ON THE UNIQUENESS OF THE SOLUTION TO THE WIENER–HOPF EQUATION WITH PROBABILITY KERNEL

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Abstract. We study the problem of uniqueness for a solution to the inhomogeneous generalized Wiener–Hopf equation whose kernel is a probability distribution with finite positive mean.

Keywords: integral equation, inhomogeneous equation, Wiener-Hopf equation, probability distribution, positive mean.

Consider the inhomogeneous generalized Wiener–Hopf equation

$$(1) \qquad z(x) = \int_{-\infty}^{x} z(x-y)\, F(dy) + f(x), \qquad x \ge 0,$$

where z is the function sought, F is a given probability distribution on $\mathbb{R}$ and f is a known function. We first consider equation (1) whose *kernel* F is a nonarithmetic probability distribution with finite positive mean $\mu = \int_{\mathbb{R}} x\, F(dx)$ and prove the uniqueness of its solution (4) (see below). If $\mu = 0$ or $\mu < 0$, then the homogeneous generalized Wiener–Hopf equation, i.e., equation (1) with $f(x) \equiv 0$, has a nontrivial solution (see [1, Theorem 1] and [2, Chapter XII, Section 3a, Example (c)]). A probability distribution F in $\mathbb{R}$ is *arithmetic* if it is concentrated on a set of points of the form $0, \pm\lambda, \pm 2\lambda, \dots$. The largest λ with this property is called the *span* of F [2, Chapter V, § 2, Definition 3]. If a distribution is not arithmetic, then we call it *nonarithmetic*. All the functions involved are assumed to be Borel measurable. Let

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ν and $\varkappa$ be finite measures on the σ-algebra $\mathscr{B}$ of Borel sets in $\mathbb{R}$. Their *convolution* is the measure

$$\nu * \varkappa(A) := \iint\limits_{\{x+y\in A\}} \nu(dx)\,\varkappa(dy) = \int_{\mathbb{R}} \nu(A-x)\,\varkappa(dx), \qquad A \in \mathscr{B};$$

here $A - x := \{y \in \mathbb{R} : x + y \in A\}$. Denote by F^{n*} the n-th convolution power of F:

$$F^{1*} := F, \quad F^{(n+1)*} := F^{n*} * F, \quad n \geq 1, \quad F^{0*} := \delta_0,$$

where δ_0 is the atomic measure of unit mass concentrated at zero. Let U be the renewal measure generated by F: $U := \sum_{n=0}^{\infty} F^{n*}$. Let ν be a measure defined on $\mathscr{B}$, and $a(x)$, $x \in \mathbb{R}$, a function. Define the convolution $\nu * a(x)$ as the function $\int_{\mathbb{R}} a(x-y)\,\nu(dy)$, $x \in \mathbb{R}$. The relation $a_n \sim b_n$ as $n \to +\infty$ means that $a_n/b_n \to 1$ as $n \to +\infty$. Let $\mathbb{R}_+$ be the set of all nonnegative numbers and $\mathbb{R}_- := \mathbb{R} \setminus \mathbb{R}_+$ be the set of all negative numbers.

Let X_k, $k \geq 1$, be independent random variables with the same distribution F not concentrated at zero. These variables generate the random walk $S_0 = 0$, $S_n = X_1 + \ldots + X_n$, $n \geq 1$. By Theorem 1 in [2, § XII.2], there exist only two types of random walks: (i) the oscillating type (S_n oscillates with probability 1 between $-\infty$ and $+\infty$); (ii) the drifting type (S_n tends to $-\infty$ or $+\infty$ with probability 1). Put $\overline{\mathscr{T}}_+ := \min\{n \geq 1 : S_n \geq 0\}$. The random variable $\overline{\mathscr{H}}_+ := S_{\overline{\mathscr{T}}_+}$ is called the *first weak ascending ladder height*. Similarly, $\overline{\mathscr{T}}_- := \min\{n \geq 1 : S_n < 0\}$ and $\overline{\mathscr{H}}_- := S_{\overline{\mathscr{T}}_-}$ is the *first strong descending ladder height*.

