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ORTHOGONAL POLYNOMIAL BASIS IN THE SPACE OF VECTOR FUNCTIONS $\mathbf{H}_0^1$ AND STOKES SYSTEM IN A BALL

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Abstract. В однородном соболевском пространстве вектор-функций $\mathbf{H}_0^1(\mathbb{B}^3)$ построен ортогональный полиномиальный базис. Некоторые из этих вектор-функций, в частности, являются векторными потенциалами для соленоидальных полей из базиса пространства $\mathbf{L}_2(\mathbb{B}^3)$. В результате решается краевая задача Дирихле для стационарной системы Стокса в шаре. Решение представлено в виде ряда по построенным полиномиальным базисным вектор-функциям.

Keywords: Vector spherical harmonics, vector fields, potential field, solenoidal field, polynomial vector functions, Sobolev space, orthogonal basis, Stokes problem

1. Introduction

In this work, we construct polynomial basis vector functions for homogeneous Sobolev space $\mathbf{H}_0^1(\mathbb{B}^3)$ in a ball $\mathbb{B}^3$. Some of which are, in particular, vector potentials for basis solenoidal fields of the space $\mathbf{L}_2(\mathbb{B}^3)$. Then we study solutions to the stationary Stokes equation in the ball emploing polynomial basis vector functions from homogeneous Sobolev space $\mathbf{H}_0^1(\mathbb{B}^3)$.

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It is required to determine the velocity vector $\mathbf{u}$ and pressure p, satisfying the system

$$-\Delta\mathbf{u} + \nabla p = \mathbf{f} \text{ in } \mathbb{B}^3 \tag{1}$$

$$\operatorname{div}\mathbf{u} = 0 \text{ in } \mathbb{B}^3, \tag{2}$$

if homogeneous Dirichlet conditions are given on the boundary of the ball

$$\mathbf{u} = 0 \text{ on } \mathbb{S}^2 = \partial\mathbb{B}^3. \tag{3}$$

Here $\mathbf{u}$ is the velocity vector, p is the ratio of pressure to constant fluid density, $\mathbf{f}$ is the vector of acceleration of external mass forces and $\nu = 1 > 0$ is the kinematic coefficient of viscosity.

Detailed analysis of problems related to solvability of the Sokes systems and the regularity of its solutions can be found, for example, in [1], [2], [3], [4], [5], [6], [7].

2. Main definitions and auxiliary results

Let $\mathbb{B}^3$ and $\mathbb{S}^2$ be the unit ball and the unit sphere in $\mathbb{R}^3$, respectively, i.e. $\mathbb{B}^3 = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| < 1\}$ and $\mathbb{S}^2 = \partial\mathbb{B}^3 = \{\boldsymbol{\xi} \in \mathbb{R}^3 : |\boldsymbol{\xi}| = 1\}$, where $|\cdot|$ denotes the Euclidean norm. Throughout the paper we adopt the convention to denote in bold type the vectors in $\mathbb{R}^3$, and in a simple type the scalars in $\mathbb{R}$.

The Legendre polynomials P_N of degree $N \in \mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}$ are given by the formula $P_N(t) = \frac{1}{N!2^N}\frac{\mathrm{d}^N}{\mathrm{d}t^N}(t^2-1)^N$. Here $\mathbb{N}$ and $\mathbb{N}_0$ are sets of the positive integers and the non-negative integers, respectively. We define as $C_N^{(3/2)}$ the Gegenbauer polynomial of degree N with parameter $\lambda = 3/2$, $C_N^{(3/2)}(t) = \frac{d}{dt}P_{N+1}(t)$. We recall that Legendre polynomials $P_N(t)$ are the orthogonal polynomials on $(-1,1)$ with weight function $w(t) = 1$. By $Y_{N\ell}$ we denote the complex normalized spherical harmonics, $N \in \mathbb{N}_0$, $|\ell| \le N$ (see e.g. [8]).

Let us recall the definition of basic differential operations related to vector analysis — the operators ∇v, div and rot. The nabla operator ∇ or gradient calculates the gradient (potential) vector field ∇v,

$$\nabla v := (\partial v/\partial x_1, \partial v/\partial x_2, \partial v/\partial x_3)^T$$

and the scalar function v is called the scalar potential.

Smooth vector field $\mathbf{f} = (f_1, f_2, f_3)^T$ called solenoidal (divergence-free), if its divergence is zero, $\operatorname{div}\mathbf{f} = 0$, where the divergence operator div is defined as

$$\operatorname{div}\mathbf{f} \equiv \nabla\boldsymbol{.}\,\mathbf{f} := \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\,.$$

Another differential operator rot is the rotor operator, briefly defined through a symbolic determinant

$$\operatorname{rot}\mathbf{f} := \nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial z_3 \\ f_1 & f_2 & f_3 \end{vmatrix},$$

here $\times$ stands for cross product. If the vector field is potential, then the rotor of such a field is equal to zero, $\operatorname{rot}\nabla v = \mathbf{0}$.

The vector Laplace operator Δ is defined by the formula through componentwise application

$$\Delta\mathbf{f} := (\Delta f_1, \Delta f_2, \Delta f_3)^T.$$

The vector Laplace operator satisfies the vector identity

$$\Delta \mathbf{f} = \nabla \mathrm{div}\, \mathbf{f} - \mathrm{rot}\,\mathrm{rot}\, \mathbf{f}.$$

We will use the standard notation for Hermitian Sobolev spaces $L_2(\mathbb{B}^3)$, $H^1(\mathbb{B}^3)$, $H_0^1(\mathbb{B}^3)$ in the scalar case and $\mathbf{L}_2(\mathbb{B}^3) \equiv (L_2(\mathbb{B}^3))^3$, $\mathbf{H}^1(\mathbb{B}^3) \equiv (H^1(\mathbb{B}^3))^3$, $\mathbf{H}_0^1(\mathbb{B}^3) \equiv (H_0^1(\mathbb{B}^3))^3$ in the vector case. Lebesgue space $L_2(\mathbb{B}^3)$ — Hermitian space of square-integrable functions, defined in the ball $\mathbb{B}^3$, with inner product and finite norm,

$$(u, v)_{L_2(\mathbb{B}^3)} = \int_{\mathbb{B}^3} u(\boldsymbol{\xi}) \cdot v^*(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi},\ \ ||u||^2_{L_2(\mathbb{B}^3)} = (u, u)_{L_2(\mathbb{B}^3)},$$

where $*$ denote the complex conjugate.

