 < q_2$, the $(1, q'_2)$-generalized triangle inequality does not hold.

We refer as a *canonical finite-dimensional Lie group* [19] to an analytical Lie group K, whose exponential mapping is the identity. Therefore, K matches to some Euclidean space $\mathbb{R}^N$ with the coordinate system $(x_1, \ldots, x_N)$, induced by the coordinate frame $(O, e_1, \ldots, e_N)$. Hence, we can match any element $u \in K$ with its coordinate notation; in particular, the unit element of the group K is

the point $O = (0, \ldots, 0)$ (the origin of the Euclidean space $\mathbb{R}^N$), and for every $u = (x_1, \ldots, x_N)$, we have $u^{-1} = (-x_1, \ldots, -x_N)$. A group operation «·» on K (in other words, the left translation $L_u^K u' = u \cdot u'$ of the element $u' \in K$ by the element $u \in K$) is defined with the help of the Campbell–Hausdorff formula [20] and the corresponding commutator table given on the orthogonal basis $\{e_i\}_{i=1,\ldots,N}$ of the Euclidean space $\mathbb{R}^N$.

A Lie algebra is called graduated [21] if it decomposes into a direct sum of vector subspaces $V = \bigoplus_{i=1}^{r} V_i$, and, moreover, $[V_i, V_k] \subset V_{i+k}$, if $i + k \leq r$, and $[V_i, V_k] = 0$, if $i + k > r$. Note that a graduated algebra is always nilpotent of degree r. An r-step *stratified* Lie algebra V [22] is a Lie algebra nilpotent of degree r, that has a *stratification*, that is,

$$V = \bigoplus_{i=1}^{r} V_i, \quad V_{i+1} = [V_1, V_i], \quad [V_1, V_r] = \{0\}.$$

An r-*step Carnot algebra* [22] is a graduated Lie algebra V, which has a stratification; a simply connected Lie group G, corresponding to an r-step Carnot algebra V, is called an r-*step Carnot group*. Let

$$(0.1) \qquad N = \sum_{i=1}^{r} n_i, \quad n_i = \dim V_i,$$

and the basis of the left-invariant vector fields $\{X_1, \ldots, X_N\}$ of the Carnot group G is ordered such that the values of the first n_1 of them form at every point $v \in G$ the basis of the subspace $V_1(v)$, the values of the next n_2 of them form at every point $v \in G$ the basis of the subspace $V_2(v)$, and so on. We assign to every vector field X_k a natural number $j = \deg X_k$, defined by the inclusion $X_k \in V_j$. A Box-quasimetric is defined as

$$(0.2) \qquad \rho_{\mathrm{Box}_G}(u, w) = \max\{|a_i|^{\frac{1}{\deg X_i}} \mid i = 1, \ldots, N\}, \quad w = \exp\Big(\sum_{i=1}^{N} a_i X_i\Big)(u).$$

The definition implies that ρ_{Box_G} satisfies the identity and symmetry axioms. Homogeneous dilatations on the Carnot group G are defined with the help of the operator δ_ε, $\varepsilon \geq 0$, acting by the rule

$$\delta_\varepsilon : (x_1, \ldots, x_N) \mapsto (\varepsilon^{\deg X_1} x_1, \ldots, \varepsilon^{\deg X_N} x_N).$$

The Box-quasimetric ρ_{Box_G} of the Carnot group G is invariant with respect to the left translations and the action of the operator of dilatations, see [11]–[14], that is,

$$\rho_{\mathrm{Box}_G}\big(L_u^G v, L_u^G w\big) = \rho_{\mathrm{Box}_G}(v, w), \quad \rho_{\mathrm{Box}_G}\big(\delta_\varepsilon v, \delta_\varepsilon w\big) = \varepsilon \rho_{\mathrm{Box}_G}(v, w) \quad \forall u, v, w \in G.$$