Denote by $F_\pm$ the distributions of the random variables $\overline{\mathscr{H}}_\pm$, respectively. The following factorization identity holds [3, Section 2]:

$$(2) \qquad 1 - \xi\mathsf{E}(e^{sX_1}) = \left[1 - \mathsf{E}\left(\xi^{\overline{\mathscr{T}}_-} e^{s\overline{\mathscr{H}}_-}\right)\right]\left[1 - \mathsf{E}\left(\xi^{\overline{\mathscr{T}}_+} e^{s\overline{\mathscr{H}}_+}\right)\right],$$

where $|\xi| \leq 1$ and $\Re s = 0$. Identity (2) with $\xi = 1$ implies

$$(3) \qquad \delta_0 - F = (\delta_0 - F_-) * (\delta_0 - F_+).$$

Let $U_\pm := \sum_{k=0}^{\infty} F_\pm^{k*}$ be the renewal measures generated by the distributions $F_\pm$, respectively. Denote by $\mathbf{1}_{\mathbb{R}_+}$ the indicator of the subset $\mathbb{R}_+$ in $\mathbb{R}$: $\mathbf{1}_{\mathbb{R}_+}(x) = 1$ for $x \in \mathbb{R}_+$ and $\mathbf{1}_{\mathbb{R}_+}(x) = 0$ for $x \in \mathbb{R}_-$. Extend the function f onto the whole line: $f(x) := 0$, $x < 0$. The following theorem has been proved in [4].

Theorem 1. *Let F be a probability distribution and $f \in L_1(\mathbb{R}_+)$. Then the function*

$$(4) \qquad z(x) = U_+ * \left[(U_- * f)\mathbf{1}_{\mathbb{R}_+}\right](x), \qquad x \in \mathbb{R}_+,$$

is the solution to (1) *coinciding with the solution obtained by successive approximations.*

The purpose of this note is to prove the uniqueness of solution (4) in the class $L_1(\mathbb{R}_+) + L_\infty(\mathbb{R}_+)$ of admissible solutions. We say that $z \in L_1(\mathbb{R}_+) + L_\infty(\mathbb{R}_+)$ if and only if $z = z_1 + z_2$ with $z_1 \in L_1(\mathbb{R}_+)$ and $z_2 \in L_\infty(\mathbb{R}_+)$. In order to justify this choice, let us give an example which shows that solution (4) can be of this form. Let F^{m*} for some $m \geq 1$ have a nonzero absolutely continuous component. Then F_+^{n*} for some $n \geq 1$ must also have a nonzero absolutely continuous component [5, Theorem 2 and Remark 2]. Stone [6] has shown that there exists a decomposition $U = U_1 + U_2$, where U_1 is a finite measure and the measure U_2 is absolutely continuous with bounded continuous density $u \geq 0$ such that $\lim_{x\to+\infty} u(x) = 1/\mu$

and $\lim_{x \to -\infty} u(x) = 0$. It follows that if $f \in L_1(\mathbb{R})$, then $U * f = U_1 * f + U_2 * f$, where $U_1 * f \in L_1(\mathbb{R})$ and $U_2 * f(x)$ is a bounded continuous function having finite limits as $x \to \pm\infty$. Indeed, by Fubini's theorem [7, Section 36, Theorem C],

$$\int_{-\infty}^{\infty} |U_1 * f(x)| \, dx \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x - y)| \, |U_1|(dy) \, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(v)| \, dv \, |U_1|(dy) = \|f\|_1 |U_1|(\mathbb{R}) < \infty,$$

$$|U_2 * f(x)| \leq \int_{-\infty}^{\infty} u(x - y)|f(y)| \, dy \leq \sup_{x \in \mathbb{R}} u(x) \|f\|_1,$$

and, by Lebesgue's bounded convergence theorem [7, Section 26, Theorem D],

$$\lim_{x \to x_0} U_2 * f(x) = \lim_{x \to x_0} \int_{-\infty}^{\infty} u(x - y) f(y) \, dy = \int_{-\infty}^{\infty} u(x_0 - y) f(y) \, dy = U_2 * f(x_0),$$

where $U_2 * f(-\infty) := 0$ and $U_2 * f(+\infty) := \int_{-\infty}^{\infty} f(y) \, dy / \mu$.