Let $H^1(\mathbb{B}^3) \equiv H(\nabla) = \{v \in L_2(\mathbb{B}^3) : \nabla v \in \mathbf{L}_2(\mathbb{B}^3)\}$ is Sobolev space of functions in the ball $\mathbb{B}^3$ with Hermitian inner product

$$(u, v)_{H^1(\mathbb{B}^3)} = \int_{\mathbb{B}^3} uv^*\, \mathrm{d}\mathbf{x} + \int_{\mathbb{B}^3} \nabla u \cdot \nabla v^*\, \mathrm{d}\mathbf{x},\ \ ||u||^2_{H^1(\mathbb{B}^3)} = (u, u)_{H^1(\mathbb{B}^3)}.$$

Here ∇u is the gradient of the function, and the corresponding norm is denoted by $||u||^2_{H^1(\mathbb{B}^3)}$.

In a homogeneous space $H_0^1(\mathbb{B}^3) = \{v \in H^1(\mathbb{B}^3) : v(\boldsymbol{\xi}) = 0,\ \boldsymbol{\xi} \in \mathbb{B}^3\}$ inner product (Dirichlet integral) and norm have the form

$$(4)\quad ((u, v))_{H_0^1(\mathbb{B}^3)} = \int_{\mathbb{B}^3} \nabla u \cdot \nabla v^*\, \mathrm{d}\mathbf{x},\ ||u||^2_{H_0^1(\mathbb{B}^3)} = ((u, u))_{H_0^1(\mathbb{B}^3)} = \int_{\mathbb{B}^3} |\nabla u|^2\, \mathrm{d}\mathbf{x}.$$

Such a norm in $H_0^1(\mathbb{B}^3)$, as a subspace of $H^1(\mathbb{B}^3)$, is equivalent to the norm of the space $H^1(\mathbb{B}^3)$.

In the vector case $\mathbf{L}_2(\mathbb{B}^3)$ — Hermitian space of vector functions, defined in the ball $\mathbb{B}^3$, with inner product and the norm,

$$(5)\qquad (\mathbf{u}, \mathbf{v})_{\mathbf{L}_2(\mathbb{B}^3)} = \int_{\mathbb{B}^3} \mathbf{u}(\boldsymbol{\xi}) \cdot \mathbf{v}^*(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi},\ ||\mathbf{u}||^2_{\mathbf{L}_2(\mathbb{B}^3)} = (\mathbf{u}, \mathbf{u})_{\mathbf{L}_2(\mathbb{B}^3)}.$$

In the Sobolev space $\mathbf{H}^1(\mathbb{B}^3)$ we use inner product $(\cdot, \cdot)_{\mathbf{H}^1}$,

$$(\mathbf{u}, \mathbf{w})_{\mathbf{H}^1} = (\mathbf{u}, \mathbf{w})_{\mathbf{L}_2(\mathbb{B}^3)} + (\mathrm{div}\, \mathbf{u}, \mathrm{div}\, \mathbf{w})_{\mathbf{L}_2(\mathbb{B}^3)} + (\mathrm{rot}\, \mathbf{u}, \mathrm{rot}\, \mathbf{w})_{\mathbf{L}_2(\mathbb{B}^3)}.$$

In a homogeneous space $\mathbf{H}_0^1(\mathbb{B}^3) = \{\mathbf{v} \in \mathbf{H^1}(\mathbb{B}^\mathbf{3}) : \mathbf{v}(\boldsymbol{\xi}) = \mathbf{0},\ \boldsymbol{\xi} \in \mathbb{B}^\mathbf{3}\}$ inner product $((\cdot, \cdot))_{\mathbf{H}_0^1}$,

$$((\mathbf{u}, \mathbf{w}))_{\mathbf{H}_0^1} = \int_{\mathbb{B}^3} \nabla \mathbf{u} : \nabla \mathbf{v}^*\, \mathrm{d}\mathbf{x} = (\mathrm{div}\, \mathbf{u}, \mathrm{div}\, \mathbf{w})_{\mathbf{L}_2(\mathbb{B}^3)} + (\mathrm{rot}\, \mathbf{u}, \mathrm{rot}\, \mathbf{w})_{\mathbf{L}_2(\mathbb{B}^3)}$$

generates an energy norm in $\mathbf{H}_0^1(\mathbb{B}^3)$, equivalent to the norm in space $\mathbf{H}^1(\mathbb{B}^3)$.

For vector fields on the sphere, we consider the Hilbert space $\mathbf{L}_2(\mathbb{S}^2) \equiv (L_2(\mathbb{S}^2))^3$, which is determined by the inner product $(\cdot, \cdot)_{\mathbf{L}_2(\mathbb{S}^2)}$ and the norm $|| \cdot ||_{\mathbf{L}_2(\mathbb{S}^2)}$

$$(\mathbf{f},\ \mathbf{g})_{\mathbf{L}_2(\mathbb{S}^2)} := \int_{\mathbb{S}^2} \mathbf{f}(\boldsymbol{\xi}) \cdot \mathbf{g}^*(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi},\ \ ||\mathbf{f}||^2_{\mathbf{L}_2(\mathbb{S}^2)} := (\mathbf{f},\ \mathbf{f})_{\mathbf{L}_2(\mathbb{S}^2)}.$$

2.1. **Helmholtz–Hodge decomposition.** A harmonic vector field is a potential field with a harmonic potential, $\mathbf{h} = \nabla h$, $\Delta h = 0$. We denote $\nabla \mathrm{Harm}(\mathbb{B}^3) = \{\mathbf{h} = \nabla h : h \in H^1(\mathbb{B}^3), \Delta h = 0\}$ — subspace of harmonic vector fields, and through $\nabla H_0^1(\mathbb{B}^3) = \{\mathbf{p} = \nabla v : v \in H_0^1(\mathbb{B}^3)\}$ — subspace of potential vector fields having only a normal component at the boundary. Then the orthogonal sum gives the whole set of potential fields,

$$\nabla H^1(\mathbb{B}^3) = \{\mathbf{u} = \nabla v : \ v \in H^1(\mathbb{B}^3)\} = \nabla H_0^1(\mathbb{B}^3) \overset{\mathbf{L}_2}{\oplus} \nabla \mathrm{Harm}(\mathbb{B}^3).$$

Also let $\mathbf{H}_0(\mathrm{div} = 0)$ be the subspace of solenoidal fields, having only a tangential component on the boundary $\mathbb{S}^2$,

$$\mathbf{H}_0(\mathrm{div} = 0) := \{\mathbf{v} \in \mathbf{L}^2(\mathbb{B}^3) : \mathrm{div}\, \mathbf{v} = 0, \ \boldsymbol{\xi} \cdot \mathbf{v}(\boldsymbol{\xi}) = 0 \text{ on } \mathbb{S}^2\}.$$

The Helmholtz–Hodge decomposition for $\mathbf{L}_2(\mathbb{B}^3)$ allows separating any vector field into the sum of three uniquely defined components: curl free (irrotational), harmonic and divergence free (solenoidal), see e.g. [1], [2], [9], [10], [11].

