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ALMOST n -ARY AND ALMOST n -ARITIZABLE THEORIES

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ABSTRACT. We study possibilities for almost n -ary and n -aritzable theories. Their dynamics both in general case, for ω -categorical theories, and with respect to operations for theories are described.

Keywords: elementary theory, almost n -ary theory, almost n -aritzable theory.

We continue to study arities of theories and of their expansions [1]. In the present paper we introduce natural notions of almost n -ary and almost n -aritzable theories, and describe their dynamics both in general case, for ω -categorical theories, and with respect to operations for theories.

1. PRELIMINARIES

Recall a series of notions related to arities and arizabilities of theories.

Definition [2]. A theory T is said to be Δ -based, where Δ is some set of formulae without parameters, if any formula of T is equivalent in T to a Boolean combination of formulae in Δ .

For Δ -based theories T , it is also said that T has *quantifier elimination* or *quantifier reduction* up to Δ .

Definition [2, 3]. Let Δ be a set of formulae of a theory T , and $p(\bar{x})$ a type of T lying in $S(T)$. The type $p(\bar{x})$ is said to be Δ -based if $p(\bar{x})$ is isolated by a set of formulas $\varphi^\delta \in p$, where $\varphi \in \Delta$, $\delta \in \{0, 1\}$.

The following lemma, being a corollary of Compactness Theorem, noticed in [2].

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Lemma 1. *A theory T is Δ -based if and only if, for any tuple \bar{a} of any (some) weakly saturated model of T , the type $\text{tp}(\bar{a})$ is Δ -based.*

Definition [1]. An elementary theory T is called *unary*, or *1-ary*, if any T -formula $\varphi(\bar{x})$ is T -equivalent to a Boolean combination of T -formulas, each of which is of one free variable, and of formulas of form $x \approx y$.

For a natural number $n \geq 1$, a formula $\varphi(\bar{x})$ of a theory T is called *n -ary*, or an *n -formula*, if $\varphi(\bar{x})$ is T -equivalent to a Boolean combination of T -formulas, each of which is of n free variables.

For a natural number $n \geq 2$, an elementary theory T is called *n -ary*, or an *n -theory*, if any T -formula $\varphi(\bar{x})$ is n -ary.

A theory T is called *binary* if T is 2-ary, it is called *ternary* if T is 3-ary, etc.

We will admit the case $n = 0$ for n -formulae $\varphi(\bar{x})$. In such a case $\varphi(\bar{x})$ is just T -equivalent to a sentence $\forall \bar{x} \varphi(\bar{x})$.

If T is a theory such that T is n -ary and not $(n - 1)$ -ary then the value n is called the arity of T and it is denoted by $\text{ar}(T)$. If T does not have any arity we put $\text{ar}(T) = \infty$.

Similarly, for a formula φ of a theory T we denote by $\text{ar}_T(\varphi)$ the natural value n if φ is n -ary and not $(n - 1)$ -ary. If a theory T is fixed we write $\text{ar}(\varphi)$ instead of $\text{ar}_T(\varphi)$.

Clearly, $\text{ar}(\varphi) \leq |\text{FV}(\varphi)|$, where $\text{FV}(\varphi)$ is the set of free variables of formula φ .

The following example illustrates the notions above, and it will be used below.

Example 1. Recall [4, 5, 6] that a *circular*, or *cyclic* order relation is described by a ternary relation K_3 satisfying the following conditions:

- (co1) $\forall x \forall y \forall z (K_3(x, y, z) \rightarrow K_3(y, z, x))$;
- (co2) $\forall x \forall y \forall z (K_3(x, y, z) \wedge K_3(y, x, z) \leftrightarrow x = y \vee y = z \vee z = x)$;
- (co3) $\forall x \forall y \forall z (K_3(x, y, z) \rightarrow \forall t [K_3(x, y, t) \vee K_3(t, y, z)])$;
- (co4) $\forall x \forall y \forall z (K_3(x, y, z) \vee K_3(y, x, z))$.

