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ON ALGEBRAIC ISOMORPHISMS OF COHOMOLOGY OF A COMPACTIFICATION OF THE NÉRON MODEL WITH MULTIPLICATIONS FROM AN IMAGINARY QUADRATIC FIELD

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ABSTRACT. It is proved that the Grothendieck standard conjecture of Lefschetz type holds for rational cohomology of degrees 2, 3 of a Künnemann compactification of the Néron minimal model of an absolutely simple principally polarized Abelian variety of non-exceptional dimension divisible by 4 over the field of rational functions of a smooth projective curve provided that the ring of endomorphisms of the generic geometric fibre is an order of an imaginary quadratic field.

Keywords: Abelian variety, Néron minimal model, Künnemann compactification, Grothendieck standard conjecture of Lefschetz type.

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Introduction

Let H be an ample divisor on a smooth complex projective d-dimensional variety X. Then, for any natural number $i \leq d$, the map

$$L^{d-i} : H^i(X, \mathbb{Q}) \xrightarrow{\smile \mathrm{cl}_X(H)^{\smile d-i}} H^{2d-i}(X, \mathbb{Q})$$

is an isomorphism by the strong Lefschetz theorem. The Grothendieck standard conjecture $B(X)$ of Lefschetz type [1] asserts that there exists an algebraic $\mathbb{Q}$-cycle Z on the Cartesian product $X \times X$ which yields the inverse *algebraic* isomorphism $H^{2d-i}(X, \mathbb{Q}) \xrightarrow[\sim]{x \mapsto \mathrm{pr}_{2*}\left(\mathrm{pr}_1^* x \smile \mathrm{cl}_{X\times X}(Z)\right)} H^i(X, \mathbb{Q})$.

A reader can find examples of varieties satisfying the standard conjecture in [1] - [17].

Due to Tankeev [15] and in order to establish notations, we recall some facts concerning Künnemann compactifications of Néron minimal models of Abelian varieties.

Let $\mathcal{M} \to C$ be the Néron minimal model of the Abelian variety $\mathcal{M}_\eta$ over the field $\kappa(\eta)$ of rational functions of a smooth complex projective curve C. Suppose that at any place $s \in C$ the reduction of the Abelian variety $\mathcal{M}_\eta$ is semi-stable in Grothendieck's sense. In this case the connected component $\mathcal{M}_s^0$ of the neutral element of the algebraic group $\mathcal{M}_s$ is an extension of an Abelian variety by a linear torus whose dimension r_s is called *the toric* (*reductive*) rank at the place s [18, Section 2.1.12].

Let R be a Dedekind domain with the fraction field K and let A_η be an Abelian variety over $\eta = \operatorname{Spec} K$ such that all reductions are semi-stable in Grothendieck's sense. As it was shown by Künnemann [19, Section 5.8], in this case there exists a finite extension K' of the field K such that the Abelian variety $A_\eta \otimes_K K'$ has (not necessarily unique) a flat projective regular model P' over the integral closure R' of the ring R in the field K'; this model P' has strict semi-stable reductions over each localization of the ring R' (in particular, every special fibre P'_s is a union of smooth divisors of multiplicity 1 with normal crossings [20, Section 1.9]), and the scheme P' contains the Néron minimal model $\mathcal{A}'$ of the variety $A_\eta \otimes_K K'$ in the case when all residue fields of the scheme $\operatorname{Spec} R'$ are perfect [20, Section 4.4, Theorem 4.6].

Using the results of Künnemann cited above we may assume after the base change determined by an appropriate ramified covering $\widetilde{C} \to C$ that, for the Néron minimal model $\mathcal{M} \to C$, there exists a smooth compactification X of the variety $\mathcal{M}$ which is flat and projective over the curve C such that the following conditions hold:

(i) the model X/C has strictly semi-stable reductions (in particular, all fibres of the structure morphism $\pi : X \to C$ are unions of smooth irreducible components of multiplicity 1 with normal crossings);

(ii) the variety X contains the variety $\mathcal{M}$ as an open dense subscheme;

(iii) the restriction $\pi|_{\mathcal{M}} : \mathcal{M} \to C$ coincides with the structure morphism of the Néron model;

(iv) the connected component $\mathcal{M}_s^0$ of the neutral element of any fibre $\mathcal{M}_s$ $(s \in C)$ is an extension of an Abelian variety by a linear torus of dimension r_s;

(v) C-group law $\mathcal{M}^0 \times_C \mathcal{M}^0 \to \mathcal{M}^0$ can be expanded to a group C-action $\mathcal{M}^0 \times_C X \to X$.

We call such compactifications of the Néron model by *Künnemann compactifications.*

By definition, the Abelian variety $\mathcal{M}_\eta$ has a *trivial* trace if, for any finite ramified covering $\widetilde{C} \to C$, the group scheme $\mathcal{M} \times_C \widetilde{C} \to \widetilde{C}$ has no non-trivial *constant* Abelian subscheme.

In this article we prove the following main result:

Theorem. *Let $\mathcal{M} \to C$ be the Néron minimal model of an absolutely simple $(d-1)$-dimensional principally polarized Abelian variety $\mathcal{M}_\eta$ over the field $\kappa(\eta)$ of rational functions of a smooth complex projective curve C.*

Assume that the trace of the Abelian variety $\mathcal{M}_\eta$ is trivial, $d \geq 5$,

$$d-1 \notin \mathrm{Ex}(4) \stackrel{\mathrm{def}}{=} \{\binom{l+2}{m} \quad (1 < m < (l+2)/2),$$
$$\binom{l+2}{m}^{n+1} \quad (1 \leq m < (l+2)/2), \quad l, m, n \in \mathbb{N}_+ = \{1,2,3,\dots\}\}$$
$$= \{9, 10, 15, 16, 21, 25, 27, 28, 35, 36, 45, 49, 55, 56, 64, 66, 78, 81, 84, 91, 100, \dots\}$$

and the ring $E_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \mathrm{End}_{\overline{\kappa(\eta)}}(\mathcal{M}_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an imaginary quadratic field.

If $d-1$ is divisible by 4,

then there exists a finite ramified covering $\widetilde{C} \to C$ such that, for any Künnemann compactification $\widetilde{X}$ of the Néron minimal model of the Abelian variety $\mathcal{M}_\eta \otimes_{\kappa(\eta)} \kappa(\widetilde{\eta})$, there exist algebraic isomorphisms

$$H^{2d-2}(\widetilde{X}, \mathbb{Q}) \xrightarrow{\sim} H^2(\widetilde{X}, \mathbb{Q}), \qquad H^{2d-3}(\widetilde{X}, \mathbb{Q}) \xrightarrow{\sim} H^3(\widetilde{X}, \mathbb{Q}).$$

Simple Abelian varieties with multiplications from imaginary quadratic fields are especially interesting because sometimes they have non-trivial *Weil cycles* [21].