Theorem 1. *(Helmholtz–Hodge decomposition). Every vector field $\mathbf{f} \in \mathbf{L}_2(\mathbb{B}^3)$ can be decomposed into an irrotational part ∇v, $v \in H_0^1(\mathbb{B}^3)$, a harmonic field ∇h with harmonic function h and a solenoidal field $\overset{\circ}{\mathbf{J}}$ with tangential flow at the boundary $\partial\mathbb{B}^3$, such that*

$$\mathbf{f} = \nabla v + \nabla h + \overset{\circ}{\mathbf{J}} = \mathbf{p} + \mathbf{h} + \overset{\circ}{\mathbf{J}} \ . \tag{6}$$

That means, the space of square integrable vector fields $\mathbf{L}_2(\mathbb{B}^3)$ is an orthogonal sum of three subspaces

$$\mathbf{L}_2(\mathbb{B}^3) = \nabla H_0^1(\mathbb{B}^3) \overset{\mathbf{L}_2}{\oplus} \nabla \mathrm{Harm}(\mathbb{B}^3) \overset{\mathbf{L}_2}{\oplus} \mathbf{H}_0(\mathrm{div} = 0; \mathbb{B}^3).$$

On the other hand, for each potential vector field $\mathbf{p}$ there is a scalar function v, called the scalar potential, $\mathbf{p} = \nabla v$. Similarly, for a solenoidal field $\mathbf{J}$ there exists a vector field $\mathbf{u}$, called the vector potential, which satisfies the equation $\mathrm{rot}\, \mathbf{u} = \mathbf{J}$. It is clear that the vector potential $\mathbf{u}$ for $\mathbf{J}$ is determined ambiguously and, in some cases, we can take as the vector potential the solenoidal vector field or impose boundary conditions on this potential.

Therefore, the Helmholtz decomposition can be written with scalar and vector potentials. For example, there are harmonic potential $h \in \mathrm{Harm}(\mathbb{B}^3)$, scalar potential $v \in H_0^1(\mathbb{B}^3)$ and vector potential $\mathbf{u} \in \mathbf{H}_0^1(\mathbb{B}^3)$, such that

$$\mathbf{h} = \nabla h, \ \mathbf{p} = \nabla v, \ \overset{\circ}{\mathbf{J}} = \mathrm{rot}\, \mathbf{u}$$

and, in this case, the potentials $\mathbf{u}$, v and h are uniquely determined.

2.2. **Zernike polynomials and generalized Zernike polynomials.** It is known that the 3D Zernike polynomials $Z_{N\ell}^{(N+2k)}$ form an orthogonal basis in the Hilbert space $L_2(\mathbb{B}^3)$, see e.g. [12].

Definition 1. *Zernike polynomials $Z_{N\ell}^{(N+2k)}$ and generalized Zernike polynomials $Z_{N\ell}^{[1](N+2k)}$ are defined by formulas*

$$Z_{N\ell}^{(N+2k)}(\mathbf{x}) := \int_{\mathbb{S}^2} Y_{N\ell}(\boldsymbol{\eta}) C_{N+2k}^{(3/2)}(\mathbf{x} \cdot \boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta}, \ N, k \in \mathbb{N}_0, \ |\ell| \le N, \tag{7}$$

$$Z_{N\ell}^{(N+2k)[1]}(\mathbf{x}) := \int_{\mathbb{S}^2} Y_{N\ell}(\boldsymbol{\eta}) P_{N+2k}(\mathbf{x} \cdot \boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta}, \ N \in \mathbb{N}, k \in \mathbb{N}_0, \ |\ell| \le N, \tag{8}$$

where $Y_{N\ell}$ are complex spherical harmonics, $C_m^{(3/2)}$ are Gegenbauer polynomials of degree m and of type $3/2$, and P_m are Legendre polynomials. The degree of polynomials (7) and (8) is determined by the superscript $N + 2k$.

With the help of Funk – Hecke formula (19) Zernike polynomials $Z_{N\ell}^{(N+2k)}$ and generalized Zernike polynomials $Z_{N\ell}^{[1](N+2k)}$ can be written in polar coordinates,

$$Z_{N\ell}^{(N+2k)}(\mathbf{x}) = 2\pi Y_{N\ell}(\boldsymbol{\xi}) \int_{-1}^{1} P_N(s) C_{N+2k}^{(3/2)}(s|\mathbf{x}|)\,\mathrm{d}s,$$

$$Z_{N\ell}^{[1](N+2k)}(\mathbf{x}) = 2\pi Y_{N\ell}(\boldsymbol{\xi}) \int_{-1}^{1} P_N(s) P_{N+2k}(s|\mathbf{x}|)\,\mathrm{d}s.$$

For $k = 0$ in (7) we obtain harmonic polynomials in the ball, $\Delta Z_{N\ell}^{(N)} = 0$. For $k = 0$ in (8) we obtain $Z_{N\ell}^{[1](N)} = \frac{Z_{N\ell}^{(N)}}{2N+1}$, i.e. $Z_{N\ell}^{[1](N)}$ are harmonic functions too.