Clearly, $\text{ar}(K_3(x, y, z)) = 3$ if the relation has at least three element domain. Hence, theories with infinite circular order relations are at least 3-ary.

The following generalization of circular order produces a *n -ball*, or *n -spherical*, or *n -circular* order relation, for $n \geq 4$, which is described by a n -ary relation K_n satisfying the following conditions:

$$\text{(nbo1)} \quad \forall x_1, \dots, x_n (K_n(x_1, x_2, \dots, x_n) \rightarrow K_n(x_2, \dots, x_n, x_1));$$

$$\text{(nbo2)} \quad \forall x_1, \dots, x_n \left(K_n(x_1, \dots, x_i, x_{i+1}, \dots, x_n) \wedge \right. \\ \left. \wedge K_n(x_1, \dots, x_{i+1}, x_i, \dots, x_n) \leftrightarrow \bigvee_{i=1}^{n-1} x_i = x_{i+1} \right);$$

$$\text{(nbo3)} \quad \forall x_1, \dots, x_n (K_n(x_1, \dots, x_n) \rightarrow \forall t [K_n(x_1, \dots, x_{n-1}, t) \vee K_n(t, x_2, \dots, x_n)]);$$

$$\text{(nbo4)} \quad \forall x_1, \dots, x_n (K_n(x_1, \dots, x_i, x_{i+1}, \dots, x_n) \vee K_n(x_1, \dots, x_{i+1}, x_i, \dots, x_n)), \quad i < n.$$

Clearly, $\text{ar}(K_n(x_1, \dots, x_n)) = n$ if the relation has at least n -element domain. Thus, theories with infinite n -ball order relations are at least n -ary.

Definition [1]. A T -formula $\varphi(\bar{x})$ is called *n -expansible*, or *n -arizable*, or *n -aritzable*, if T has an expansion T' such that $\varphi(\bar{x})$ is T' -equivalent to a Boolean combination of T' -formulas with n free variables.

A theory T is called *n-expansible*, or *n-arizable*, or *n-aritizable*, if there is an n -ary expansion T' of T .

A theory T is called *arizable* or *aritizable*, if T is n -aritizable for some n .

A 1-aritizable theory is called *unary-able*, or *unary-tizable*. A 2-aritizable theory is called *binary-tizable* or *binarizable*, a 3-aritizable theory is called *ternary-tizable* or *ternarizable*, etc.

Definition. [8] The *disjoint union* $\bigsqcup_{n \in \omega} \mathcal{M}_n$ of pairwise disjoint structures \mathcal{M}_n for pairwise disjoint predicate languages Σ_n , $n \in \omega$, is the structure of language $\bigcup_{n \in \omega} \Sigma_n \cup \{P_n^{(1)} \mid n \in \omega\}$ with the universe $\bigsqcup_{n \in \omega} M_n$, $P_n = M_n$, and interpretations of predicate symbols in Σ_n coinciding with their interpretations in \mathcal{M}_n , $n \in \omega$. The *disjoint union of theories* T_n for pairwise disjoint languages Σ_n accordingly, $n \in \omega$, is the theory

$$\bigsqcup_{n \in \omega} T_n \equiv \text{Th} \left(\bigsqcup_{n \in \omega} \mathcal{M}_n \right),$$

where $\mathcal{M}_n \models T_n$, $n \in \omega$. Taking empty sets instead of some structures \mathcal{M}_k we obtain disjoint unions of finitely many structures and theories. In particular, we have the disjoint unions $\mathcal{M}_0 \sqcup \dots \sqcup \mathcal{M}_n$ and their theories $T_0 \sqcup \dots \sqcup T_n$.