The condition $\mu \in (0, \infty)$ and the law of large numbers [2, Section VII.7, Theorem 2] imply that the random walk $\{S_n\}$ drifts to $+\infty$, i.e., $S_n \sim n\mu \to +\infty$ as $n \to +\infty$ with probability one. Moreover, as stated in [2, Section XVIII.4 (a)], drift to $+\infty$ occurs if F_- is *defective*, i.e., if $F_-(\mathbb{R}) < 1$. Hence U_- is a finite measure: $U_-(\mathbb{R}) = 1/(1 - \mathbb{F}_-(\mathbb{R})) < \infty$. Obviously, $U_- * f \in L_1(\mathbb{R})$ and the more so $(U_- * f)\mathbf{1}_{\mathbb{R}_+} \in L_1(\mathbb{R})$. Applying the established already to U_+ and $(U_- * f)\mathbf{1}_{\mathbb{R}_+}$ instead of U and f, respectively, we see that

$$z(x) = U_+ * \big[(U_- * f)\mathbf{1}_{\mathbb{R}_+}\big](x) = z_1(x) + z_2(x),$$

where $z_1 \in L_1(\mathbb{R}_+)$ and the function $z_2(x)$ is continuous and has a finite limit as $x \to +\infty$, which entails $z_2 \in L_\infty(\mathbb{R}_+)$. Thus, $z \in L_1(\mathbb{R}_+) + L_\infty(\mathbb{R}_+)$.

Theorem 2. *Let F be a nonarithmetic probability distribution with finite positive mean μ and $f \in L_1(\mathbb{R}_+)$. Then the solution z to equation (1) defined by (4) is unique in the class $L_1(\mathbb{R}_+) + L_\infty(\mathbb{R}_+)$.*

Proof. Denote

$$(5) \qquad v(x) = \begin{cases} z(x) & \text{for } x \geq 0, \\ 0 & \text{for } x < 0, \end{cases} \quad n(x) = \begin{cases} 0 & \text{for } x \geq 0, \\ -\int_{-\infty}^{x} z(x - y) \, F(dy) & \text{for } x < 0. \end{cases}$$

First, let us prove that

$$z \in L_1(\mathbb{R}_+) + L_\infty(\mathbb{R}_+) \Rightarrow n \in L_1(\mathbb{R}).$$

Obviously, this assertion follows by synthesis from both implications

$$z \in L_1(\mathbb{R}_+) \Rightarrow n \in L_1(\mathbb{R}_-) \quad \text{and} \quad z \in L_\infty(\mathbb{R}_+) \Rightarrow n \in L_1(\mathbb{R}_-).$$

If $z \in L_1(\mathbb{R}_+)$, then, by Fubini's theorem,

$$\int_{-\infty}^{0} |n(x)| \, dx \leq \int_{-\infty}^{0} \int_{-\infty}^{x} |z(x - y)| \, F(dy) \, dx$$

$$= \int_{-\infty}^{0} \int_{0}^{|y|} |z(u)| \, du \, F(dy) \leq \|z\|_1 F(\mathbb{R}_-) < \infty.$$

If $z \in L_\infty(\mathbb{R}_+)$, then

$$\int_{-\infty}^{0} |n(x)|\, dx \leq \|z\|_\infty \int_{-\infty}^{0} |y|\, F(dy) < \infty,$$

since the mean μ exists by assumption. Consider now the renewal equation

$$v(x) = \int_{-\infty}^{\infty} v(x - y)\, F(dy) + f(x) + n(x), \qquad x \in \mathbb{R},$$