A canonical 2-step group $\mathbb{D}_n$ with a one-dimensional centre is defined in the standard Euclidean space $\mathbb{R}^{n+1}$ with the coordinate system $(x_1, \ldots, x_n, t)$ and the coordinate frame $(O', e_1, \ldots, e_n, e_{n+1})$ with the help of the following commutator table

$$(0.3) \qquad [e_i, e_j] = \alpha_{ij} e_{n+1}, \quad \sum_{i,j=1}^{n} \alpha_{ij}^2 \neq 0,$$

the rest of possible commutators $e_1, \ldots, e_{n+1}$ equal 0. Suppose that

$$x = (x_1, \ldots, x_n, t), \quad x = (x'_1, \ldots, x'_n, t').$$

Using the Campbell–Hausdorff formula [20], with the help of (0.3), we obtain

$$(0.4)\quad L_x^{\mathbb{D}_n} x' = x \cdot x' = \Big(x_1 + x_1', \ldots, x_n + x_n',\ t + t' + \sum_{i,j=1,\ldots,n,\ i<j} \frac{\alpha_{ij}}{2}(x_i x_j' - x_j x_i')\Big).$$

The values of basis left-invariant vector fields $X_1, \ldots, X_n, T$ of the group $\mathbb{D}_n$ at every point $u = (x_1, \ldots, x_n, t)$ are defined as

$$(X_1, \ldots, X_n, T)(u) = \frac{\partial L_u^{\mathbb{D}_n}(x_1', \ldots, x_n', t')}{\partial(x_1', \ldots, x_n', t')}\Big|_{(x_1', \ldots, x_n', t') = (0, \ldots, 0)}.$$

If in (0.3) we put $n = 2m$, $m \in \mathbb{N}$, $\sum_{i=1}^{m-1} \alpha_{2i,2i+1}^2 = 0$ and $\alpha_{2j-1,2j} = \alpha \neq 0$, $j = 1, \ldots, m$, then we obtain a commutator table that defines the canonical Heisenberg group $\mathbb{H}_\alpha^m$ [18]. In particular, $\mathbb{D}_2 = \mathbb{H}_\alpha^1$.

For every point $u \in \mathbb{D}_n$, consider the mapping $\theta_u : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, acting by the rule

$$\theta_u(x_1, \ldots, x_n, t) = \exp\big(x_1 X_1 + \ldots x_n X_n + tT\big)(u).$$

According to (0.2), $(1, q_2)$-quasimetric $\rho_{\mathrm{Box}_{\mathbb{D}_n}}$ is defined by the rule

$$\rho_{\mathrm{Box}_{\mathbb{D}_n}}(u, v) = \max\big\{|a_1|, \ldots, |a_n|, |b|^{\frac{1}{2}}\ \big|\ \theta_u(a_1, \ldots, a_n, b) = v\big\}.$$

To find the exact value of the constant q_2 for the Carnot group $\mathbb{D}_3$, we more or less follow the methods of work [18], where calculation of the exact value was based on the following simple observation: the expression

$$x_1 x_2' - x_2 x_1', \quad x_1, x_2', x_2, x_1' \in [-1, 1],$$

reaches its maximal value, equal 2, when $x_1 = 1$, $x_2' = 1$, $x_2 = 1$, $x_1' = -1$. However, when we turn to the general group $\mathbb{D}_n$, it becomes complicated to «guess» the values of $x_1, \ldots, x_n, x_1', \ldots, x_n' \in [-1, 1]$, for which the expression

$$\sum_{i,j=1,\ldots,n,\ i<j} \frac{\alpha_{ij}}{2}(x_i x_j' - x_j x_i')$$

from (0.4) has the maximal value. In §1, we provide the proof of the fact that both minimal and maximal values of a bi-linear function in $\mathbb{R}^n$, considered on a standard unit n-dimensional cube

$$Q(n) = \{(x_1, \ldots, x_n) \mid \max_{i=1,\ldots,n} |x_i| \le 1\},$$

are reached on pairs of vectors, each of which has coordinates of some vertex of the cube $Q(n)$ (Theorem 1.1). This provides us with a method of defining the exact value of the constant q_2 for an arbitrary group $\mathbb{D}_n$, which will help to obtain the exact value of the constant q_2 for the group $\mathbb{D}_3$, see (2.2) and Corollary 2.2. In §3, we find the exact value of the constant q_2 for some 2-step groups related to $\mathbb{D}_3$.