Theorem 1. [1]. 1. For any theories T_m , $m \in \omega$, and their disjoint union $\bigsqcup_{m \in \omega} T_m$, all T_m are n -theories iff $\bigsqcup_{m \in \omega} T_m$ is an n -theory, moreover,

$$\text{ar} \left(\bigsqcup_{m \in \omega} T_m \right) = \max \{ \text{ar}(T_m) \mid m \in \omega \}.$$

2. For any theories T_m , $m \in \omega$, and their disjoint union $\bigsqcup_{m \in \omega} T_m$, all T_m are n -aritizable iff $\bigsqcup_{m \in \omega} T_m$ is n -aritizable.

Definition [7]. Let \mathcal{M} and \mathcal{N} be structures of relational languages $\Sigma_{\mathcal{M}}$ and $\Sigma_{\mathcal{N}}$ respectively. We define the *composition* $\mathcal{M}[\mathcal{N}]$ of \mathcal{M} and \mathcal{N} satisfying the following conditions:

- 1) $\Sigma_{\mathcal{M}[\mathcal{N}]} = \Sigma_{\mathcal{M}} \cup \Sigma_{\mathcal{N}}$;
- 2) $M[\mathcal{N}] = M \times N$, where $M[\mathcal{N}]$, M , N are universes of $\mathcal{M}[\mathcal{N}]$, \mathcal{M} , and \mathcal{N} respectively;
- 3) if $R \in \Sigma_{\mathcal{M}} \setminus \Sigma_{\mathcal{N}}$, $\mu(R) = n$, then $((a_1, b_1), \dots, (a_n, b_n)) \in R_{\mathcal{M}[\mathcal{N}]}$ if and only if $(a_1, \dots, a_n) \in R_{\mathcal{M}}$;
- 4) if $R \in \Sigma_{\mathcal{N}} \setminus \Sigma_{\mathcal{M}}$, $\mu(R) = n$, then $((a_1, b_1), \dots, (a_n, b_n)) \in R_{\mathcal{M}[\mathcal{N}]}$ if and only if $a_1 = \dots = a_n$ and $(b_1, \dots, b_n) \in R_{\mathcal{N}}$;
- 5) if $R \in \Sigma_{\mathcal{M}} \cap \Sigma_{\mathcal{N}}$, $\mu(R) = n$, then $((a_1, b_1), \dots, (a_n, b_n)) \in R_{\mathcal{M}[\mathcal{N}]}$ if and only if $(a_1, \dots, a_n) \in R_{\mathcal{M}}$, or $a_1 = \dots = a_n$ and $(b_1, \dots, b_n) \in R_{\mathcal{N}}$.

The composition $\mathcal{M}[\mathcal{N}]$ is called *e-definable*, or *equ-definable*, if $\mathcal{M}[\mathcal{N}]$ has an \emptyset -definable equivalence relation E whose E -classes are universes of the copies of \mathcal{N} forming $\mathcal{M}[\mathcal{N}]$. If the equivalence relation E is fixed, the *e-definable composition* is called *E-definable*.

Using a nice basedness of E -definable compositions $T_1[T_2]$ (see [7]) till the formulas of form $E(x, y)$ and generating formulas for T_1 and T_2 we have the following:

Theorem 2. [1]. 1. For any theories T_1 and T_2 and their E -definable composition $T_1[T_2]$, T_1 and T_2 are n -theories, for $n \geq 2$, iff $T_1[T_2]$ is an n -theory, moreover, $\text{ar}(T_1[T_2]) = \max\{\text{ar}(T_1), \text{ar}(T_2)\}$, if models of T_1 and of T_2 have at least two elements, and $\text{ar}(T_1[T_2]) = \max\{\text{ar}(T_1), \text{ar}(T_2), 2\}$, if a model of T_1 or T_2 is a singleton.

2. For any theories T_1 and T_2 and their E -definable composition $T_1[T_2]$, T_1 and T_2 are n -aritizable iff $T_1[T_2]$ is n -aritizable.