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§ 1. Some remarks on Hodge groups, global monodromy and cohomology of local systems

1.1. We may assume that

$$\{s \in C \mid \text{ the fibre } \mathcal{M}_s \text{ is non-compact}\} = \Delta$$
$$\stackrel{\text{def}}{=} \{\delta \in C \mid \operatorname{Sing}(X_\delta) \neq \varnothing\}.$$

Set $C' = C \setminus \Delta$, $\quad C' \stackrel{j}{\hookrightarrow} C$ the canonical embedding,

$$X' = X \setminus \pi^{-1}(\Delta), \quad \pi' = \pi|_{X'} : X' \to C'.$$

Considering, if necessary, a ramified covering $\widetilde{C} \to C$ and a projective Künnemann model $\widetilde{X} \to \widetilde{C}$ of the corresponding Néron model $\widetilde{\mathcal{M}} \to \widetilde{C}$ of the generic scheme fibre of the canonical projection $X \times_C \widetilde{C} \to \widetilde{C}$, we may assume in virtue of [19, Section 5.8]; [20, Section 4.4, Theorem 4.6] that any singular fibre X_δ is a union of smooth irreducible components of multiplicity 1 with normal crossings. One may also assume that there is a *countable* subset $\Delta_{\text{countable}} \subset C'$ such that, for any point $s \in C' \setminus \Delta_{\text{countable}}$, the closure G of the image of the monodromy representation $\pi_1(C', s) \to \operatorname{GL}(H^1(X_s, \mathbb{Q}))$ in the Zariski topology of the algebraic group $\operatorname{GL}(H^1(X_s, \mathbb{Q}))$ is a connected semi-simple [22, Corollary 4.2.9] *normal* [23, Theorem 7.3] subgroup of the Hodge group

$$\operatorname{Hg}(X_s) \stackrel{\text{def}}{=} \operatorname{Hg}(H^1(X_s, \mathbb{Q}))$$

of the Abelian variety X_s. We may also assume that local monodromies (Picard - Lefschetz transformations) *are unipotent* and

$$\operatorname{End}_{\kappa(\eta)}(X_\eta) = \operatorname{End}_{\overline{\kappa(\eta)}}(X_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}).$$

1.2. Consider the normalization $f : Z \to \pi^{-1}(\Delta)$ of the scheme $\pi^{-1}(\Delta)$. Then Z is a disjoint union of smooth irreducible components of the divisor $\pi^{-1}(\Delta)$. Since $f : Z \to \pi^{-1}(\Delta)$ is a resolution of singularities of the subscheme $i_\Delta : \pi^{-1}(\Delta) \hookrightarrow X$, there is a canonical exact sequence of mixed Hodge $\mathbb{Q}$-structures [24, Corollary (8.2.8)]:

$$H^{n-2}(Z, \mathbb{Q}) \xrightarrow{(i_\Delta f)_*} H^n(X, \mathbb{Q}) \xrightarrow{\varphi_n} H^n(X', \mathbb{Q}),$$

where $(i_\Delta f)_*$ is a morphism of bidegree $(1, 1)$ of pure Hodge structures and φ_n is the restriction map. In particular,

$$(i_\Delta f)_* H^{n-2}(Z, \mathbb{Q}) = \operatorname{Ker}[H^n(X, \mathbb{Q}) \xrightarrow{\varphi_n} H^n(X', \mathbb{Q})]. \tag{1.1}$$

1.3. Since $\mathrm{End}_{\overline{\kappa(\eta)}}(X_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an imaginary quadratic field, for any embedding of fields $\kappa(\eta) \hookrightarrow \mathbb{C}$, the semi-simple part $\mathrm{Hg}^{\mathrm{ss}}(X_\eta \otimes_{\kappa(\eta)} \mathbb{C})$ of the reductive Hodge group $\mathrm{Hg}(X_\eta \otimes_{\kappa(\eta)} \mathbb{C})$ [25, Proposition B57] is a $\mathbb{Q}$-simple algebraic group by Borovoi's theorem [26]. Therefore, by the triviality of the trace of the Abelian variety X_η, we have the equality

$$G = \mathrm{Hg}^{\mathrm{ss}}(X_\eta \otimes_{\kappa(\eta)} \mathbb{C}). \tag{1.2}$$

By the same reasons there is a canonical isomorphism [22, Proposition 4.4.11]

$$\mathrm{End}_{C'}(X') \widetilde{\to} \mathrm{End}_{C'}(R^1\pi'_*\mathbb{Z}). \tag{1.3}$$

A choice of a point $s \in C' \setminus \Delta_{\mathrm{countable}}$ determines the canonical embeddings

$$\mathrm{Im}[\pi_1(C', s) \to \mathrm{GL}(H^1(X_s, \mathbb{Q}))] \ \subset \ G \ \subset \ \mathrm{Hg}(X_s).$$

Therefore it follows from (1.3) and from the well known equality [25, Lemma B.60]

$$\mathrm{End}_{\mathrm{Hg}(X_s)} H^1(X_s, \mathbb{Q}) = \mathrm{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}$$

that there are canonical maps

$$\begin{aligned} &\mathrm{End}_{\kappa(\eta)}(X_\eta) \otimes_{\mathbb{Z}} \mathbb{Q} \widetilde{\to} \mathrm{End}_{\pi_1(C',s)} H^1(X_s, \mathbb{Q}) \\ &\hookleftarrow \mathrm{End}_{\mathrm{Hg}(X_s)} H^1(X_s, \mathbb{Q}) = \mathrm{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}. \end{aligned} \tag{1.4}$$

The restriction map $\mathrm{End}_{\kappa(\eta)}(X_\eta) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective, hence it follows from (1.4) that there exists a canonical isomorphism

$$\mathrm{End}_{\kappa(\eta)}(X_\eta) \otimes_{\mathbb{Z}} \mathbb{Q} \widetilde{\to} \mathrm{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{1.5}$$

In particular, the Abelian variety X_s is simple and $\mathrm{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an imaginary quadratic field. Hence, by Borovoi's theorem [26], the semi-simple part $\mathrm{Hg}^{\mathrm{ss}}(X_s)$ of the Hodge group $\mathrm{Hg}(X_s)$ is a $\mathbb{Q}$-simple algebraic group, so that we have the equality

$$G = \mathrm{Hg}^{\mathrm{ss}}(X_s) \quad \text{for all} \quad s \in C' \setminus \Delta_{\mathrm{countable}}. \tag{1.6}$$

Besides, the canonical representation of the Lie algebra

$$\mathrm{Lie}\,\mathrm{Hg}^{\mathrm{ss}}(X_s) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$$

in the space $H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is determined by *minuscule weights* [27], [23, Theorem 0.5.1] in Bourbaki's sense [28]. Since $\dim_{\mathbb{C}}(X_s) = d - 1 \notin \mathrm{Ex}(4)$, it follows that $\mathrm{Lie}\,\mathrm{Hg}^{\mathrm{ss}}(X_s) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is a simple Lie algebra of type

A_{d-2} and there exists a canonical decomposition of $\mathrm{Lie}\,\mathrm{Hg}^{\mathrm{ss}}(X_s)\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}$-modules [29, Section 2.9]

$$H^1(X_s,\mathbb{Q})\otimes_{\mathbb{Q}}\overline{\mathbb{Q}} = E(\omega_1)\,\oplus\,E(\omega_1)^{\vee} = E(\omega_1)\,\oplus\,E(\omega_{d-2}). \tag{1.7}$$