1. ON EXTREMES OF BI-LINEAR FUNCTIONS

Let Q_i be vertices of a cube $Q(n)$; such are all the points, whose coordinates consist only of the numbers ± 1. Consider the bi-linear function

$$P(x, y) = P(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

$$= \left\langle \begin{pmatrix} A_{11} & A_{21} & \ldots & A_{n1} \\ A_{12} & A_{22} & \ldots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \ldots & A_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\rangle$$

$$= \sum_{i,j=1}^{n} A_{ij} x_i y_j, \quad \sum_{i,j=1}^{n} A_{ij}^2 \neq 0,$$

where $\langle \cdot, \cdot \rangle$ is a standard dot product in the Euclidean space $\mathbb{R}^n$

Theorem 1.1. *There exist vertices Q_i, Q_j, Q_k, Q_l of the cube $Q(n)$, such that*

$$\max_{x,y \in Q(n)} P(x, y) = P(Q_i, Q_j), \quad \min_{x,y \in Q(n)} P(x, y) = P(Q_k, Q_l).$$

Proof. Elementary geometric considerations show that pairs of points x, y, on which the minimum and maximum of the function $P(x, y)$ are reached, belong to the boundary $\partial Q(n)$. Suppose that the maximum of the function $P(x, y)$ is reached on the pair

$$(x^0, y^0) = (x_1^0, \ldots, x_n^0, y_1^0, \ldots, y_n^0).$$

We will prove Theorem 1.1 for the maximum of the function $P(x, y)$ (the arguments for the minimum are similar). Assume that

$$\sum_{j=1}^{n} A_{1j} y_j^0 \neq 0,$$

and, moreover, $x_1^0 \neq \pm 1$. Then, shifting the coordinate x_1 on some interval $(-\varepsilon + x_1^0, \varepsilon + x_1^0)$, we obtain the values of the function $P(x, y)$ that are both larger and smaller than $P(x^0, y^0)$, which contradicts the fact that on the pair (x^0, y^0) the maximum of the function $P(x, y)$ is reached. Therefore, $x_1^0 = \pm 1$. If

$$\sum_{j=1}^{n} A_{1j} y_j^0 = 0,$$

then we consider the situation where

$$\sum_{j=1}^{n} A_{2j} y_j^0 \neq 0,$$

and so on. From the mentioned above, it follows that for every k such that

$$\sum_{j=1}^{n} A_{kj} y_j^0 \neq 0,$$

we obtain that $x_k^0 = \pm 1$, and for k such that

$$\sum_{j=1}^{n} A_{kj} y_j^0 = 0,$$

for x_k^0 one can take any number from $[-1, 1]$, in particular, ± 1. Hence, we obtain that we can always consider as x^0 some vertex of the cube $Q(n)$.

We have

$$P(x^0, y^0) = \sum_{i,j=1}^{n} A_{ij} \delta_{ij} y_j^0,$$

where $\delta_{ij} = \pm 1$.

Now we fix some vertex $Q_k = (\delta_{k1}, \ldots, \delta_{kn})$ and consider the linear mapping

$$P(Q_k, y) = \sum_{i,j=1}^{n} A_{ij} \delta_{ki} y_j : \mathbb{R}^n \to \mathbb{R}.$$

We denote by A the matrix build of $\{A_{ij}\}$. If

$$(\delta_{k1}, \ldots, \delta_{kn}) \in ker A,$$

then $P(Q_k, y) \equiv 0$. But since the matrix A is nonzero, then $\dim ker A < n$; on the other hand, it is easy to see that the vector set $\{(\delta_{k1}, \ldots, \delta_{kn})\}$ forms a complete system in $\mathbb{R}^n$. Suppose that

$$(\delta_{k1}, \ldots, \delta_{kn}) \notin ker A.$$

In this case, the equations

$$(1.1) \qquad\qquad \sum_{i,j=1}^{n} A_{ij} \delta_{ki} y_j + D = 0$$

for all possible $D \in \mathbb{R}$ are the equations of parallel hyper-planes Π_D. The maximum of the expression

$$\sum_{i,j=1}^{n} A_{ij} \delta_{ki} y_j$$

on the cube $Q(n)$ will be reached in the case when the hyper-plane (1.1) has a non-empty intersection with the boundary $\partial Q(n)$, but at the same time has no common points with the interior of the cube $Q(n)$. But in this case, there always exists a vertex $Q_l \in (\Pi_D \cap \partial Q(n))$. Then

$$\max_{x,y \in Q(n)} P(x, y) = \max_{k,l} P(Q_k, Q_l).$$

$\square$

Consider an arbitrary parallelepiped $\bar{P} \subset \mathbb{R}^n$ with edges parallel to the coordinate axes and the center of symmetry at the origin.