2. ALMOST n -ARY AND n -ARITIZABLE THEORIES, THEIR DYNAMICS

Definition. (Cf. [5, 6]) A theory T is called *almost n -ary* if there are finitely many formulae $\varphi_1(\bar{x}), \dots, \varphi_m(\bar{x})$ such that each T -formula is T -equivalent to a Boolean combination of n -formulae and formulae obtained by substitutions of free variables in $\varphi_1(\bar{x}), \dots, \varphi_m(\bar{x})$.

In such a case we say that the formulae $\varphi_1(\bar{x}), \dots, \varphi_m(\bar{x})$ witness that T is almost n -ary.

Almost 1-ary theories are called *almost unary*, almost 2-ary theories are called *almost binary*, almost 3-ary theories are called *almost ternary*, etc.

A theory T is called *almost n -aritizable* if some expansion T' of T is almost n -ary.

Almost 1-aritizable theories are called *almost unary-tizable*, almost 2-aritizable theories are called *almost binarizable*, almost 3-aritizable theories are called *almost ternarizable*, etc.

The following properties are obvious.

1. Any n -ary (respectively, n -aritizable) theory is almost n -ary (almost n -aritizable).

2. Any almost n -ary (respectively, n -aritizable) theory is almost k -ary (almost k -aritizable) for any $k \geq n$.

3. Any theory of a finite structure is almost unary.

Families of weakly circularly minimal structures produce examples of almost binary theories which are not binary [5, 6]. Similarly natural generalizations of weakly circularly minimal structures till n -circular orders give examples of almost $(n-1)$ -ary theories T_n with $\text{ar}(T_n) = n$, $n \geq 4$.

Assuming that the witnessing set $\{\varphi_1(\bar{x}), \dots, \varphi_m(\bar{x})\}$ is minimal for the almost n -ary theory T we have either $m = 0$ or $l(\bar{x}) > n$.

Thus we have two minimal characteristics witnessing the almost n -arity of T : m and $l(\bar{x})$. The pair $(m, l(\bar{x}))$ is called the *degree* of the almost n -arity of T , or the *aar-degree* of T , denoted by $\text{deg}_{\text{aar}}(T)$. Here we assume that n is minimal with almost n -arity of T , this n is denoted by $\text{aar}(T)$. Clearly, $\text{aar}(T) \leq \text{ar}(T)$, and if $m = 0$, i.e., $n = \text{ar}(T) = \text{aar}(T)$ then it is supposed that $l(\bar{x}) = 0$, too.

We have $\text{aar}(T) \in \omega$ if and only if $\text{ar}(T) \in \omega$. So if $\text{ar}(T) = \infty$ then it is natural to put $\text{aar}(T) = \infty$.

Besides, $n = \text{ar}(T) = \text{aar}(T) \in \omega$ means that the set $\Delta_n(T)$ of T -formulae with n free variables allows to express all \emptyset -definable sets for T by Boolean combinations and taking any set $\Delta_k(T)$ of T -formulae with $k < n$ free variables all \emptyset -definable sets for T can be expressed by Boolean combinations of formulae in Δ_k and substitutions of infinitely many formulae $\varphi(\bar{x})$ only, where $l(\bar{x}) = n$.

By the definition if $\text{aar}(T) = n$ then

$$(1) \quad \text{deg}_{\text{ar}}(T) \in \{(0, 0)\} \cup \{(m, r) \mid m \in \omega \setminus \{0\}, r \in \omega, r > n\}.$$

The described pairs in the relation (1) are called *admissible*.