We will not subsequently write out and use the explicit form of the Zernike polynomials. In the integral form it is more convenient to study the properties of these polynomials. We will make an exception only for harmonic polynomials $Z_{N\ell}^{(N)}$,

$$\begin{aligned} Z_{N\ell}^{(N)}(\mathbf{x}) &= \int_{\mathbb{S}^2} Y_{N\ell}(\boldsymbol{\eta}) C_N^{(3/2)}(\mathbf{x}\cdot\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta} \\ &= 2\pi Y_{N\ell}(\boldsymbol{\xi})|\mathbf{x}|^N \int_{-1}^{1} C_N^{(3/2)}(s) P_N(s)\mathrm{d\,s} = 4\pi|\mathbf{x}|^{\mathrm{N}}\mathrm{Y}_{\mathrm{N}\ell}(\xi). \end{aligned}$$

Harmonic Zernike polynomials of the first degree are

$$Z_{1,-1}^{(1)}(\mathbf{x}) = 4\pi\sqrt{\frac{3}{8\pi}}(x_1 - \boldsymbol{i}x_2),\ Z_{10}^{(1)}(\mathbf{x}) = 4\pi\sqrt{\frac{3}{4\pi}}x_3,\ Z_{11}^{(1)}(\mathbf{x}) = -4\pi\sqrt{\frac{3}{8\pi}}(x_1 + \boldsymbol{i}x_2).$$

The following theorem lists the basic properties of the Zernike polynomials.

Theorem 2. *Zernike polynomial system (7) and generalized Zernike polynomial system (8) are complete in $L_2(\mathbb{B}^3)$. Zernike polynomial system (7) forms an orthogonal basis of the space $L_2(\mathbb{B}^3)$. The generalized Zernike polynomials $\{Z_{N\ell}^{[1](N+2k)}, |\ell| \leq N\}_{N\in\mathbb{N}_0, k\in\mathbb{N}}$ will be a basis for the homogeneous space $H_0^1(\mathbb{B}^3)$ and orthogonal with respect to inner product (4). Following are the properties of the Zerike polynomials,*

$$\left\{\begin{array}{l} \textit{Harmonic Zernike polynomials}\ ,\ N \geq 0 \\ Z_{N\ell}^{(N)}(\mathbf{x}) = 4\pi|\mathbf{x}|^N Y_{N\ell}(\boldsymbol{\xi}),\ \mathbf{x} = |\mathbf{x}|\boldsymbol{\xi} \\ Z_{N\ell}^{(N)}|_{\mathbb{S}^2} = 4\pi Y_{N\ell} \\ \Delta Z_{N\ell}^{(N)} = 0 \\ ||Z_{N\ell}^{(N)}||_{L_2} = ||\mathbf{A}_{N\ell}^{(N-1)}||_{\mathbf{L}_2} = \dfrac{4\pi\sqrt{N}}{2N+1} \end{array}\right. \tag{9}$$

$$\left\{\begin{array}{l} \textit{Zernike polynomials},\ N \geq 0,\ k \geq 1 \\ Z_{N\ell}^{(N+2k)} \\ Z_{N\ell}^{(N+2k)}|_{\mathbb{S}^2} = 4\pi Y_{N\ell} \\ ||Z_{N\ell}^{(N+2k)}||_{L_2} = \dfrac{4\pi}{\sqrt{2N+4k+3}} \end{array}\right. \tag{10}$$

$$(11)\quad \begin{cases} \textit{Harmonic Generalized Zernike polynomials},\ N\geq 0\\ Z^{[1](N)}_{N\ell}=\dfrac{Z^{(N)}_{N\ell}}{2N+1}\\ Z^{[1](N)}_{N\ell}|_{\mathbb{S}^2}=\dfrac{4\pi}{2N+1}Y_{N\ell}\\ ||Z^{[1](N)}_{N\ell}||_{L_2}=||\mathbf{B}^{(N-1)}_{N\ell}||_{\mathbf{L}_2}=\dfrac{4\pi(N+1)\sqrt{N}}{2N+1}\end{cases}$$

$$(12)\quad \begin{cases} \textit{Generalized Zernike polynomials},\ N\geq 0, k\geq 1\\ Z^{[1](N+2k)}_{N\ell}=\dfrac{Z^{(N+2k)}_{N\ell}-Z^{(N+2k-2)}_{N\ell}}{2N+4k+1}\\ \nabla Z^{[1](N+2k)}_{N\ell}(\mathbf{x})=\int_{\mathbb{S}^2}\mathbf{y}^{(1)}_{N\ell}(\boldsymbol{\eta})C^{(3/2)}_{N+2k-1}(\mathbf{x}\cdot\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}=:\mathbf{A}^{(N+2k-1)}_{N\ell}(\mathbf{x}),\\ Z^{[1](N+2k)}_{N\ell}|_{\mathbb{S}^2}=0\\ ||Z^{[1](N+2k)}_{N\ell}||_{L_2}=4\pi\sqrt{\dfrac{2(2N+4k+1)}{(2N+4k-1)(2N+4k+3)}}\\ ||Z^{[1](N+2k)}_{N\ell}||_{H^1_0}=||\mathbf{A}^{(N+2k-1)}_{N\ell}||_{\mathbf{L}_2}=\dfrac{4\pi}{\sqrt{2N+4k+1}}.\end{cases}\quad .$$

2.3. **Vector spherical harmonics (VSHs).** There are vectorial analogues of scalar spherical harmonics called vector spherical harmonics (VSHs). In this section we give definitions and properties of the vector spherical harmonics, which are needed in our work. We refer to [13, 14, 15, 16] for more details in this theme. Here we recall the definitions and some properties of covariant derivatives on the surface of a sphere $\mathbb{S}^2$.

We introduce standard polar coordinates (θ,φ) on the unit sphere $\mathbb{S}^2$, $0<\varphi<2\pi$, $0<\theta<\pi$, θ — latitude, φ — longitude and the unit vector $\boldsymbol{\xi}\equiv(\theta,\varphi)\in\mathbb{S}^2$ be represented in polar coordinates

$$\boldsymbol{\xi}=(\xi_1,\xi_2,\xi_3)=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)=\mathbf{i}\sin\theta\cos\varphi+\mathbf{j}\sin\theta\sin\varphi+\mathbf{k}\cos\theta,$$

where $\mathbf{i},\mathbf{j},\mathbf{k}$ is the canonical orthonormal Cartesian basis. The plane $\boldsymbol{\xi}^\perp=\{\mathbf{x}\in\mathbb{R}^3:\mathbf{x}\cdot\boldsymbol{\xi}=0\}$ is spanned by the two orthonormal vectors $\mathbf{e}_\theta$, $\mathbf{e}_\varphi$ with representations in polar coordinates $\mathbf{e}_\theta$, $\mathbf{e}_\varphi$ are the standard unit vectors tangent to the sphere,

$$(13)\quad \mathbf{e}_\theta=\frac{\partial\boldsymbol{\xi}}{\partial\theta}=(\cos\theta\cos\varphi,\cos\theta\sin\varphi,-\sin\theta),\ \ \mathbf{e}_\varphi=\frac{1}{\sin\theta}\frac{\partial\boldsymbol{\xi}}{\partial\varphi}=(-\sin\varphi,\cos\varphi,0).$$

The $\mathbf{e}_r=\boldsymbol{\xi}$, $\mathbf{e}_\theta$, $\mathbf{e}_\varphi$ form the so called local moving triad $\boldsymbol{\xi}\cdot\mathbf{e}_\theta=0$, $\boldsymbol{\xi}\cdot\mathbf{e}_\theta=0$, $\mathbf{e}_\varphi\cdot\mathbf{e}_\theta=0$ and $\mathbf{e}_\theta\times\mathbf{e}_\varphi=\boldsymbol{\xi}$.