and its solution $v(x) = U * (f + n)(x)$, $x \in \mathbb{R}$. Suppose there exist two solutions z_1, $z_2 \in L_1(\mathbb{R}_+) + L_\infty(\mathbb{R}_+)$ to equation (1). Let v_1, v_2 and n_1, n_2 correspond to z_1, z_2, respectively (see (5)). Then the function $V = v_1 - v_2 \in L_1(\mathbb{R}_+) + L_\infty(\mathbb{R}_+)$ satisfies the equation $V(x) = F * V(x) + n_1(x) - n_2(x)$, $x \in \mathbb{R}$, or, by (3),

$$(\delta_0 - F) * V(x) = (\delta_0 - F_-) * (\delta_0 - F_+) * V(x) = n_1(x) - n_2(x), \qquad x \in \mathbb{R}.$$

Since the distribution F_- is defective, we have $U_- * (\delta_0 - F) = \delta_0$. Hence

(6) $$(\delta_0 - F_+) * V(x) = U_- * n_1(x) - U_- * n_2(x), \qquad x \in \mathbb{R}.$$

The left-hand side is identically zero on $\mathbb{R}_-$ and so is the right-hand side. Moreover, U_- is concentrated on $\mathbb{R}_- \cup \{0\}$ and $n_1(x) - n_2(x) \equiv 0$ on $\mathbb{R}_+$. Hence the right-hand side of (6) is identically zero on $\mathbb{R}$. It follows that

$$n_1(x) - n_2(x) = (\delta_0 - F_-) * [U_- * n_1(x) - U_- * n_2(x)] \equiv 0$$

and we have $V(x) = F * V(x)$, $x \in \mathbb{R}$. By [8, Theorem 1], there exists a constant c such that $V(x) = c$ a.e. (almost everywhere). However, $V(x) = 0$ on $\mathbb{R}_-$. Hence $V(x) = 0$ a.e. It follows that $z_1(x) = z_2(x)$ a.e. The proof of the theorem is complete. $\qquad\square$

For the sake of completeness, we give a discrete version of Theorem 2. Let $\mathbb{Z}$ be the set of all integers, let $\mathbb{Z}_+$ be the set of all nonnegative integers, and let $\mathbb{Z}_- := \mathbb{Z} \setminus \mathbb{Z}_+$ be the set of all negative integers. The discrete analog of equation (1) is the infinite system of equations

(7) $$z_j = \sum_{k=-\infty}^{j} z_{j-k}\, f_k + g_j, \qquad j \in \mathbb{Z}_+,$$

where $\{f_j\}_{j=-\infty}^{\infty}$ and $\{g_j\}_{j=0}^{\infty}$ are known numeric sequences, and $\{z_j\}_{j=0}^{\infty}$ is a sequence of sought quantities (see [9, Section 13.1]). Moreover, we assume that $F = \{f_j\}_{j=-\infty}^{\infty}$ is an arithmetic probability distribution with span 1.

Denote the space of all two-sided summable sequences $a = \{a_n\}_{n=-\infty}^{\infty} \subset \mathbb{C}$ by l_1 and the space of all bounded sequences by l_∞. Agree to write sequences in the form of functions of integer argument if their notations contain more than one letter. Define the *convolution* $a * b$ of sequences $a = \{a_n\}_{n=-\infty}^{\infty}$ and $b = \{b_n\}_{n=-\infty}^{\infty}$ in l_1 as the sequence $c = \{c_n\}_{n=-\infty}^{\infty} \in l_1$, where

$$c_n := a * b(n) := \sum_{j=-\infty}^{\infty} a_{n-j} b_j, \qquad n \in \mathbb{Z}.$$

This definition of convolution will also be used in the cases when the sum makes sense for all $n \in \mathbb{Z}$ despite the fact that the initial sequences a or b may fail to belong to l_1, e.g., when $a \in l_1$ and $b \in l_\infty$. The space l_1 is a Banach algebra with the norm $\|a\|_1 = \sum_{n=-\infty}^{\infty} |a_n|$ and the convolution $a * b$ as multiplication; the addition, subtraction, and multiplication of sequences by numbers are carried

out termwise: $a \pm b = \{a_n \pm b_n\}$, $\alpha a = \{\alpha a_n\}$, $\alpha \in \mathbb{C}$, and δ_0 is the unity of the algebra l_1 (see [10, Appendix, § 1, Subsection 1, Example 4]). Since complex-valued measures concentrated on $\mathbb{Z}$ can be regarded as sequences, by the convolution of these measures we will mean the convolution of sequences. We will not distinguish such measures and the corresponding sequences.