Theorem 3. *For any $m, n \in \omega \setminus \{0\}$ with $m \leq n$ there is a theory T_{mn} with $\text{aar}(T_{mn}) = m$ and $\text{ar}(T_{mn}) = n$.*

Proof. We use disjoint unions of *dense* n -spherically ordered theories T_n with infinite orders, i.e., theories, generated by n -spherical orders $K_n(x_1, x_2, x_3, \dots, x_n)$ satisfying the axioms

$$\begin{aligned} \forall x_1, x_2, \dots, x_n \left(K_n(x_1, x_2, x_3, \dots, x_n) \wedge \bigwedge_{i \neq j} \neg x_i \approx x_j \rightarrow \right. \\ \left. \rightarrow \exists y \left(\bigwedge_{i \leq n} \neg x_i \approx y \wedge K_n(x_1, y, x_3, \dots, x_n) \right) \right). \end{aligned}$$

For $n = 2$ we take the theory T_2 of dense linear order $K_2(x_1, x_2)$ without endpoints, having $\text{ar}(T_2) = 2$, and for $n = 1$ — the theory T_1 of the empty languages, having $\text{ar}(T_1) = 1$.

Similarly to dense linear orders and dense circular orders, dense n -spherical orders produce quantifier eliminations with $\text{ar}(T_n) = n$, $n \in \omega \setminus \{0\}$.

Using Theorem 1 we obtain $m = \text{ar}(T) = \text{aar}(T)$ taking $T_{mm} = \bigsqcup_{r \in \omega} T_m^r$, where T_m^r are copies of T_m in disjoint languages $\{K_m^r\}$, $r \in \omega$. Indeed, disjoint predicates K_m^r producing $\text{ar}(T_m) = m$ witness that $\text{ar}(T_{mm}) = m$. Finitely many these predicates can not define all definable sets for T_{mm} since there are infinitely many of them. Thus, $\text{aar}(T_{mm}) = m$, too.

Now for any $m < n$ we form $T_{mn} = T_n \sqcup \bigsqcup_{r \in \omega} T_m^r$. By T_n in T_{mn} and $m < n$ we have $\text{ar}(T_{mn}) = n$. And $\text{aar}(T_{mn}) = m$ since there are infinitely many disjoint predicates of arity m .

The following theorem shows that all admissible pairs are realized.

Theorem 4. *For any admissible pair (m, r) and $n \in \omega \setminus \{0, 1\}$ there is a theory T with $\text{aar}(T) = n$ and $\text{deg}_{\text{ar}}(T) = (m, r)$.*

Proof. The admissible pair $(0, 0)$ with $\text{ar}(T) = n$ is realized by Theorem 3. Now for an admissible pair $(m, r) \neq (0, 0)$ we can take a disjoint union T of countably many dense n -spherically ordered theories and of m dense r -spherically ordered theories. Using arguments for Theorem 3 we obtain $\text{aar}(T) = n$ and $\text{deg}_{\text{ar}}(T) = (m, r)$. \square

Proposition 1. *Any almost n -ary theory T is k -ary for some $k \geq n$.*

Proof. Let T be an almost n -ary theory witnessed by the formulae $\varphi_1(\bar{x}), \dots, \varphi_m(\bar{x})$. Then taking the set Δ_k of all formulae with $k = \max\{n, l(\bar{x})\}$ we observe, using $\varphi_1(\bar{x}), \dots, \varphi_m(\bar{x})$, that T is Δ_k -based, i.e., T is k -ary.

Corollary 1. *Any theory T is n -ary for some n iff T is almost m -ary for some m .*

Corollary 2. *Any theory T is n -aritzable for some n iff T is almost m -aritzable for some m .*

3. ω -CATEGORICAL ALMOST n -ARY AND n -ARITIZABLE THEORIES

Proposition 2. *If T is an almost n -ary ω -categorical theory, for some n , then T is almost k -ary for any $k \in \omega \setminus \{0\}$, i.e., $\text{aar}(T) = 1$.*

Proof. If $k \geq n$ then T is almost m -ary, as noticed above. If $k < n$ then we collect in a set Z all formulae $\varphi_1(\bar{x}), \dots, \varphi_m(\bar{x})$ witnessing the almost n -arity of T and, by Ryll-Nardzewski Theorem, all non-equivalent formulae $\psi_1(\bar{y}), \dots, \psi_r(\bar{y})$ with $l(\bar{y}) = n$. Clearly, the set Z witnesses that T is almost k -ary. Taking $k = 1$ we obtain $\text{aar}(T) = 1$.