Definition 2. *The tangential gradient or the surface gradient, denoted by* $\nabla\equiv\nabla_{\boldsymbol{\xi}}$ *and the tangential rotated gradient (the surface curl–gradient), denoted by* $\nabla^\perp\equiv\nabla^\perp_{\boldsymbol{\xi}}$*, are defined accordingly as*

$$(14)\quad \nabla_{\boldsymbol{\xi}}u=\frac{\partial u}{\partial\theta}\mathbf{e}_\theta(\boldsymbol{\xi})+\frac{1}{\sin\theta}\frac{\partial u}{\partial\varphi}\mathbf{e}_\varphi(\boldsymbol{\xi}),$$

$$(15)\quad \nabla^\perp_{\boldsymbol{\xi}}u=\boldsymbol{\xi}\times\nabla_{\boldsymbol{\xi}}u=-\frac{1}{\sin\theta}\frac{\partial u}{\partial\varphi}\mathbf{e}_\theta(\boldsymbol{\xi})+\frac{\partial u}{\partial\theta}\mathbf{e}_\varphi(\boldsymbol{\xi}).$$

Obviously, we have $\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{\xi}}u(\boldsymbol{\xi})=0$, $\boldsymbol{\xi}\cdot\nabla^\perp_{\boldsymbol{\xi}}u(\boldsymbol{\xi})=0$ and $\nabla u\cdot\nabla^\perp u=0$. Therefore, ∇u and $\nabla^\perp u$ are tangent (tangential) vector fields on $\mathbb{S}^2$. The vector field $\nabla^\perp u$ on the sphere is obtained by rotation of ∇u by the angle $\pi/2$ in the tangent plane and $\nabla^{\perp} \equiv \mathrm{i}\mathbf{L}$, where $\mathbf{L}$ is angular momentum operator. In this article we work with the pure–spin VSHs given in [16] with the exception that the normalization factors are neglected here.

The vector spherical harmonics are defined for $\boldsymbol{\xi} \in \mathbb{S}^2$ by

Definition 3. *The vector spherical harmonics (pure–spin VSHs or Hansen spherical harmonics) are arranged in three families:* $\mathbf{y}^{(1)}_{N\ell}(\boldsymbol{\xi})$, $\mathbf{y}^{(2)}_{N\ell}(\boldsymbol{\xi})$ *and* $\mathbf{y}^{(3)}_{N\ell}(\boldsymbol{\xi})$. *For* $\boldsymbol{\xi} \in \mathbb{S}^2$ *and given a scalar spherical harmonic* $Y_{N\ell}(\boldsymbol{\xi})$ *the unnormalized vector spherical harmonics are the set*

$$\mathbf{y}^{(1)}_{N\ell}(\boldsymbol{\xi}) = \boldsymbol{\xi} Y_{N\ell}(\boldsymbol{\xi}), \quad N \in \mathbb{N}_0, \tag{16}$$

$$\mathbf{y}^{(2)}_{N\ell}(\boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}} Y_{N\ell}(\boldsymbol{\xi}), \quad N \in \mathbb{N}, \tag{17}$$

$$\mathbf{y}^{(3)}_{N\ell}(\boldsymbol{\xi}) = \boldsymbol{\xi} \times \mathbf{y}^{(2)}_{N\ell}(\boldsymbol{\xi}) = \nabla^{\perp}_{\boldsymbol{\xi}} Y_{N\ell}(\boldsymbol{\xi}), \quad N \in \mathbb{N}. \tag{18}$$

The connections between the VSHs of indices $+\ell$ and $-\ell$ are the following

$$\begin{aligned}
\mathbf{y}^{(1)}_{N,-\ell}(\boldsymbol{\xi}) &= (-1)^{\ell} \mathbf{y}^{(1)*}_{N\ell}(\boldsymbol{\xi}) \\
\mathbf{y}^{(2)}_{N,-\ell}(\boldsymbol{\xi}) &= (-1)^{\ell} \mathbf{y}^{(2)*}_{N\ell}(\boldsymbol{\xi}) \\
\mathbf{y}^{(3)}_{N,-\ell}(\boldsymbol{\xi}) &= (-1)^{\ell} \mathbf{y}^{(3)*}_{N\ell}(\boldsymbol{\xi}),
\end{aligned}$$

where $*$ denote the complex conjugate. We will say that $\mathbf{y}^{(2)}_{N\ell}$ and $\mathbf{y}^{(3)}_{N\ell}$ are the grad and curl spherical harmonics. The field $\mathbf{y}^{(2)}_{N\ell}$ is called poloidal, electric, or potential; while $\mathbf{y}^{(3)}_{N\ell}$ — toroidal, magnetic, or solenoidal.

The pure–spin VSHs form a complete set of orthogonal vector functions on the surface of a sphere $\mathbb{S}^2$ with the inner product of the $\mathbf{L}_2(\mathbb{S}^2)$ space. The vector spherical harmonics, defined as above, are orthogonal but not normalized,

$$(\mathbf{y}^{(1)}_{N\ell}, \mathbf{y}^{(1)}_{N'\ell'})_{\mathbf{L}_2} = \delta^{N'}_{N} \delta^{\ell'}_{\ell}, \ (\mathbf{y}^{(2)}_{N\ell}, \mathbf{y}^{(2)}_{N'\ell'})_{\mathbf{L}_2} = \lambda^2_N \delta^{N'}_{N} \delta^{\ell'}_{\ell}, \ (\mathbf{y}^{(3)}_{N\ell}, \mathbf{y}^{(3)}_{N'\ell'})_{\mathbf{L}_2} = \lambda^2_N \delta^{N'}_{N} \delta^{\ell'}_{\ell}.$$

The normalizing pure–spin vector spherical harmonics or orthonormal system of VSHs are

$$\widehat{\mathbf{y}}^{(1)}_{N\ell} = \mathbf{y}^{(1)}_{N\ell}, \quad \widehat{\mathbf{y}}^{(2)}_{N\ell} = \lambda^{-1}_N \mathbf{y}^{(2)}_{N\ell}, \quad \widehat{\mathbf{y}}^{(3)}_{N\ell} = \lambda^{-1}_N \mathbf{y}^{(3)}_{N\ell}, \quad \lambda_N = \sqrt{N(N+1)}.$$

The scalar Funk-Hecke formula is contained in the next theorem, see e.g. [8], [16].