Denote by $F_+ := \{f_j^+\}_{j=0}^\infty$ and by $F_- := \{f_j^-\}_{j=-\infty}^{-1}$ the distributions of the random variables $\overline{\mathscr{H}}_+$ and $\overline{\mathscr{H}}_-$, respectively. It is convenient to extend these sequences as follows: $f_j^+ = 0$ for $j \in \mathbb{Z}_-$ and $f_j^- = 0$ for $j \in \mathbb{Z}_+$. Then the distributions $F_+ := \{f_j^+\}_{j=-\infty}^\infty$ and $F_- := \{f_j^-\}_{j=-\infty}^\infty$ are elements of the Banach algebra l_1. Let

$$U = \sum_{k=0}^\infty F^{k*} = \{U(\{n\})\}_{n=-\infty}^\infty \quad \text{and} \quad U_\pm = \sum_{k=0}^\infty F_\pm^{k*} = \{U_\pm(\{n\})\}_{n=-\infty}^\infty$$

be the discrete renewal measures generated by the distributions F and $F_\pm$, respectively. Denote by $\mathbf{1}_{\mathbb{Z}_-}$ the indicator of the subset $\mathbb{Z}_-$ in $\mathbb{Z}$: $\mathbf{1}_{\mathbb{Z}_-}(j) = 1$ for $j \in \mathbb{Z}_-$ and $\mathbf{1}_{\mathbb{Z}_-}(j) = 0$ for $j \in \mathbb{Z}_+$. The notation $\mathbf{1}_{\mathbb{Z}_+}$ has a similar sense. Extend the sequence $g = \{g_j\}_{j=0}^\infty$ by setting $g_j = 0$ for $j < 0$. The following theorem has been proved in [3, Theorem 1].

Theorem 3. *Let F be an arithmetic probability distribution with span 1 and let $g \in l_1$. Then the sequence*

$$(8) \qquad\qquad z_j = U_+ * \big[(U_- * g)\mathbf{1}_{\mathbb{Z}_+}\big](j), \qquad j \in \mathbb{Z}_+,$$

is the solution to (7) coinciding with the solution obtained by successive approximations.

We will also need the following lemma [2, Lemma 1].

Lemma 1. *Let F be a probability distribution not concentrated at the origin, and let ζ be a bounded continuous solution of the convolution equation*

$$\zeta(x) = \int_{-\infty}^\infty \zeta(x - y)\, F(dy).$$

Then $\zeta(x) = \mathrm{const}$ except if F is arithmetic. If F has span λ then ζ is periodic with period λ.

Theorem 4. *Let $F = \{f_j\}_{j=-\infty}^\infty$ be an arithmetic probability distribution with step 1 and finite positive mean $\mu = \sum_{-\infty}^\infty j f_j$, and let $g \in l_1$. Then the solution $z = \{z_j\}_{j=0}^\infty$ to equation (7) defined by (8) is unique in the class $l_\infty(\mathbb{Z}_+)$.*

Proof. Suppose that there exist two solutions $z^{(1)} = \{z_j^{(1)}\}_{j=0}^\infty$ and $z^{(2)} = \{z_j^{(2)}\}_{j=0}^\infty$ to equation (7) and that both solutions are in $l_\infty(\mathbb{Z}_+)$. Consider the following sequences:

$$(9) \qquad v_j = \begin{cases} z_j & \text{for } j \geq 0, \\ 0 & \text{for } j < 0, \end{cases} \quad n_j = \begin{cases} 0 & \text{for } j \geq 0, \\ -\displaystyle\sum_{k=-\infty}^j z_{j-k} f_k & \text{for } j < 0. \end{cases}$$