Corollary 3. *For any ω -categorical theory T either $\text{aar}(T) = 1$ with $\text{ar}(T) \in \omega$, or $\text{aar}(T) = \infty$ with $\text{ar}(T) = \infty$.*

The following example illustrates Corollary 3.

Example 2. Taking a dense linear order K_2 without endpoint we can step-by-step extend it to a chain of dense n -spherical orders K_n , $n \geq 2$, in the following way.

We put $(a, b, c) \in K_3$ if $(a, b) \in K_2$ and $(b, c) \in K_2$, or $(b, c) \in K_2$ and $(c, a) \in K_2$, or $(c, a) \in K_2$ and $(a, b) \in K_2$. If K_n , $n \geq 3$, is defined then we put $(a_1, \dots, a_{n+1}) \in K_{n+1}$ if $(a_1, \dots, a_n) \in K_n$ and $(a_2, \dots, a_{n+1}) \in K_n$, or $a_2, \dots, a_{n+1} \in K_n$ and $(a_3, \dots, a_{n+1}, a_1) \in K_n$, or $(a_3, \dots, a_{n+1}, a_1) \in K_n$ and $(a_4, \dots, a_{n+1}, a_1, a_2) \in K_n$. The obtained structure \mathcal{M}_∞ in the language $\Sigma_\infty = \{K_n \mid n \in \omega \setminus \{0, 1\}\}$ has an ω -categorical theory T_∞ with $\text{aar}(T_\infty) = \text{ar}(T_\infty) = \infty$, since each new K_n increases the arity. The same characteristics have restrictions of T_∞ to any infinite sublanguages.

At the same time each restriction \mathcal{M} of \mathcal{M}_∞ to a finite nonempty sublanguage $\{K_{n_1}, \dots, K_{n_m}\}$ produces a theory T with $\text{aar}(T) = 1$ with

$$\text{ar}(T) = \max\{n_1, \dots, n_m\}.$$

By Corollary 3 and the definition of almost aritizability we immediately have:

Corollary 4. *Any restriction T of n -ary ω -categorical theory T' is almost unary-tizable.*

4. OPERATIONS FOR ALMOST n -ARY AND n -ARITIZABLE THEORIES

In this section we consider links for arities of theories with respect to disjoint unions of theories and E -definable compositions of theories.

Theorem 5. 1. *For any theories T_1, T_2 and their disjoint union $T_1 \sqcup T_2$, both T_1 and T_2 are almost n -ary iff $T_1 \sqcup T_2$ is almost n -ary, moreover, $\text{aar}(T_1 \sqcup T_2) = \max\{\text{aar}(T_1), \text{aar}(T_2)\}$.*

2. *For any theories T_1, T_2 and their disjoint union $T_1 \sqcup T_2$, both T_1 and T_2 are almost n -aritizable iff $T_1 \sqcup T_2$ is almost n -aritizable.*

Proof. 1. Let Φ_1 and Φ_2 be finite sets of formulas witnessing that T_1 and T_2 are almost n -ary, respectively. Using the definition of disjoint union and Theorem 1 we obtain that the finite set $\Phi_1 \cup \Phi_2$ witnesses that $T_1 \sqcup T_2$ is almost n -ary. Conversely, if a finite set Φ of formulas witnesses that $T_1 \sqcup T_2$ is almost n -ary then Φ witnesses that both T_1 and T_2 are almost n -ary.

The equality $\text{aar}(T_1 \sqcup T_2) = \max\{\text{aar}(T_1), \text{aar}(T_2)\}$ follows from the definition of disjoint union since if $\text{aar}(T_1 \sqcup T_2) = k$ then the maximal value of $\text{aar}(T_1)$ and $\text{aar}(T_2)$ is responsible for this equality.