Theorem 1.2. *There exist vertices P_i, P_j, P_k, P_l of the parallelepiped $\bar{P}$ such that*

$$\max_{x,y \in \bar{P}} P(x, y) = P(P_i, P_j), \quad \min_{x,y \in \bar{P}} P(x, y) = P(P_k, P_l).$$

Proof. Theorem 1.2 is proved similarly to Theorem 1.1.

$\square$

2. A METHOD OF FINDING THE EXACT CONSTANTS IN THE $(1, q_2)$-GENERALIZED TRIANGLE INEQUALITY FOR 2-STEP CARNOT GROUPS WITH A ONE-DIMENSIONAL CENTRE

We will provide some considerations that follow from work [18], which will help us to define the exact value of the constant q_2 in the $(1, q_2)$-generalized triangle inequality for Box-quasimetrics of the Carnot groups $\mathbb{D}_n$.

Consider an arbitrary canonical Carnot group G. To define the exact value of the constant q_2 in the $(1, q_2)$-generalized triangle inequality, for every triple of points $u, v, w \in G$, we must find a number $q = q(u, v, w)$ such that $\rho_{Box_G}(u, w) = \rho_{Box_G}(u, v) + q\rho_{Box_G}(v, w)$, and then $q_2 = \sup\limits_{u,v,w \in G} q(u, v, w)$. But, taking into account the fact that the $(1, q_2)$-quasimetrics $\rho_{Box_{\mathbb{D}_n}}$ is invariant with respect to the left translations and action of dilatations, it suffices to consider only such triples of points u, v, w, where

$$u = 0, \quad \rho_{Box_G}(0, v) = 1, \quad w = v \cdot \delta_\varepsilon w', \quad \rho_{Box_G}(0, w') = 1, \quad \varepsilon > 0,$$

and therefore, search for $q = q(0, v, w)$ from the equality

$$\rho_{Box_G}(0, w) = \rho_{Box_G}(0, v) + q\rho_{Box_G}(v, w) = 1 + q\varepsilon.$$

We denote $S_G(0, 1) = \{x \in G \mid \rho_{Box_G}(0, x) = 1\} = \partial Q(N)$, see (0.1).

We transfer our considerations on the Carnot group $\mathbb{D}_n$. Let

$$v = (x_1, \ldots, x_n, t) \in S_{\mathbb{D}_n}(0, 1), \quad w' = (x'_1, \ldots, x'_n, t') \in S_{\mathbb{D}_n}(0, 1).$$

Using (0.4), we obtain

$$v \cdot \delta_\varepsilon w' = \left(x_1 + \varepsilon x'_1, \ldots, x_n + \varepsilon x'_n, \ t + \varepsilon^2 t' + \varepsilon \sum_{i,j=1,\ldots,n,\ i<j} \frac{\alpha_{ij}}{2}(x_i x'_j - x_j x'_i) \right).$$

We denote

$$(2.1) \qquad M_{\mathbb{D}_n} = \sup_{(x_1,\ldots,x_n),\, (x'_1,\ldots,x'_n) \in Q(n)} \left| \sum_{i,j=1,\ldots,n,\ i<j} \frac{\alpha_{ij}}{2}(x_i x'_j - x_j x'_i) \right|.$$

Note that by Theorem 1.1, we can consider as $(x_1, \ldots, x_n)$, $(x'_1, \ldots, x'_n)$ in (2.1), the vertices of the cube $Q(n)$.

We have

$$\left| t + \varepsilon^2 t' + \varepsilon \sum_{i,j=1,\ldots,n,\ i<j} \frac{\alpha_{ij}}{2}(x_i x'_j - x_j x'_i) \right| \leq 1 + M_{\mathbb{D}_n}\varepsilon + \varepsilon^2.$$

Then

$$(2.2) \qquad q_2 = \begin{cases} 1, & M_{\mathbb{D}_n} \leq 2, \\ \frac{M_{\mathbb{D}_n}}{2}, & M_{\mathbb{D}_n} > 2. \end{cases}$$

It is formula (2.2) that provides the exact value of the constant q_2 for the Carnot group $\mathbb{D}_n$.