First, let us prove that $z \in l_\infty(\mathbb{Z}_+) \Rightarrow n \in l_1$. If $z \in l_\infty(\mathbb{Z}_+)$, then

$$\sum_{j=-\infty}^0 |n_j| \leq \|z\|_\infty \sum_{j=-\infty}^{-1} |j| f_j < \infty,$$

since the mean μ exists by assumption. Consider now the renewal equation

$$v_j = \sum_{k-\infty}^{\infty} v_{j-k} f_k + g_j + n_j, \qquad j \in \mathbb{Z},$$

and its solution $v_j = U * (g + n)(j)$, $j \in \mathbb{Z}$. Suppose there exist two solutions $z^{(1)}$, $z^{(2)} \in l_\infty(\mathbb{Z}_+)$ to equation (7). Let $v^{(1)}$, $v^{(2)}$ and $n^{(1)}$, $n^{(2)}$ correspond to $z^{(1)}$, $z^{(2)}$, respectively (see (9)). Then the sequence $V = v^{(1)} - v^{(2)} \in l_\infty$ satisfies the equation $V = F * V + n^{(1)} - n^{(2)}$ or, by (3),

$$(\delta_0 - F) * V = (\delta_0 - F_-) * (\delta_0 - F_+) * V = n^{(1)} - n^{(2)}.$$

Since the distribution F_- is defective, we have $U_- * (\delta_0 - F) = \delta_0$. Hence

(10) $\qquad (\delta_0 - F_+) * V(j) = (U_- * n^{(1)})(j) - (U_- * n^{(2)})(j), \qquad j \in \mathbb{Z}.$

The left-hand side is identically zero on $\mathbb{Z}_-$ and so is the right-hand side. Moreover, U_- is concentrated on $\mathbb{Z}_- \cup \{0\}$ and $n^{(1)}(j) - n^{(2)}(j) \equiv 0$ on $\mathbb{Z}_+$. Hence the right-hand side of (10) is identically zero on $\mathbb{Z}$. It follows that

$$n^{(1)}(j) - n^{(2)}(j) = (\delta_0 - F_-) * [(U_- * n^{(1)})(j) - (U_- * n^{(2)})(j)] \equiv 0$$

and we have

$$V_j = F * V(j) = \sum_{k=-\infty}^{\infty} V_{j-k} f_k, \qquad j \in \mathbb{Z}.$$

Let us consider the following function on $\mathbb{R}$:

$$V(x) := V_j + (x - j)(V_{j+1} - V_j), \qquad x \in [j, j+1], \quad j \in \mathbb{Z}.$$

We have $V(j) = V_j$, $j \in \mathbb{Z}$. Moreover, the function $V(x)$, $x \in \mathbb{R}$, is continuous and bounded: $\sup_{x \in \mathbb{R}} |V(x)| = \sup_{j \in \mathbb{Z}} |V_j| < \infty$ since $\{V_j\}_{j=-\infty}^{\infty} \in l_\infty$. Its graph is an infinite broken line. We now show that $V(x) = F * V(x)$, $x \in \mathbb{R}$. Let $x \in [j, j+1]$. Then, for $k \in \mathbb{Z}$, $x - k \in [j-k, j+1-k]$ and

$$F * V(x) = \int_{-\infty}^{\infty} V(x - y) \, F(dy) = \sum_{k=-\infty}^{\infty} V(x - k) f_k$$

$$= \sum_{k=-\infty}^{\infty} \{V_{j-k} + [x - k - (j-k)](V_{j+1-k} - V_{j-k})\} f_k = V_j + (x - j)(V_{j+1} - V_j) = V(x).$$

Applying Lemma 1, we see that $V(x)$ is periodic with period 1. However, $V_j = 0$ for $j \in \mathbb{Z}_-$. Hence $V_j = 0$ for all $j \in \mathbb{Z}$ and $z^{(1)} = z^{(2)}$, which proves the theorem. $\quad\square$

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