Item 2 follows from Item 1 since expansions of T_1 and T_2 correspond expansions of $T_1 \sqcup T_2$: some expansions of T'_1 and T'_2 of T_1 and T_2 , respectively, are almost n -ary iff $T'_1 \sqcup T'_2$ produces an almost n -ary expansion of $T_1 \sqcup T_2$.

Using induction we obtain the following:

Corollary 5. 1. For any theories T_1, T_2, \dots, T_m and their disjoint union $\bigsqcup_{i=1}^m T_i$, all T_1, T_2, \dots, T_m are almost n -ary iff $\bigsqcup_{i=1}^m T_i$ is almost n -ary, moreover,

$$\text{aar} \left(\bigsqcup_{i=1}^m T_i \right) = \max\{\text{aar}(T_i) \mid i \leq m\}.$$

2. For any theories T_1, T_2, \dots, T_m and their disjoint union $\bigsqcup_{i=1}^m T_i$, all T_1, T_2, \dots, T_m are almost n -arizable iff $\bigsqcup_{i=1}^m T_i$ is almost n -arizable.

Remark 1. Both almost n -arity and almost n -arizability can fail taking disjoint unions of infinitely many theories T_i , $i \in I$. Indeed, each theory T_i can have its own finite set Φ_i of formulas witnessing the almost n -arity/ n -arizability, say in disjoint languages, whereas finite unions $\bigcup \Phi_i$ can not witness the almost n -arity/ n -arizability for $\bigsqcup_{i \in I} T_i$.

Generalizing Theorem 2 we obtain:

Theorem 6. 1. For any theories T_1 and T_2 and their E -definable composition $T_1[T_2]$, T_1 and T_2 are almost n -ary iff $T_1[T_2]$ is almost n -ary, moreover, $\text{aar}(T_1[T_2]) = \max\{\text{aar}(T_1), \text{aar}(T_2), 2\}$, if models of T_1 and of T_2 have at least two elements, and $\text{aar}(T_1[T_2]) = \max\{\text{aar}(T_1), \text{aar}(T_2)\}$, if a model of T_1 or T_2 is a singleton.

2. For any theories T_1 and T_2 and their E -definable composition $T_1[T_2]$, T_1 and T_2 are almost n -arizable iff $T_1[T_2]$ is almost n -arizable.

Proof. 1. Let T_i be Δ_i -based for $i = 1, 2$. Since $T_1[T_2]$ is E -definable it is Δ -based, where Δ consists of formulae in $\Delta_1 \cup \Delta_2$ and $E(x, y)$ [7]. Now assuming that T_1 and T_2 are almost n -ary we can choose Δ_i consisting of n -formulae and finitely many formulae forming Φ_i , $i = 1, 2$. Hence $T_1[T_2]$ is almost n -ary.

Conversely, if $T_1[T_2]$ is almost n -ary and it is witnessed by a set Φ of formulae then Φ witnesses that T_1 and T_2 are almost n -ary.

If models of T_1 and of T_2 have at least two elements then $T_1[T_2]$ is at least binary that witnessed by the formula $E(x, y)$. Thus since $T_1[T_2]$ is Δ -based we have $\text{aar}(T_1[T_2]) = \max\{\text{aar}(T_1), \text{aar}(T_2), 2\}$. If T_1 or T_2 is a theory of singleton then $T_1[T_2]$ is $(\Delta_1 \cup \Delta_2)$ -based implying $\text{aar}(T_1[T_2]) = \max\{\text{aar}(T_1), \text{aar}(T_2)\}$.

Item 2 follows from Item 1 repeating the arguments for Item 2 of Theorem 5.

Theorem 6 immediately implies

Corollary 6. Any composition of finitely many almost n -ary (almost n -arizable) theories, for $n \geq 2$, is again many almost n -ary (almost n -arizable).

5. CONCLUSION

We considered possibilities for almost arities and almost aritizabilities of theories and their dynamics both in general case, for ω -categorical theories, and with respect to operations for theories. It would be interesting to describe values of almost arities and almost aritizabilities for natural classes of theories.

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