Theorem 3. *(scalar Funk-Hecke theorem) Suppose* $F(t)$ *is continuous for* $-1 \leq t \leq 1$. *Then for every spherical harmonics of degree* N

$$\int_{\mathbb{S}^2} F(\boldsymbol{\xi} \cdot \boldsymbol{\eta}) Y_{N\ell}(\boldsymbol{\eta}) \, \mathrm{d}\,\boldsymbol{\xi} = 2\pi Y_{N\ell}(\boldsymbol{\xi}) \int_{-1}^{1} F(t) P_N(t) \mathrm{d}t. \tag{19}$$

Her $\boldsymbol{\xi} \cdot \boldsymbol{\eta}$ *is the usual inner product in* $\mathbb{R}^3$.

Now we formulate vectorial version of the Funk–Hecke formula (see [15, 16]).

Theorem 4. *(Vector Funk–Hecke theorem) Let* $F(t) \in L_1(-1,1)$ *— an integrable function of one variable on the segment* $-1 \leq t \leq 1$. *Then, for vector spherical harmonics, the following equalities hold:*

$$\begin{aligned}
&\int_{\mathbb{S}^2} \mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\eta}) F(\boldsymbol{\xi} \cdot \boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta} = \alpha_{11} \mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\xi}) + \alpha_{12} \mathbf{y}_{N\ell}^{(2)}(\boldsymbol{\xi}),\\
&\int_{\mathbb{S}^2} \mathbf{y}_{N\ell}^{(2)}(\boldsymbol{\eta}) F(\boldsymbol{\xi} \cdot \boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta} = \alpha_{21} \mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\xi}) + \alpha_{22} \mathbf{y}_{N\ell}^{(2)}(\boldsymbol{\xi}),\\
&\int_{\mathbb{S}^2} \mathbf{y}_{N\ell}^{(3)}(\boldsymbol{\eta}) F(\boldsymbol{\xi} \cdot \boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta} = \alpha_{33} \mathbf{y}_{N\ell}^{(3)}(\boldsymbol{\xi}),
\end{aligned}$$

where the coefficients are calculated by the formulas

$$\begin{aligned}
\alpha_{11} &= \frac{2\pi}{2N+1}\left(N\int_{-1}^{1} F(s)P_{N-1}(s)\,\mathrm{d}s + (N+1)\int_{-1}^{1} F(s)P_{N+1}(s)\,\mathrm{d}s\right),\\
\alpha_{12} &= \frac{2\pi}{2N+1}\left(\int_{-1}^{1} F(s)P_{N-1}(s)\,\mathrm{d}s - \int_{-1}^{1} F(s)P_{N+1}(s)\,\mathrm{d}s\right),\\
\alpha_{21} &= N(N+1)\alpha_{12},\\
\alpha_{22} &= \frac{2\pi}{2N+1}\left((N+1)\int_{-1}^{1} F(s)P_{N-1}(s)\,\mathrm{d}s + N\int_{-1}^{1} F(s)P_{N+1}(s)\,\mathrm{d}s\right),\\
\alpha_{33} &= 2\pi\int_{-1}^{1} F(s)P_{N}(s)\,\mathrm{d}s.
\end{aligned}$$

These formulas imply the following formula

$$\begin{aligned}
&\int_{\mathbb{S}^2}\Big(N\mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\eta}) + \mathbf{y}_{N\ell}^{(2)}(\boldsymbol{\eta})\Big)F(\boldsymbol{\xi} \cdot \boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta} = 2\pi\Big(N\mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\xi}) + \mathbf{y}_{N\ell}^{(2)}(\boldsymbol{\xi})\Big)\int_{-1}^{1} F(s)P_{N-1}(s)\,\mathrm{d}s,\\
&\int_{\mathbb{S}^2}\Big(-(N+1)\mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\eta}) + \mathbf{y}_{N\ell}^{(2)}(\boldsymbol{\eta})\Big)F(\boldsymbol{\xi} \cdot \boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}\\
&= 2\pi\Big(-(N+1)\mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\xi}) + \mathbf{y}_{N\ell}^{(2)}(\boldsymbol{\xi})\Big)\int_{-1}^{1} F(s)P_{N+1}(s)\,\mathrm{d}s.
\end{aligned}$$

The Funk – Hecke vector formulas will also be used to calculate spherical integrals, in particular, for computing the boundary values of vector fields in the ball $\mathbb{B}^3$.

2.4. **The basis for the object space $\mathbf{L}_2(\mathbb{B}^3)$. Solenoidal and potential polynomial vector fields.** This part contains information from the work [15], [17]. We recall the definitions and the main properties of solenoidal and potential basis vector fields of the space $\mathbf{L}^2(\mathbb{B}^3)$.

Definition 4.

$$\mathbf{A}_{N\ell}^{(N+2k-1)}(\mathbf{x}) := \int_{\mathbb{S}^2} \mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\eta}) C_{N+2k-1}^{(3/2)}(\mathbf{x} \cdot \boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta},\ N,k \in \mathbb{N}_0\ , \tag{20}$$

$$\mathbf{H}_{N\ell}^{(N-1)}(\mathbf{x}) := \int_{\mathbb{S}^2}\Big(N\mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\eta}) + \mathbf{y}_{N\ell}^{(2)}(\boldsymbol{\eta})\Big) C_{N-1}^{(3/2)}(\mathbf{x} \cdot \boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta},\ N \in \mathbb{N}, \tag{21}$$

$$\mathbf{B}_{N\ell}^{(N+2k-1)}(\mathbf{x}) := \int_{\mathbb{S}^2} \mathbf{y}_{N\ell}^{(2)}(\boldsymbol{\eta}) C_{N+2k-1}^{(3/2)}(\mathbf{x} \cdot \boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta},\ N \in \mathbb{N},\ k \in \mathbb{N}_0\ , \tag{22}$$

$$\mathbf{C}_{N\ell}^{(N+2k)}(\mathbf{x}) := \int_{\mathbb{S}^2} \mathbf{y}_{N\ell}^{(3)}(\boldsymbol{\eta}) C_{N+2k}^{(3/2)}(\mathbf{x} \cdot \boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta},\qquad N \in \mathbb{N}\ ,\ k \in \mathbb{N}_0, \tag{23}$$