1.4. It follows from the existence of the natural embedding

$$R^2\pi'_*\mathbb{Q} = \wedge^2 R^1\pi'_*\mathbb{Q}\ \hookrightarrow\ R^1\pi'_*\mathbb{Q}\otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}$$

and from (1.3) that $H^0(C', R^2\pi'_*\mathbb{Q})$ is a rational Hodge structure of type $(1,1)$ because a polarization on X determines an isomorphism of families of Hodge structures [22, Section 4.2.3]

$$[R^1\pi'_*\mathbb{Q}]^{\vee}\,\widetilde{\rightarrow}\,R^1\pi'_*\mathbb{Q}(1),$$

the ring $\mathrm{End}_{C'}(X')\otimes_{\mathbb{Z}}\mathbb{Q}$ coincides with the component of type $(0,0)$ of the Hodge $\mathbb{Q}$-structure $\mathrm{End}_{C'}(R^1\pi'_*\mathbb{Q})$ [22, Section 4.4.6] and there are morphisms of rational Hodge structures

$$\begin{gathered} H^0(C', R^2\pi'_*\mathbb{Q})\ \hookrightarrow\ H^0(C', R^1\pi'_*\mathbb{Q}\otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}) \\ \widetilde{\rightarrow}\,H^0(C',[R^1\pi'_*\mathbb{Q}]^{\vee}\otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q})(-1) = \mathrm{End}_{C'}(R^1\pi'_*\mathbb{Q})\otimes_{\mathbb{Z}}\mathbb{Q}(-1) \\ \widetilde{\rightarrow}\ \mathrm{End}_{C'}(X')\otimes_{\mathbb{Z}}\mathbb{Q}(-1). \end{gathered}$$

Taking into account this fact and arguments of [15, § 2], it is easy to show the existence of an algebraic isomorphism

$$H^{2d-2}(X,\mathbb{Q})\,\widetilde{\rightarrow}\,H^2(X,\mathbb{Q}).$$

§ 2. Some canonical decompositions of rational Hodge structures of odd weights

2.1. By the assumption of the theorem, the generic scheme fibre $\mathcal{M}_\eta$ of the Néron model is a principally polarized Abelian variety; consequently, for any point $s\in C'$, the Abelian variety X_s has a principal polarization determined by certain ample divisor H_s on the variety X_s.

It is known that the Poincaré bundle $\mathcal{P}'_s$ on the variety $X_s\times\overset{\vee}{X}_s$ is uniquely determined (up to an isomorphism) by the following properties ([30], Ch. 2, § 5):

(a) $\mathcal{P}'_s|_{X_s\times\{L_s\}}\,\widetilde{\rightarrow}\,L_s$ for all $L_s\in\overset{\vee}{X}_s=\mathrm{Pic}(X_s)$,
(b) $\mathcal{P}'_s|_{\{0\}\times\overset{\vee}{X}_s}\,\widetilde{\rightarrow}\,\mathcal{O}_{\overset{\vee}{X}_s}$.

Since X_s is an Abelian variety with a principal polarization, then we have the equalities $X_s=\mathrm{Pic}^0(X_s)=\overset{\vee}{X}_s$. It follows easily from the properties (a) and (b) that an element $\mathrm{c}_1(\mathcal{P}'_s) \in H^2(X_s \times X_s, \mathbb{Q})$ has Künneth type (1,1) [30, Ch. 14, Lemma 14.1.9], so that

$$\begin{aligned}\mathrm{c}_1(\mathcal{P}'_s) &\in [H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_s, \mathbb{Q})] \cap H^{1,1}(X_s \times X_s, \mathbb{C}) \\ &= [H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_s, \mathbb{Q})]^{\mathrm{Hg}(X_s)}.\end{aligned}$$

Finitely, the correspondence $\mathrm{c}_1(\mathcal{P}'_s)$ induces the algebraic isomorphism ([2], Section 2A1(ii), Theorem 2A9; [30], Ch. 16, § 16.4, P. 532))

$$H^{2d-3}(X_s, \mathbb{Q}) \xrightarrow[\smile]{x \mapsto \mathrm{pr}_{2s*}\left(pr^*_{1s}(x) \smile \mathrm{c}_1(\mathcal{P}'_s)\right)} H^1(\mathrm{Pic}^0(X_s), \mathbb{Q}) = H^1(X_s, \mathbb{Q}).$$

Besides, for any point $s \in C'$ outside some countable subset $\Delta_{\mathrm{countable}}$, the group G (defined in Section 1.1) is a normal subgroup of the Hodge group $\mathrm{Hg}(X_s) = \mathrm{Hg}(H^1(X_s, \mathbb{Q}))$ of a rational Hodge structure $H^1(X_s, \mathbb{Q})$ [23, Theorem 7.3]. We fixe such a point s. It follows from the existence of an inclusion $G \hookrightarrow \mathrm{Hg}(X_s)$ that the correspondence $\mathrm{c}_1(\mathcal{P}'_s)$ determines the section

$$\Lambda'_{1,1} \in H^0(C', R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}) \xrightarrow{\sim} [H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_s, \mathbb{Q})]^{\pi_1(C', s)}$$

of type $(1,1)$ of a local system of Hodge structures $R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}$ inducing the correspondence $\mathrm{c}_1(\mathcal{P}'_t)$ for any point $t \in C'$.

By Deligne's theorem, the canonical map $H^2(Y, \mathbb{Q}) \to H^0(C', R^2\tau'_*\mathbb{Q})$ is a surjective morphism of Hodge $\mathbb{Q}$-structures ([22], Theorem 4.1.1, a proof of Corollary 4.1.2). Since $\Lambda'_{1,1} \in H^0(C', R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}) \subset H^0(C', R^2\tau'_*\mathbb{Q})$ is an element of Hodge type $(1,1)$, it follows from the Lefschetz theorem on divisors that there exists an algebraic $\mathbb{Q}$-cycle $D^{(1)}$ on the variety Y such that the image of the class $\mathrm{cl}_Y(D^{(1)}) \in H^2(Y, \mathbb{Q}) \cap H^{1,1}(Y, \mathbb{C})$ with respect to the canonical surjective morphism $H^2(Y, \mathbb{Q}) \to H^0(C', R^2\tau'_*\mathbb{Q})$ coincides with the section $\Lambda'_{1,1}$.

