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PROPERTIES OF RANKS FOR FAMILIES OF STRONGLY MINIMAL THEORIES

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Abstract. We study rank properties for families of strongly minimal theories. A criterion for e-total transcendence of families of strongly minimal theories is obtained in terms of the description of the language symbols.

Keywords: strongly minimal theory, sentence, family of theories, rank for a family of theories.

1. Introduction

In the study of elementary theories, an important role is played by the description of their interrelationships and derived objects, including main characteristics. Essential interrelationships between theories arise when considering their various combinations [1, 2], closures and generating sets for families of theories [3, 4], approximations of theories [5]. Along with combinations of theories in general, various combinations of ordered theories are considered, which allow describing their essential structural properties [6, 7, 8, 9]. Among various approximations of theories, one of the central places is occupied by approximations by theories of finite structures, which define pseudofinite structures [10, 11, 12, 13]. Important derived objects are algebras of definable subfamilies [14, 15]. Main characteristics related to the families of theories are the e-spectra [1, 16], the ranks for the families of theories [14, 17, 18] and for the formulas [19, 20]. Rank values allow one to determine how complex certain types of families of theories are, both in general case and in relation to natural classes of theories, including classes of substitution theories [21], theories of abelian groups [22], ordered theories [23, 24].

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The present article is concerned the notion of *strong minimality* initially deeply investigated in [25]. We study rank properties for families of strongly minimal theories. A criterion for e-total transcendence of families of strongly minimal theories is obtained in terms of the description of the language symbols.

2. Preliminary notions

Definition 1. [25] Let T be a complete theory without finite models. The theory T is said to be *strongly minimal* if for any model $\mathcal{M} \models T$, any formula $\phi(x, \bar{y})$ and any tuple $\bar{a} \in M$ either the set $\phi(\mathcal{M}, \bar{a})$ is finite or the set $\neg\phi(\mathcal{M}, \bar{a})$ is finite.

Definition 2. [17] Let $\mathcal{T}$ be a family of complete theories in a fixed language Σ, ϕ be an arbitrary Σ-sentence. Then the set $\mathcal{T}_\phi := \{T \in \mathcal{T} \mid T \models \phi\}$ is said to be the *ϕ-neighbourhood* of the family $\mathcal{T}$, or the *s-definable* subfamily of $\mathcal{T}$ defined by the sentence ϕ.

We define the *rank* RS for the family $\mathcal{T}$ as follows:

(1) $\mathrm{RS}(\mathcal{T}) = -1$, if $\mathcal{T} = \emptyset$.

(2) $\mathrm{RS}(\mathcal{T}) = 0$, if $\mathcal{T}$ is a finite nonempty family.

(3) $\mathrm{RS}(\mathcal{T}) \geq 1$, if $\mathcal{T}$ is infinite.

(4) $\mathrm{RS}(\mathcal{T}) \geq \alpha + 1$, if there are pairwise inconsistent Σ-sentences $\phi_n, n \in \omega$, such that $\mathrm{RS}(\mathcal{T}_{\phi_n}) \geq \alpha$.

(5) If δ is a limit ordinal, then $\mathrm{RS}(\mathcal{T}) \geq \delta$, if $\mathrm{RS}(\mathcal{T}) \geq \beta$ for any $\beta < \delta$.

We put $\mathrm{RS}(\mathcal{T}) = \alpha$, if $\mathrm{RS}(\mathcal{T}) \geq \alpha$ and $\neg[\mathrm{RS}(\mathcal{T}) \geq \alpha + 1]$.

If $\mathrm{RS}(\mathcal{T}) \geq \alpha$ for any α, then we put $\mathrm{RS}(\mathcal{T}) = \infty$.

A family $\mathcal{T}$ is called *e-totally transcendental*, or *totally transcendental*, if $\mathrm{RS}(\mathcal{T})$ is an ordinal.

If a family $\mathcal{T}$ is e-totally transcendental, with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, then the *degree* $\mathrm{ds}(\mathcal{T})$ of $\mathcal{T}$ is the maximal number of pairwise inconsistent sentences ϕ_i with $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

By the definition, if $\mathrm{RS}(\mathcal{T}) = \alpha \in \mathrm{Ord}$ then $\mathrm{ds}(\mathcal{T}) \in \omega \setminus \{0\}$.

3. Families of strongly minimal theories and their ranks

Throughout this article we denote by $\mathcal{T}_\Sigma$ the family of all strongly minimal theories in a language Σ.

Proposition 1. *Let $\Sigma^1_\kappa := \{P^1_i\}_{i<\kappa}$, where κ is a cardinal. Then the following holds:*

(1) *If $\kappa < \omega$ then $\mathrm{RS}(\mathcal{T}_{\Sigma^1_\kappa}) = \kappa$.*

(2) *If $\kappa \geq \omega$ then $\mathrm{RS}(\mathcal{T}_{\Sigma^1_\kappa}) = \infty$.*

Proof. (1) By virtue of strong minimality, each unary predicate selects a finite or cofinite set in any model of an arbitrary strongly minimal theory.