For convenience, we put

$$\mathbf{A}_{00}^{(-1)}(\mathbf{x}) \equiv 0.$$

Remark, that if we put $k=0$ in (20) and (22), then we get

$$\mathbf{A}_{N\ell}^{(N-1)}(\mathbf{x}) = \frac{1}{2N+1}\mathbf{H}_{N\ell}^{(N-1)}(\mathbf{x}),\ \mathbf{B}_{N\ell}^{(N-1)}(\mathbf{x}) = \frac{N+1}{2N+1}\mathbf{H}_{N\ell}^{(N-1)}(\mathbf{x}),$$

which in the end gives

$$\mathbf{H}_{N\ell}^{N-1}(\mathbf{x}) = N\mathbf{A}_{N\ell}^{(N-1)} + \mathbf{B}_{N\ell}^{(N-1)}(\mathbf{x}).$$

The following two theorems reflect the results obtained in [15], [17].

Theorem 5. *Following are the properties of the polynomial vector fields (20) – (23). Action results of operators* div, rot, *the values of these vector functions on the boundary (on the sphere)* $\mathbb{S}^2$ *are considered. All of these properties are listed below for each group of polynomial vector fields (20) – (23) separately.*

$$\begin{cases} \textit{harmonic, poloidal fields},\ N\geq 1 \\ \mathbf{A}_{N\ell}^{(N-1)} = \nabla Z_{N\ell}^{[1](N)} = \dfrac{\nabla Z_{N\ell}^{(N)}}{2N+1} = \dfrac{\mathbf{H}_{N\ell}^{(N-1)}}{2N+1} \\ \operatorname{div}\mathbf{A}_{N\ell}^{(N-1)} = 0 \\ \operatorname{rot}\mathbf{A}_{N\ell}^{(N-1)} = \mathbf{0} \\ \mathbf{A}_{N\ell}^{(N-1)}|_{\mathbb{S}^2} = \dfrac{4\pi\left(N\mathbf{y}_{N\ell}^{(1)} + \mathbf{y}_{N\ell}^{(2)}\right)}{2N+1} \\ ||\mathbf{A}_{N\ell}^{(N-1)}||_{\mathbf{L}_2} = \dfrac{4\pi\sqrt{N}}{2N+1} \end{cases} \tag{24}$$

$$\begin{cases} \textit{potential fields},\ N\geq 0,\ k\geq 1 \\ \mathbf{A}_{N\ell}^{(N+2k-1)} = \nabla Z_{N\ell}^{[1](N+2k)} \\ \operatorname{div}\mathbf{A}_{N\ell}^{(N+2k-1)} = \sum_{s=0}^{k-1}(2N+4s+3)Z_{N\ell}^{(N+2s)} \\ \operatorname{rot}\mathbf{A}_{N\ell}^{(N+2k-1)} = \mathbf{0} \\ \mathbf{A}_{N\ell}^{(N+2k-1)}|_{\mathbb{S}^2} = 4\pi\mathbf{y}_{N\ell}^{(1)} \\ ||\mathbf{A}_{N\ell}^{(N+2k-1)}||_{\mathbf{L}_2} = \dfrac{4\pi}{\sqrt{2N+4k+1}} \end{cases} \tag{25}$$

$$\begin{cases} \textit{harmonic, poloidal fields}, N\geq 1 \\ \mathbf{B}_{N\ell}^{(N-1)} = (N+1)\nabla Z_{N\ell}^{[1](N)} = \dfrac{N+1}{2N+1}\mathbf{H}_{N\ell}^{(N-1)} \\ \operatorname{div}\mathbf{B}_{N\ell}^{(N-1)} = 0 \\ \operatorname{rot}\mathbf{B}_{N\ell}^{(N-1)} = \mathbf{0} \\ \mathbf{B}_{N\ell}^{(N-1)}|_{\mathbb{S}^2} = \dfrac{4\pi(N+1)\left(N\mathbf{y}_{N\ell}^{(1)} + \mathbf{y}_{N\ell}^{(2)}\right)}{2N+1} \\ ||\mathbf{B}_{N\ell}^{(N-1)}||_{\mathbf{L}_2} = \dfrac{4\pi(N+1)\sqrt{N}}{2N+1} \\ \mathbf{B}_{N\ell}^{(N-1)}(\mathbf{x}) = \operatorname{rot}\operatorname{rot}\left(\mathbf{x}Z_{N\ell}^{[1](N)}(\mathbf{x})\right) \end{cases} \tag{26}$$

$$(27)\quad \begin{cases} \textit{solenoidal, poloidal fields}, N \geq 1, k \geq 1 \\ \mathbf{B}_{N\ell}^{(N+2k-1)} = -\mathrm{rot} \int_{\mathbb{S}^2} \mathbf{y}_{N\ell}^{(3)}(\boldsymbol{\eta}) P_{N+2k}(\mathbf{x} \cdot \boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta} \\ \mathrm{div}\, \mathbf{B}_{N\ell}^{(N+2k-1)} = 0 \\ \mathrm{rot}\, \mathbf{B}_{N\ell}^{(N+2k-1)} = \sum_{s=0}^{k-1} (2N+4s+3) \mathbf{C}_{N\ell}^{(N+2s)} \\ \mathbf{B}_{N\ell}^{(N+2k-1)}|_{\mathbb{S}^2} = 4\pi \mathbf{y}_{N\ell}^{(2)} \\ ||\mathbf{B}_{N\ell}^{(N+2k-1)}||_{\mathbf{L}_2} = 4\pi \sqrt{\dfrac{N(N+1)}{2N+4k+1}} \\ \mathbf{B}_{N\ell}^{(N+2k-1)}(\mathbf{x}) = \mathrm{rot}\,\mathrm{rot} \left(\mathbf{x} Z_{N\ell}^{[1](N+2k)}(\mathbf{x}) \right) \end{cases}$$