It is known that, for any point $s \in C'$, the algebraic correspondence $\mathrm{c}_1(\mathcal{P}'_s)^{\smile 2}$ yields the algebraic isomorphism $H^{2d-4}(X_s, \mathbb{Q}) \xrightarrow{\sim} H^2(X_s, \mathbb{Q})$ ([2], Lemma 2A12, Remark 2A13; [30], Ch. 16, § 16.4, P. 532). Using arguments of [15, Sections 2.1 and 2.3], the canonical commutative diagram

$$\begin{array}{ccc} X \times_C X & \xrightarrow{p_1} & X \\ \downarrow^{p_2} & \searrow^{\tau} & \downarrow^{\pi} \\ X & \xrightarrow{\pi} & C, \end{array}$$

the canonical embedding $\iota : X \times_C X \hookrightarrow X \times X$, a resolution of singularities $\sigma : Y \to X \times_C X$ of the fibre product $X \times_C X$ such that σ induces an isomorphism over C' and $(\tau\sigma)^{-1}(\Delta)$ is a union of smooth divisors (of some positive multiplicities) with normal crossings, we see that the element $\Lambda'_{1,1}$ is induced by the class of certain divisor $D^{(1)}$ on the smooth projective compactification Y of the variety $X' \times_{C'} X'$ and the element ${\Lambda'_{1,1}}^{\smile 2}$ yields the isomorphism of local systems $R^{2d-4}\pi'_*\mathbb{Q} \widetilde{\to} R^2\pi'_*\mathbb{Q}$ determined by the composite of morphisms of sheaves

$$R^{2d-4}\pi'_*\mathbb{Q} \xrightarrow{(p'_1)^*} R^{2d-4}\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} \pi'_*\mathbb{Q} \xrightarrow{\smile {\Lambda'_{1,1}}^{\smile 2}} R^{2d-2}\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^2\pi'_*\mathbb{Q}$$
$$\xrightarrow{(p'_2)_*} R^2\pi'_*\mathbb{Q}. \tag{2.1}$$

We may also expand (2.1) to a sequence of morphisms of sheaves

$$R^{2d-4}\pi_*\mathbb{Q} \xrightarrow{(p_1\sigma)^*} R^{2d-4}(\tau\sigma)_*\mathbb{Q} \xrightarrow{\smile \mathrm{cl}_Y(D^{(1)})^{\smile 2}} R^{2d-2}(\tau\sigma)_*\mathbb{Q}$$
$$\xrightarrow{(p_2\sigma)_*} R^2\pi_*\mathbb{Q},$$

whose composite is an isomorphism outside the finite set Δ. Therefore there exists an isomorphism of bidegree $(3-d, 3-d)$ of mixed Hodge structures

(2.2)
$$H^1(C, R^{2d-4}\pi_*\mathbb{Q}) \xrightarrow[\sim]{[x \mapsto (p_2\sigma)_*((p_1\sigma)^* x \smile \mathrm{cl}_Y(D^{(1)})^{\smile 2})]_1} H^1(C, R^2\pi_*\mathbb{Q}).$$

2.2. Lemma. *For any odd natural number n, there are the equalities*

$$H^0(C', R^n\tau'_*\mathbb{Q}) = H^0(C', R^n\pi'_*\mathbb{Q})$$
$$= H^2(C, R^n(\tau\sigma)_*\mathbb{Q}) = H^2(C, R^n\pi_*\mathbb{Q}) = 0.$$

Proof. By the condition of the theorem, $d-1$ is divisible by 4. Then, by (1.5) - (1.7), for all $s \in C' \setminus \Delta_{\mathrm{countable}}$, the Lie algebra $\mathrm{Lie}\, G \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = \mathrm{Lie}\,\mathrm{Hg}^{\mathrm{ss}}(X_s) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is a simple Lie algebra of type A_{d-2} and there is a decomposition of $\mathrm{Lie}\, G \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$-modules

$$H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = E(\omega_1) \oplus E(\omega_1)^\vee = E(\omega_1) \oplus E(\omega_{d-2}).$$

Let $E(\widetilde{\omega_1})$ be the standard irreducible representation of degree 2 of the Lie algebra $sl_2(\overline{\mathbb{Q}})$ of type A_1. Since $\dim_{\overline{\mathbb{Q}}} E(\omega_1) = d-1$ is divisible by 4, there exist (non-canonical) identifications of $sl_2(\overline{\mathbb{Q}})$-modules

$$E(\widetilde{\omega}_1)^{\frac{d-1}{2}} \widetilde{\to} [\overline{\mathbb{Q}}^2]^{\frac{d-1}{2}} = \overline{\mathbb{Q}}^{d-1},$$

so a choice of an appropriate basis of the space $E(\omega_1)$ determine an embedding of Lie algebras $sl_2(\overline{\mathbb{Q}}) \hookrightarrow sl(E(\omega_1))$.

For natural numbers p, q, the Klebsh - Gordon formula [28, Ch. VIII, § 9, n^0 4] yields the decomposition

$$E(p\,\widetilde{\omega}_1) \otimes_{\mathbb{Q}} E(q\,\widetilde{\omega}_1)$$
$$= E((p+q)\,\widetilde{\omega}_1) \oplus E((p+q-2)\,\widetilde{\omega}_1) \oplus E((p+q-4)\,\widetilde{\omega}_1) \oplus \dots$$

Therefore, if p is an odd natural number, then the representation of $sl_2(\overline{\mathbb{Q}})$ in $\wedge^p(E(\omega_1))$ is a sum of representations of type $E(\widetilde{\omega}_1)$, $E(3\,\widetilde{\omega}_1)$, $\dots$, $E(p\,\widetilde{\omega}_1)$. On the other hand, if q is an even natural number, then the representation of $sl_2(\overline{\mathbb{Q}})$ in $\wedge^q(E(\omega_1))$ is a sum of representations of type $E(0), E(2\,\widetilde{\omega}_1)$, $E(4\,\widetilde{\omega}_1)$, $\dots$, $E(q\,\widetilde{\omega}_1)$. Consequently, by the Schur lemma, one has for odd $p+q$:

$$\begin{aligned}&\mathrm{Hom}_{G\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}}(\wedge^p E(\omega_1), \wedge^q(E(\omega_1)))\\ &= \mathrm{Hom}_{G\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}}(\wedge^p E(\omega_1), \wedge^q(E(\omega_1))^\vee) = 0.\end{aligned}$$

Finally, in virtue of the theorem on local invariant cycles, the canonical maps

$$R^n(\tau\sigma)_*\mathbb{Q} \to j_*R^n(\tau'\sigma')_*\mathbb{Q}, \qquad R^n\pi_*\mathbb{Q} \to j_*R^n\pi'_*\mathbb{Q}$$

are surjective with the kernels concentrated on the finite set Δ ([31], Proposition (15.12); [32], Section (3.7)). Therefore it remains to note that by [31, Proposition (10.5)], one has

$$\begin{aligned}H^2(C, R^n(\tau\sigma)_*\mathbb{Q}) &= H^2(C, j_*R^n(\tau'\sigma')_*\mathbb{Q})\\ &\widetilde{\to} H^0(C', R^n(\tau'\sigma')_*\mathbb{Q})^\vee = H^0(C', R^n\tau'_*\mathbb{Q})^\vee;\\ H^2(C, R^n\pi_*\mathbb{Q}) &= H^2(C, j_*R^n\pi'_*\mathbb{Q}) \widetilde{\to} H^0(C', R^n\pi'_*\mathbb{Q})^\vee.\end{aligned}$$

The lemma is proved.