We construct an infinitely branching tree of length κ for the family $\mathcal{T}_{\Sigma^1_\kappa}$ of theories. At the first level, we distinguish theories in which the first predicate P_1 selects in any model a finite set consisting of n elements for each $n < \omega$. Further, at the i-th level ($i \leq \kappa$), we distinguish theories in which the predicate P_i selects a finite set consisting of n elements for each $n < \omega$. Therefore, we obtain $\mathrm{RS}(\mathcal{T}_{\Sigma^1_\kappa}) = \kappa$.

(2) Since there exist infinitely many unary predicates, then one can construct an infinitely branching tree of infinite length, whence $\mathrm{RS}(\mathcal{T}_{\Sigma^1_\kappa}) = \infty$. □

Let $\Sigma_1^f := \{f^1\}$, where f is a unary functional symbol, i.e. any theory T of the language Σ_1^f satisfies the following sentence: $T \models (\forall x)(\exists! y) f(x) = y$.

Let T be a complete theory of the language Σ_1^f, $M \models T$.

We say that an element $a \in M$ *forms a cycle of length* 1 if $f(a) = a$.

We say that elements $a_1, a_2, \ldots, a_n \in M$ *form a cycle of length* n if $a_i \neq a_j$ for any $1 \leq i, j \leq n$ with $i \neq j$, $f(a_1) = a_2, f(a_2) = a_3, \ldots, f(a_{n-1}) = a_n$ and $f(a_n) = a_1$.

Consider the following sentences:

$$\theta_n^1 := \exists x_1 \ldots \exists x_n [\wedge_{i \neq j} x_i \neq x_j \wedge \wedge_{i=1}^{n-1} f(x_i) = x_{i+1} \wedge f(x_n) = x_1 \wedge$$
$$\forall y_1 \ldots \forall y_n (\wedge_{i \neq j} y_i \neq y_j \wedge \wedge_{i=1}^{n-1} f(y_i) = y_{i+1} \wedge f(y_n) = y_1 \to \vee_{i=1}^{n} y_1 = x_i)],$$
$$\theta_n^2 := \exists x_1 \ldots \exists x_n \exists y_1 \ldots \exists y_n [\wedge_{i \neq j} x_i \neq x_j \wedge \wedge_{i=1}^{n-1} f(x_i) = x_{i+1} \wedge f(x_n) = x_1 \wedge$$
$$\wedge_{i \neq j} y_i \neq y_j \wedge \wedge_{i=1}^{n-1} f(y_i) = y_{i+1} \wedge f(y_n) = y_1 \wedge$$
$$\forall z_1 \ldots \forall z_n (\wedge_{i \neq j} z_i \neq z_j \wedge \wedge_{i=1}^{n-1} f(z_i) = z_{i+1} \wedge f(z_n) = z_1 \to \vee_{i=1}^{n} (z_1 = x_i \vee z_1 = y_i))].$$

The sentence θ_n^1 asserts that there exists a unique cycle of length n, and the sentence θ_n^2 asserts that there exist exactly two cycles of length n. Similarly, we can construct a sentence θ_n^k asserting that there exist exactly k cycles of length n, for any $k, n < \omega$.

Proposition 2. *Let* $\Sigma_1^f := \{f^1\}$, *where* f *is a unary functional symbol. Then* $\mathrm{RS}(\mathcal{T}_{\Sigma_1^f}) = \infty$.

Proof. We construct an infinitely branching tree of infinite length for $\mathcal{T}_{\Sigma_1^f}$.

At the first level, we distinguish theories in which there are exactly n cycles of length 1 for each $n < \omega$. At the second level, we distinguish theories in which there are exactly n cycles of length 2 for each $n < \omega$. At the k-th level ($k < \omega$), we distinguish theories in which there are exactly n cycles of length k for each $n < \omega$. Thus, we obtain $\mathrm{RS}(\mathcal{T}_{\Sigma_1^f}) = \infty$. □

Since k-ary predicates and k-ary operations, for $k \geq 2$, allow to interpret unary operations, Proposition 2 implies:

Corollary 1. *If a language* Σ *contains a* k*-ary predicate symbol or a* k*-ary functional symbol, where* $k \geq 2$, *then* $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$.

Proposition 3. [18] *If* Σ *is a language consisting of constant symbols then for the family* $\mathcal{T}$ *of all theories in the language* Σ *either* $\mathrm{RS}(\mathcal{T}) = 1$ *and* $\mathrm{ds}(\mathcal{T}) = P(n)$, *where* $P(n)$ *is the number of partitions of* n*-element sets, if* Σ *consists of* n *symbols, where* $n < \omega$, *or* $\mathrm{RS}(\mathcal{T}) = \infty$, *if* Σ *has infinitely many symbols.*

Proposition 3 implies:

Fact 1. *Let* $\Sigma_\kappa^c := \{c_i\}_{i<\kappa}$, *where* κ *is a cardinal. Then* $\mathrm{RS}(\mathcal{T}_{\Sigma_\kappa^c}) = 0$ *and* $\mathrm{ds}(\mathcal{T}_{\Sigma_\kappa^c}) = P(\kappa)$, *where* $P(\kappa)$ *is the number of partitions of* n*-element set, if* $\kappa < \omega$, *or* $\mathrm{RS}(\mathcal{T}_{\Sigma_\kappa^c}) = \infty$, *if* $\kappa \geq \omega$.