We apply the method described above to find the exact value of the constant q_2 for the Carnot group $\mathbb{D}_3$, that is defined with the help of the following commutator table:

$$(2.3) \qquad \begin{cases} [e_1, e_2] = a_{12} e_4, \\ [e_2, e_3] = a_{23} e_4, \\ [e_3, e_1] = a_{31} e_4. \end{cases}$$

The group operation, see (0.4), on $\mathbb{D}_3$ is written in the form

$$(2.4) \quad (x, y, z, t)(x', y', z', t') = \Big(x + x', y + y', z + z',$$

$$t + t' + \frac{1}{2}\big(a_{12}(xy' - x'y) + a_{23}(yz' - y'z) + a_{31}(zx' - z'x))\big)\Big).$$

We introduce notations $(x, y, z) = u$, $(x', y', z') = v$, $a = (a_{12}, a_{23}, a_{31})$; then

$$a_{12}(xy' - x'y) + a_{23}(yz' - y'z) + a_{31}(zx' - z'x) = \langle a, u \otimes v \rangle.$$

Lemma 2.1. *Let u, v, $u \neq v$, be non-collinear vectors whose coordinates coincide with the coordinates of the vertices of the cube $Q(3)$. We denote*

$$u \otimes v = (q_1, q_2, q_3).$$

Then the triples of numbers q_1, q_2, q_3 satisfy the following conditions: one of the numbers equals 0, two others have the values ± 2.

Proof. It is easy to obtain the proof of Lemma 2.1 by direct calculations. $\qquad\square$

Corollary 2.2.

$$M_{\mathbb{D}_3} = \max\{|a_{12} \pm a_{23}|, |a_{31} \pm a_{12}|, |a_{23} \pm a_{31}|\}.$$

3. SOME COROLLARIES

Consider the 2-step Carnot group G_1, which is defined by the following commutator table:

$$\begin{cases} [e_1, e_2] = a_{12}e_4, \\ [e_2, e_3] = a_{23}e_5, \\ [e_3, e_1] = a_{31}e_6. \end{cases}$$

The group operation, see (0.4), on G_1 is written in the form

$$(x, y, z, t_4, t_5, t_6)(x', y', z', t_4', t_5', t_6') = \Big(x + x', y + y', z + z',$$

$$t_4 + t_4' + \frac{a_{12}}{2}(xy' - x'y), \; t_5 + t_5' + \frac{a_{23}}{2}(yz' - y'z), \; t_6 + t_6' + \frac{a_{31}}{2}(zx' - z'x)\Big).$$

We denote $a = \max\{|a_{12}|, |a_{23}|, |a_{31}|\}$. The following theorem follows from the results of work [18].

Theorem 3.1. *For a canonical Carnot group G_1, the exact value of the constant q_2 is defined by the formula*

$$q_2 = \begin{cases} 1, & a \leq 2, \\ \frac{a}{2}, & a > 2. \end{cases}$$

Consider a 2-step Carnot group G_2, which is defined by the following commutator table:

$$\begin{cases} [e_1, e_2] = a_{12}e_4, \\ [e_2, e_3] = a_{23}e_4, \\ [e_3, e_1] = a_{31}e_5. \end{cases}$$

The group operation, see (0.4), on G_2 is written in the form

$$(x, y, z, t_4, t_5)(x', y', z', t'_4, t'_5) = \Big(x + x', y + y', z + z',$$
$$t_4 + t'_4 + \frac{a_{12}}{2}(xy' - x'y) + \frac{a_{23}}{2}(yz' - y'z), t_3 + t'_3 + \frac{a_{31}}{2}(zx' - z'x) \Big).$$

We denote

$$b = \max\{|a_{31}|, |a_{12} \pm a_{23}|\}.$$

Using the results of § 2 and the results of work [18], we obtain the following theorem.

Theorem 3.2. *For a canonical Carnot group G_2, the exact value of the constant q_2 is defined by the formula*

$$q_2 = \begin{cases} 1, & b \le 2, \\ \frac{b}{2}, & b > 2. \end{cases}$$

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