2.3. Let

$$\begin{aligned}K_{nX} &= \mathrm{Ker}[H^n(X,\mathbb{Q}) \to H^0(C, R^n\pi_*\mathbb{Q})],\\ K_{nY} &= \mathrm{Ker}[H^n(Y,\mathbb{Q}) \to H^0(C, R^n(\tau\sigma)_*\mathbb{Q})].\end{aligned}$$

The Leray spectral sequences $E_2^{p,q}(\pi) = H^p(C, R^q\pi_*\mathbb{Q})$ and $E_2^{p,q}(\tau\sigma) = H^p(C, R^q(\tau\sigma)_*\mathbb{Q})$ are degenerated: $E_2^{p,q} = E_\infty^{p,q}$ [31, Corollary (15.15)]. Hence, for any natural number n, there are exact sequences of Hodge $\mathbb{Q}$-structures ([13], Formula (2.4))

$$0 \to H^2(C, R^{n-2}\pi_*\mathbb{Q}) \to K_{nX} \xrightarrow{\alpha_{nX}} H^1(C, R^{n-1}\pi_*\mathbb{Q}) \to 0, \tag{2.3}$$

$$0 \to H^2(C, R^{n-2}(\tau\sigma)_*\mathbb{Q}) \to K_{nY} \xrightarrow{\alpha_{nY}} H^1(C, R^{n-1}(\tau\sigma)_*\mathbb{Q}) \to 0. \tag{2.4}$$

It follows from Lemma 2.2, (2.3), (2.4), the theorem on local invariant cycles and the strong Lefschetz theorem for fibres of a smooth morphism

π' that

$$K_{3X} = H^1(C, R^2\pi_*\mathbb{Q}); \tag{2.5}$$

$$K_{(2d-3)X} = H^1(C, R^{2d-4}\pi_*\mathbb{Q}); \tag{2.6}$$

$$\begin{aligned} \mathrm{cl}_X(H)^{\smile d-3} \smile K_{3X} &= \mathrm{cl}_X(H)^{\smile d-3} \smile H^1(C, R^2\pi_*\mathbb{Q}) \\ &= \mathrm{cl}_X(H)^{\smile d-3} \smile H^1(C, j_*R^2\pi'_*\mathbb{Q}) = K_{(2d-3)X}. \end{aligned} \tag{2.7}$$

Moreover, taking into account the non-degeneracy of the canonical pairing [31, Proposition (10.5)]

$$\begin{aligned} &H^1(C, j_*R^{2d-4}\pi'_*\mathbb{Q}) \times H^1(C, j_*R^2\pi'_*\mathbb{Q}) \\ &\xrightarrow{x \times x' \mapsto x \smile x'} H^2(C, j_*R^{2d-2}\pi_*\mathbb{Q}) = H^{2d}(X, \mathbb{Q}) \end{aligned}$$

we see, that the canonical pairing

$$K_{(2d-3)X} \times K_{3X} \xrightarrow{x \times x' \mapsto x \smile x'} H^{2d}(X, \mathbb{Q})$$

is non-degenerate. Therefore the restriction of the *non-degenerate* [2, Section 1.2.A] bilinear form

$$\begin{aligned} \Phi : H^3(X, \mathbb{Q}) \times H^3(X, \mathbb{Q}) &\xrightarrow{x \times y \mapsto x \smile y \smile \mathrm{cl}_X(H)^{\smile d-3}} \\ H^{2d}(X, \mathbb{Q}) &= \mathbb{Q}(-d) \xrightarrow[\sim]{(2\pi i)^d} \mathbb{Q}. \end{aligned}$$

to the subspace $K_{3X} \subset H^3(X, \mathbb{Q})$ is non-degenerate. Hence, as it was noticed in Section 1.2 of [15], there exists the decomposition of rational Hodge structures

$$H^3(X, \mathbb{Q}) = K_{3X} \oplus K_{3X}^{\perp}, \tag{2.8}$$

where $K_{3X}^{\perp}$ is the orthogonal complement of the subspace $K_{3X} \hookrightarrow H^3(X, \mathbb{Q})$ with respect to the form Φ. In virtue of (2.7) we have:

$$\begin{aligned} K_{3X}^{\perp} &= \{x \in H^3(X, \mathbb{Q}) \mid x \smile \mathrm{cl}_X(H)^{\smile d-3} \smile K_{3X} = 0\} \\ &\{x \in H^3(X, \mathbb{Q}) \mid x \smile K_{(2d-3)X} = 0\}. \end{aligned}$$

Consequently, the decomposition (2.8) is *canonical* and it does not depend on the choice of an ample divisor H; moreover, by (2.7), (2.8) and by the strong Lefschetz theorem, there is a *canonical* decomposition of rational Hodge structures

$$H^{2d-3}(X, \mathbb{Q}) = K_{(2d-3)X} \oplus K_{(2d-3)X}^{\perp}, \tag{2.9}$$

where

$$\begin{aligned} K_{(2d-3)X}^{\perp} &\overset{\mathrm{def}}{=} \mathrm{cl}_X(H)^{\smile d-3} \smile K_{3X}^{\perp} \\ &= \{x \in H^{2d-3}(X, \mathbb{Q}) \mid K_{3X} \smile x = 0\}. \end{aligned}$$

2.4. For any point $s \in C$, we denote by $\iota_{X_s/X} : X_s \hookrightarrow X$ the canonical embedding. The morphism π is proper, therefore the fibre of the sheaf $R^n\pi_*\mathbb{Q}$ over a point $s \in C$ coincides with the space $H^n(X_s, \mathbb{Q})$ ([33], Ch. II, § 4, Remark 4.17.1; [34], Ch. VI, § 2, Corollary 2.5). Consequently, the restriction map $\iota^*_{X_s/X}$ coincides with the composite ([35], Vol. II, Ch. 4, Section 4.3.1)

$$H^n(X, \mathbb{Q}) \to E^{0,n}_\infty(\pi) \to E^{0,n}_2(\pi) = H^0(C, R^n\pi_*\mathbb{Q}) \to H^n(X_s, \mathbb{Q}).$$

Thus the map $\iota^*_{X_s/X}$ is the composite of canonical maps

$$H^n(X, \mathbb{Q}) \to H^0(C, R^n\pi_*\mathbb{Q}) \hookrightarrow \prod_{s\in C} H^n(X_s, \mathbb{Q}) \to H^n(X_s, \mathbb{Q}),$$

where the $\mathbb{Q}$-space $\prod_{s\in C} H^n(X_s, \mathbb{Q})$ is identified with the $\mathbb{Q}$-space of *discontinuous* global sections of the sheaf $R^n\pi_*\mathbb{Q}$ ([33], Ch. II, § 4, Section 4.4.4). Evidently

$$\omega \in K_{nX} \;\Leftrightarrow\; (\forall s \in C) \quad \iota^*_{X_s/X}(\omega) = 0. \tag{2.10}$$

§ 3. Constructions of algebraic isomorphisms of cohomology of odd weights

3.1. Lemma. *There is the equality* $(i_\Delta f)_* H^1(Z, \mathbb{Q}) = K^\perp_{3X}$.

Proof. In virtue of Lemma 2.2 and the surjectivity [31, Corollary (15.14)] of the canonical edge map $H^3(X, \mathbb{Q}) \to H^0(C, R^3\pi_*\mathbb{Q})$ of the Leray spectral sequence $E^{p,q}_2(\pi)$ one may assume that $\Delta \neq \varnothing$ because (2.8) induces a canonical isomorphism $K^\perp_{3X} \xrightarrow{\sim} H^0(C, R^3\pi_*\mathbb{Q})$.