Proposition 4. *Let* $\Sigma_{k,m}^{1,c} := \{P_1^1, \ldots, P_k^1, c_1, \ldots, c_m\}$, *where* $k, m < \omega$. *Then* $\mathrm{RS}(\mathcal{T}_{\Sigma_{k,m}^{1,c}}) = k$ *and* $\mathrm{ds}(\mathcal{T}_{\Sigma_{k,m}^{1,c}}) = P'(k,m)$, *where* $P'(k,m)$ *is the total number of actions by predicates* $P_1^1, \ldots, P_k^1$ *on each of* $P(m)$ *partitions of the set* $\{c_1, \ldots, c_m\}$ *on* r *parts,* $1 \leq r \leq m$: $P'(k,m) = \sum\limits_r 2^{kr}$.

Proof. By virtue of Proposition 1 and Fact 1, we assert that $\mathrm{RS}(\mathcal{T}_{\Sigma^{1,c}_{k,m}}) = k$. Let us now understand that $\mathrm{ds}(\mathcal{T}_{\Sigma^{1,c}_{k,m}}) = 2^{km}$. Each of the unary predicates P^1_i can act on the set of identified constants c_j in two ways, i.e. either $P^1_i(c_j)$, for all identified constants c_j, or $\neg P^1_i(c_j)$ (each constant has only two possibilities for a chosen unary predicate). Therefore, taking an arbitrary partition of the set $\{c_1, \ldots, c_m\}$ into r parts, we obtain the multiplication r times 2^k of the actions of the predicates $P^1_1, \ldots, P^1_k$ on each element partition equal to 2^{kr}. Running through all the partitions of constants and all variants of the influence of the predicates $P^1_1, \ldots, P^1_k$ on the elements of the partitions of the set $\{c_1, \ldots, c_m\}$, we obtain the desired equality $\mathrm{ds}(\mathcal{T}_{\Sigma^{1,c}_{k,m}}) = P'(k, m)$. □

Proposition 5. [18] *If Σ is a language of 0-ary predicates, then for the family $\mathcal{T}$ of all theories in the language Σ either $\mathrm{RS}(\mathcal{T}) = 0$ and $\mathrm{ds}(\mathcal{T}) = 2^m$, if Σ consists of m 0-ary predicates, where $m < \omega$, or $\mathrm{RS}(\mathcal{T}) = \infty$, if Σ has infinitely many symbols.*

Proposition 5 implies:

Fact 2. *Let $\Sigma^0_\kappa := \{P^0_i\}_{i<\kappa}$, where κ is a cardinal. Then either $\mathrm{RS}(\mathcal{T}_{\Sigma^0_\kappa}) = 0$ and $\mathrm{ds}(\mathcal{T}_{\Sigma^c_\kappa}) = 2^\kappa$, if $\kappa < \omega$, or $\mathrm{RS}(\mathcal{T}_{\Sigma^0_\kappa}) = \infty$, if $\kappa \geq \omega$.*

In view of Fact 2 if we add to the language Σ finitely many 0-ary predicate symbols then we preserve the e-total transcendence of the family of all strongly minimal theories in the expanded language Σ', whereas adding to the language Σ infinitely many 0-ary predicate symbols we obtain the family $\mathcal{T}'_\Sigma$ with $\mathrm{RS}(\mathcal{T}'_\Sigma) = \infty$. Thus, Propositions 1, 2, 4, Corollary 1, and Facts 1, 2 imply the following:

Theorem 1. *A family $\mathcal{T}_\Sigma$ of strongly minimal theories is e-totally transcendental if and only if the language Σ consists of finitely many constant symbols as well as of finitely many 0-ary and unary predicate symbols.*

Remark 1. Assuming that distinct constant symbols are interpreted by distinct elements, e-totally transcendental families of strongly minimal theories in Theorem 1 admit arbitrarily many constant symbols in the language.

Remark 2. Since $\mathrm{RS}(\mathcal{T}) \leq \mathrm{RS}(\mathcal{T}')$ for $\mathcal{T} \subseteq \mathcal{T}'$ then the criterion for e-total transcendence of families of strongly minimal theories in Theorem 1 can be transformed for any family $\mathcal{T}'$ containing the family $\mathcal{T}_\Sigma$ of strongly minimal theories of given language Σ. In particular, it holds for families of ω-stable, superstable, and stable theories.

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