For any irreducible smooth projective variety W, we denote by $<>$: $H^{2\dim_{\mathbb{C}} W}(W, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}$ the orientation isomorphism of Weil cohomology ([2], Section 1.2.A) determined by a choice of an element $\sqrt{-1} \in \mathbb{C}$.

Irreducible components of a smooth divisor Z are naturally identified with irreducible components $X_{\delta i_\delta}$ ($i_\delta \in \{1, \dots, m_\delta\}$) of the divisor $\pi^{-1}(\Delta) = \sum_{\delta\in\Delta} X_\delta$. Denote by $\iota_{X_{\delta i_\delta}/X} : X_{\delta i_\delta} \hookrightarrow X$, $\iota_{X_{\delta i_\delta}/X_\delta} : X_{\delta i_\delta} \hookrightarrow X_\delta$, $\iota_{X_{\delta i_\delta}/Z} : X_{\delta i_\delta} \hookrightarrow Z$ the canonical embeddings. In accordance with the functoriality of cohomology with coefficients in the field $\mathbb{Q}$ there is the equality $\iota^*_{X_{\delta i_\delta}/X} = \iota^*_{X_{\delta i_\delta}/X_\delta}\, \iota^*_{X_\delta/X}$. On the other hand, $\iota^*_{X_\delta/X} K_{(2d-3)X} = 0$ by (2.10). Hence $\iota^*_{X_{\delta i_\delta}/X} K_{(2d-3)X} = 0$. By definition, the operator $\iota_{X_{\delta i_\delta}/X*}$ is conjugate to the operator $\iota^*_{X_{\delta i_\delta}/X}$ ([2], Section 1.2.A; [34], Ch. VI, § 11, Remark 11.6), therefore

$$\begin{aligned} &< \iota_{X_{\delta i_\delta}/X*} H^1(X_{\delta i_\delta}, \mathbb{Q}) \smile K_{(2d-3)X} > \\ &=< H^1(X_{\delta i_\delta}, \mathbb{Q}) \smile \iota^*_{X_{\delta i_\delta}/X} K_{(2d-3)X} > . \end{aligned}$$

Consequently, $\iota_{X_{\delta i_\delta}/X*}\, H^1(X_{\delta i_\delta}, \mathbb{Q}) \smile K_{(2d-3)X} = 0$. From the commutativity of the diagram

$$\begin{array}{ccc} X_{\delta i_\delta} & \xrightarrow{\iota_{X_{\delta i_\delta}/X}} & X \\ || & & \uparrow^{i_\Delta f} \\ X_{\delta i_\delta} & \overset{\iota_{X_{\delta i_\delta}/Z}}{\hookrightarrow} & Z \end{array}$$

of canonical morphisms and ([34], Ch. VI, § 11, Remark 11.6 (c)) we obtain the equality

$$(3.1) \qquad (i_\Delta f)_*(\iota_{X_{\delta i_\delta}/Z})_*|_{H^1(X_{\delta i_\delta},\mathbb{Q})} = \iota_{X_{\delta i_\delta}/X*}|_{H^1(X_{\delta i_\delta},\mathbb{Q})}.$$

Thus $(i_\Delta f)_*\, H^1(Z, \mathbb{Q}) \smile K_{(2d-3)X} = 0$, so that from (2.7) we obtain the equality

$$(i_\Delta f)_*\, H^1(Z, \mathbb{Q}) \smile \mathrm{cl}_X(H)^{\smile d-3} \smile K_{3X} = 0$$

and the existence of the inclusion

$$(3.2) \qquad (i_\Delta f)_* H^1(Z, \mathbb{Q}) \subset K_{3X}^{\perp}.$$

In accordance with Lemma 2.2 we have the equalities

$$H^2(C, R^1\pi_*\mathbb{Q}) = H^0(C', R^3\pi'_*\mathbb{Q}) = 0.$$

On the other hand, the theorem on local invariant cycles and the Leray spectral sequence for the embedding $j : C' \subset C$ yield the embedding of mixed Hodge $\mathbb{Q}$-structures $H^1(C, R^2\pi_*\mathbb{Q}) \hookrightarrow H^1(C', R^2\pi'_*\mathbb{Q})$ ([31], P. 457, Corollary (13.10), Remark (14.5)). Besides, the canonical map

$$H^3(X, \mathbb{Q}) \to H^0(C, R^3\pi_*\mathbb{Q})$$

is surjective ([31], Corollary (15.14)). Consequently, taking into account (1.1), (2.3), the diagram (15.1) in [31], the degeneracy of the Leray spectral sequence $E_2^{p,q}(\pi') = H^p(C', R^q\pi'_*\mathbb{Q})$ ([22], Theorem 4.1.1), the equality $H^2(C', R^1\pi'_*\mathbb{Q}) = 0$ (because the cohomological dimension of the affine curve C' equals 1 ([34], Ch. VI, § 7, Theorem 7.2)) and the commutativity of the diagram of morphisms

$$\begin{array}{ccc} X' & \subset & X \\ \downarrow^{\pi'} & & \downarrow^{\pi} \\ C' & \subset & C, \end{array}$$

we obtain the commutative diagram of mixed Hodge $\mathbb{Q}$-structures with exact rows

$$\begin{array}{ccccccccc} 0 & \to & H^1(C,R^2\pi_*\mathbb{Q}) & \to & H^3(X,\mathbb{Q}) & \to & H^0(C,R^3\pi_*\mathbb{Q}) & \to & 0 \\ & & \bigcap & & \downarrow^{\varphi_3} & & \downarrow & & \\ 0 & \to & H^1(C',R^2\pi'_*\mathbb{Q}) & \to & H^3(X',\mathbb{Q}) & \to & 0. & & \end{array}$$

Therefore (2.5), (2.8) yield the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & K_{3X} & \to & K_{3X} \oplus K_{3X}^{\perp} & \to & K_{3X}^{\perp} & \to & 0 \\ & & \bigcap & & \downarrow^{\varphi_3} & & \downarrow & & \\ 0 & \to & H^1(C', R^2\pi'_*\mathbb{Q}) & \to & H^3(X', \mathbb{Q}) & \to & 0. & & \end{array} \tag{3.3}$$

Evidently $\mathrm{Im}(\varphi_3) = K_{3X}$, therefore (3.3) yields the commutative diagram of canonical morphisms of Hodge $\mathbb{Q}$-structures

$$\begin{array}{ccccccccc} 0 & \to & K_{3X} & \to & K_{3X} \oplus K_{3X}^{\perp} & \to & K_{3X}^{\perp} & \to & 0 \\ & & || & & \downarrow^{\varphi_3} & & \downarrow & & \\ 0 & \to & K_{3X} & \to & K_{3X} & \to & 0. & & \end{array}$$

The corresponding exact sequence of Hodge $\mathbb{Q}$-structures ([13], Section 2.6) of the snake-like diagram ([36], § 1, Proposition 2) and (1.1) yield the canonical identifications

$$(i_\Delta f)_* H^1(Z, \mathbb{Q}) = \mathrm{Ker}(\varphi_3) = K_{3X}^{\perp}.$$

The lemma is proved.

3.2. Lemma. *There is a canonical embedding $(p_2\sigma)_*(K_{(2d+1)Y}) \subset K_{3X}$.*

Proof. Since

$$K_{3X} = \{x \in H^3(X, \mathbb{Q}) \mid x \smile K_{3X}^{\perp} \smile \mathrm{cl}_X(H)^{\smile d-3} = 0\},$$

it suffices to check the equality

$$(p_2\sigma)_*(K_{(2d+1)Y}) \smile K_{3X}^{\perp} \smile \mathrm{cl}_X(H)^{\smile d-3} = 0,$$

which is equivalent ([2], Section 1.2.A; [34], Ch. VI, § 11, Remark 11.6) to the equalities

$$\begin{aligned} &< (p_2\sigma)_*(K_{(2d+1)Y}) \smile K_{3X}^{\perp} \smile \mathrm{cl}_X(H)^{\smile d-3} >= \\ &< K_{(2d+1)Y} \smile (p_2\sigma)^*(K_{3X}^{\perp} \smile \mathrm{cl}_X(H)^{\smile d-3}) >= 0. \end{aligned} \tag{3.4}$$

If $\Delta = \varnothing$, then the formula (3.4) is true in virtue of Lemma 3.1. From now on we assume that $\Delta \neq \varnothing$.

By (2.4) and Lemma 2.2 we have the canonical identification

$$K_{(2d+1)Y} = H^1(C, R^{2d}(\tau\sigma)_*\mathbb{Q}). \tag{3.5}$$

It is known that the Gysin map $\iota_{X_{\delta i_\delta}/X*} : H^k(X_{\delta i_\delta}, \mathbb{Q}) \to H^{k+2}(X, \mathbb{Q})$ is given by $\alpha \mapsto \alpha \smile \mathrm{cl}_X(X_{\delta i_\delta})$ ([15], Formula (4.20)). On the other hand, the strong Lefschetz theorem for the variety $X_{\delta i_\delta}$ yields the existence of the embedding

$$H^1(X_{\delta i_\delta}, \mathbb{Q}) \smile \iota^*_{X_{\delta i_\delta}/X}\, \mathrm{cl}_X(H)^{\smile d-3} \subset H^{2d-5}(X_{\delta i_\delta}, \mathbb{Q}).$$

Therefore the projection formula ([2], Section 1.2.A) yields the inclusion

$$\begin{gathered}\iota_{X_{\delta i_\delta}/X*} H^1(X_{\delta i_\delta}, \mathbb{Q}) \smile \mathrm{cl}_X(H)^{\smile d-3} \\ \subset \iota_{X_{\delta i_\delta}/X*} H^{2d-5}(X_{\delta i_\delta}, \mathbb{Q}) = H^{2d-5}(X_{\delta i_\delta}, \mathbb{Q}) \smile \mathrm{cl}_X(X_{\delta i_\delta}).\end{gathered}$$

Finitely, it follows from (3.1) that there is the embedding

$$\begin{gathered}(i_\Delta f)_* H^1(Z, \mathbb{Q}) \smile \mathrm{cl}_X(H)^{\smile d-3} \\ \subset \sum_{\delta\in\Delta,\ i_\delta\in\{1,\dots,m_\delta\}} H^{2d-5}(X_{\delta i_\delta}, \mathbb{Q}) \smile \mathrm{cl}_X(X_{\delta i_\delta}).\end{gathered} \tag{3.6}$$

By definition ([35], Vol. II, Ch. 4, Section 4.2.1), for any point $s \in C'$, the $\smile$-product by the class $\mathrm{cl}_X(X_{\delta i_\delta}) \in H^2(X, \mathbb{Q})$ acts on the fibre $H^q(X_s, \mathbb{Q}) = [j_* R^q \pi'_* \mathbb{Q}]_s$ of the sheaf $j_* R^q \pi'_* \mathbb{Q}$ as the $\smile$-product by the class $\iota^*_{X_s/X}(\mathrm{cl}_X(X_{\delta i_\delta}))$. It follows from the evident equality

$$\iota^*_{X_s/X}(\mathrm{cl}_X(X_{\delta i_\delta})) = 0$$

that

$$j_* R^q \pi'_* \mathbb{Q} \smile \mathrm{cl}_X(X_{\delta i_\delta}) = 0. \tag{3.7}$$

In virtue of the theorem on local invariant cycles and by Künneth's formula on fibres of a smooth morphism $\tau' : Y' = X' \times_{C'} X' \to C'$ there is a canonical decomposition

$$H^1(C, R^{2d}(\tau\sigma)_* \mathbb{Q}) = \oplus_{p+q=2d} H^1(C, j_*(R^p \pi'_* \mathbb{Q} \otimes_{\mathbb{Q}} R^q \pi'_* \mathbb{Q})). \tag{3.8}$$

It follows from (3.7) that

$$H^1(C, j_*(R^p \pi'_* \mathbb{Q} \otimes_{\mathbb{Q}} R^q \pi'_* \mathbb{Q})) \smile [1 \otimes_{\mathbb{Q}} \mathrm{cl}_X(X_{\delta i_\delta})] = 0,$$

so that (3.4) follows from Lemma 3.1, (3.5), (3.6), (3.8). The lemma is proved.

3.3. There is a non-degenerate canonical pairing [31, Proposition (10.5)]

$$\begin{gathered}H^1(C, j_* R^{2d-4} \pi'_* \mathbb{Q}) \times H^1(C, j_* R^2 \pi'_* \mathbb{Q}) \xrightarrow{x \times x' \mapsto x \smile x'} \\ H^2(C, j_* R^{2d-2} \pi'_* \mathbb{Q}) = H^2(C, R^{2d-2} \pi_* \mathbb{Q}) = H^{2d}(X, \mathbb{Q}) \widetilde{\to} \mathbb{Q},\end{gathered}$$

identifying (in accordance with the theorem on local invariant cycles) the space $H^1(C, R^{2d-4}\pi_*\mathbb{Q})^\vee$ with $H^1(C, R^2\pi_*\mathbb{Q})$. By the similar reason we have the identification $H^1(C, R^{2d-4}(\tau\sigma)_*\mathbb{Q})^\vee = H^1(C, R^{2d}(\tau\sigma)_*\mathbb{Q})$. Therefore by the theorem on local invariant cycles, (2.5), (3.5) and

Lemma 3.2 we obtain a commutative diagram

(3.9)

$$\begin{array}{ccc} K_{3X} & \xleftarrow{(p_2\sigma)_*|_{K_{(2d+1)Y}}} & K_{(2d+1)Y} \\ \| & & \| \\ H^1(C,j_*R^2\pi'_*\mathbb{Q}) = H^1(C,R^2\pi_*\mathbb{Q}) & \xleftarrow{[(p_2\sigma)_*]_1=[p'_{2*}]_1} & H^1(C,R^{2d}(\tau\sigma)_*\mathbb{Q}) = H^1(C,j_*R^{2d}\tau'_*\mathbb{Q}. \end{array}$$

Finally, in accordance with ([37], Ch. 2, § 8, Formula (5)) and by (2.10) we obtain the equalities

$$\begin{aligned} &\iota^*_{X_s/X}(K_{(2d-3)Y} \smile \mathrm{cl}_Y(D^{(1)})^{\smile 2}) \\ &= \iota^*_{X_s/X}(K_{(2d-3)Y}) \smile \iota^*_{X_s/X}(\mathrm{cl}_Y(D^{(1)})^{\smile 2}) = 0, \end{aligned}$$

so in virtue of (2.6), (2.10) there is a commutative diagram

$$(3.10)\qquad \begin{array}{ccc} K_{(2d-3)Y} & \xrightarrow{\smile\ \mathrm{cl}_Y(D^{(1)})^{\smile 2}} & K_{(2d+1)Y} \\ \| & & \| \\ H^1(C,R^{2d-4}(\tau\sigma)_*\mathbb{Q}) & \xrightarrow{[\smile\ \mathrm{cl}_Y(D^{(1)})^{\smile 2}]_1} & H^1(C,R^{2d}(\tau\sigma)_*\mathbb{Q}). \end{array}$$

On the other hand, the commutative diagram of morphisms

$$\begin{array}{ccc} X & \xleftarrow{p_1\sigma} & Y \\ \uparrow{\scriptstyle \iota_{X_s/X}} & & \uparrow{\scriptstyle \iota_{Y_s/Y}} \\ X_s & \xleftarrow{p_{1s}\sigma_s} & Y_s \end{array}$$

yields the commutative diagram of canonical maps

$$\begin{array}{ccc} H^{2d-3}(X,\mathbb{Q}) & \xrightarrow{(p_1\sigma)^*} & H^{2d-3}(Y,\mathbb{Q}) \\ \downarrow{\scriptstyle \iota^*_{X_s/X}} & & \downarrow{\scriptstyle \iota^*_{Y_s/Y}} \\ H^{2d-3}(X_s,\mathbb{Q}) & \xrightarrow{(p_{1s}\sigma_s)^*} & H^{2d-3}(Y_s,\mathbb{Q}), \end{array}$$

so that, for any point $s \in C$, it follows from (2.10) that

$$\iota^*_{X_s/X}(K_{(2d-3)X}) = 0$$

and

$$\iota^*_{Y_s/Y}(p_1\sigma)^*(K_{(2d-3)X}) = 0.$$

The morphism $\tau\sigma$ is proper, therefore by arguments of Section 2.4 the canonical restriction map

$$\iota^*_{Y_s/Y} : H^{2d-3}(Y,\mathbb{Q}) \to H^{2d-3}(Y_s,\mathbb{Q})$$

is the composite of canonical maps

$$\begin{aligned} H^{2d-3}(Y,\mathbb{Q}) \to H^0(C,R^{2d-3}(\tau\sigma)_*\mathbb{Q}) \hookrightarrow \prod_{s\in C} H^{2d-3}(Y_s,\mathbb{Q}) \\ \to H^{2d-3}(Y_s,\mathbb{Q}). \end{aligned}$$

Thus, it follows from the equalities $\iota^*_{Y_s/Y}(p_1\sigma)^*(K_{(2d-3)X}) = 0\ \ (s \in C)$ and from (2.10) that $(p_1\sigma)^* K_{(2d-3)X} \hookrightarrow K_{(2d-3)Y}$. By (2.4), (2.6) and Lemma 2.2 we have a commutative diagram

$$\begin{array}{ccc} K_{(2d-3)X} & \xrightarrow{(p_1\sigma)^*} & K_{(2d-3)Y} \\ \| & & \| \\ H^1(C, R^{2d-4}\pi_*\mathbb{Q}) & \xrightarrow{[(p_1\sigma)^*]_1} & H^1(C, R^{2d-4}(\tau\sigma)_*\mathbb{Q}). \end{array} \tag{3.11}$$

Gluing together diagrams (3.9) - (3.11), we see that (2.2) takes the form

$$K_{(2d-3)X} \xrightarrow[\sim]{x \mapsto (p_2\sigma)_*((p_1\sigma)^* x \smile \mathrm{cl}_Y(D^{(1)})^{\smile 2})} K_{3X}. \tag{3.12}$$

For $x \in H^{2d-3}(X, \mathbb{Q})$, the projection formula [2, Section 1.2.A] yields the equalities

$$\begin{aligned} &(p_2\sigma)_* \left((p_1\sigma)^* x \smile [\mathrm{cl}_Y(D^{(1)})]^{\smile 2}\right) \\ &= [\mathrm{pr}_2\, \iota\sigma]_* \left([\mathrm{pr}_1\, \iota\sigma]^* x \smile [\mathrm{cl}_Y(D^{(1)})]^{\smile 2}\right) \\ &= \mathrm{pr}_{2*}(\iota\sigma)_* \left((\iota\sigma)^* \, \mathrm{pr}_1^* \, x \smile [\mathrm{cl}_Y(D^{(1)})]^{\smile 2}\right) \\ &= \mathrm{pr}_{2*} \left(\mathrm{pr}_1^* \, x \smile (\iota\sigma)_* \left[[\mathrm{cl}_Y(D^{(1)})]^{\smile 2}\right]\right). \end{aligned}$$

Therefore an algebraic class $(\iota\sigma)_* \left[[\mathrm{cl}_Y(D^{(1)})]^{\smile 2}\right] \in H^6(X \times X, \mathbb{Q})$ determines the isomorphism (3.12), which takes the form

$$K_{(2d-3)X} \xrightarrow[\sim]{x \mapsto \mathrm{pr}_{2*}\left(\mathrm{pr}_1^* \, x \smile (\iota\sigma)_* \left[[\mathrm{cl}_Y(D^{(1)})]^{\smile 2}\right]\right)} K_{3X}. \tag{3.13}$$

3.4. By arguments of [15, Section 3.5], we have the isomorphism

$$K^{\perp}_{(2d-3)X} \xrightarrow[\sim]{x \mapsto \mathrm{pr}_{2*}(\mathrm{pr}_1^*(x) \smile \wp(K^{\perp}_{3X}))} K^{\perp}_{3X}, \tag{3.14}$$

where $\wp(K^{\perp}_{3X})$ is an *algebraic* (by Lemma 3.1 and [15, Lemma 3.8]) Poincaré class.

On the other hand, taking into account decompositions (2.8), (2.9), the existence of *algebraic* isomorphisms (3.13), (3.14) and arguments of [15, Sections 4.1 - 4.4], it is easy to show the existence of an *algebraic* isomorphism

$$H^{2d-3}(X, \mathbb{Q}) \xrightarrow[\sim]{x \mapsto \mathrm{pr}_{2*}(\mathrm{pr}_1^* \, x \smile (u_{3,3} + u_{3,3^{\perp}} + h_{10} + \wp(K^{\perp}_{3X})))} H^3(X, \mathbb{Q}),$$

where $u_{3,3} \in K_{3X} \otimes_{\mathbb{Q}} K_{3X}$, $\ u_{3,3^{\perp}} \in K_{3X} \otimes_{\mathbb{Q}} K^{\perp}_{3X}$ are components of an algebraic class $u \stackrel{\mathrm{def}}{=} (\iota\sigma)_* \left[[\mathrm{cl}_Y(D^{(1)})]^{\smile 2}\right] \in H^6(X \times X, \mathbb{Q})$ and h_{10} is an appropriate element of the $\mathbb{Q}$-space $\oplus_{p+q=6,\ p \neq 3} H^p(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^q(X, \mathbb{Q})$. The theorem is proved.

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