$$(28)\quad \begin{cases} \textit{solenoidal, toroidal fields}, \; N \geq 1, k \geq 0 \\ \mathbf{C}_{N\ell}^{(N+2k)} = \mathrm{rot} \int_{\mathbb{S}^2} \mathbf{y}_{N\ell}^{(2)}(\boldsymbol{\eta}) P_{N+2k+1}(\mathbf{x} \cdot \boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta} \\ \mathrm{div}\, \mathbf{C}_{N\ell}^{(N+2k)} = 0 \\ \mathrm{rot}\, \mathbf{C}_{N\ell}^{(N+2k)} = -\sum_{s=0}^{k} (2N+4s+1) \mathbf{B}_{N\ell}^{(N+2s-1)} \\ \mathbf{C}_{N\ell}^{(N+2k)}|_{\mathbb{S}^2} = 4\pi \mathbf{y}_{N\ell}^{(3)} \\ ||\mathbf{C}_{N\ell}^{(N+2k)}||_{\mathbf{L}_2} = 4\pi \sqrt{\dfrac{N(N+1)}{2N+4k+3}} \\ \mathbf{C}_{N\ell}^{(N+2k)}(\mathbf{x}) = -\mathrm{rot} \left(\mathbf{x} Z_{N\ell}^{(N+2k)}(\mathbf{x}) \right) = \mathbf{x} \times \nabla Z_{N\ell}^{(N+2k)}(\mathbf{x}) \end{cases}$$

$$(29)\quad \begin{cases} \textit{harmonic, poloidal fields}, \; N \geq 1 \\ \mathbf{H}_{N\ell}^{(N-1)} = \nabla Z_{N\ell}^{(N)} \\ \mathrm{div}\, \mathbf{H}_{N\ell}^{(N-1)} = 0 \\ \mathrm{rot}\, \mathbf{H}_{N\ell}^{(N-1)} = \mathbf{0} \\ \mathbf{H}_{N\ell}^{(N-1)}|_{\mathbb{S}^2} = 4\pi \left(N \mathbf{y}_{N\ell}^{(1)} + \mathbf{y}_{N\ell}^{(2)} \right) \\ ||\mathbf{H}_{N\ell}^{(N-1)}||_{\mathbf{L}_2} = 4\pi \sqrt{N} \\ \mathbf{H}_{N\ell}^{(N-1)}(\mathbf{x}) = \dfrac{1}{N+1} \mathrm{rot}\,\mathrm{rot} \left(\mathbf{x} Z_{N\ell}^{(N)}(\mathbf{x}) \right) \end{cases}$$

In [17] was shown that $\mathbf{B}_{N\ell}^{(N+2k-1)}$ and $\mathbf{H}_{N\ell}^{(N-1)}$ are poloidal, while $\mathbf{C}_{N\ell}^{(N+2k)}$ are toroida vector fields. Decompose a solenoidal vector field in the ball with respect to this basis we obtain the so–called poloidal–toroidal decomposition.

Theorem 6. *Polynomial vector field system (20) − (23)*

$$\left\{ \mathbf{A}_{N\ell}^{(N+2k-1)} \right\}_{N \in \mathbb{N}_0,\; k \in \mathbb{N}}$$
$$\bigcup \left\{ \mathbf{H}_{N\ell}^{(N-1)} \right\}_{N \in \mathbb{N}}, \bigcup \left\{ \mathbf{B}_{N\ell}^{(N+2k+1)} \right\}_{N \in \mathbb{N},\; k \in \mathbb{N}_0} \bigcup \left\{ \mathbf{C}_{N\ell}^{(N+2k)} \right\}_{N \in \mathbb{N},\; k \in \mathbb{N}_0}$$

forms a polynomial orthogonal basis of the space $\mathbf{L}_2(\mathbb{B}^3)$. *Thus we have an orthogonal polynomial basis and for each vector functions* $\mathbf{f} \in \mathbf{L}_2(\mathbb{B}^3)$ *there is orthogonal expansion in Fourier series*

$$(30)\qquad \mathbf{f} = \underbrace{\sum_{N=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell} f_{N\ell}^{(N+2k-1)A}\mathbf{A}_{N\ell}^{(N+2k-1)}}_{potential\ part}$$
$$+\underbrace{\sum_{N=1}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell} f_{N\ell}^{(N+2k+1)B}\mathbf{B}_{N\ell}^{(N+2k+1)} + \sum_{N=1}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell} f_{N\ell}^{(N+2k)C}\mathbf{C}_{N\ell}^{(N+2k)}}_{=\mathring{\mathbf{J}},\ solenoidal\ part}.$$

If we need to highlight the harmonic part, then we get

$$\mathbf{f} = \underbrace{\sum_{N=0}^{\infty}\sum_{k=1}^{\infty}\sum_{\ell} f_{N\ell}^{(N+2k-1)A}\mathbf{A}_{N\ell}^{(N+2k-1)}}_{=\mathbf{p}=\nabla v,\ potential\ part} + \underbrace{\sum_{N=1}^{\infty}\sum_{\ell} f_{N\ell}^{(N-1)H}\mathbf{H}_{N\ell}^{(N-1)}}_{=\mathbf{h},\ harmonic\ part}$$
$$+\underbrace{\sum_{N=1}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell} f_{N\ell}^{(N+2k+1)B}\mathbf{B}_{N\ell}^{(N+2k+1)} + \sum_{N=1}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell} f_{N\ell}^{(N+2k)C}\mathbf{C}_{N\ell}^{(N+2k)}}_{=\mathring{\mathbf{J}},\ solenoid\ part}.$$

In the future for the sake of simplicity the summation over ℓ will be omitted.

Exactly, $\left\{\mathbf{H}_{N\ell}^{(N-1)},\ |\ell|\le N\right\}_{N\in\mathbb{N}}$ — an orthogonal basis of the subspace of harmonic fields $\nabla\mathrm{Harm}(\mathbb{B}^3)$, $\left\{\mathbf{A}_{N\ell}^{(N+2k-1)},\ |\ell|\le N\right\}_{N\in\mathbb{N}_0,k\in\mathbb{N}}$ — an orthogonal basis of the subspace of potential fields $\nabla H_0^1(\mathbb{B}^3)$, $\left\{\mathbf{B}_{N\ell}^{(N+2k+1)},\mathbf{C}_{N\ell}^{(N+2k)},\ |\ell|\le N\right\}_{N,k\in\mathbb{N}_0}$ — an orthogonal basis of the subspace of solenoidal vector fields $\mathbf{H}_0(\mathrm{div}=0)$.

As an example, we consider the harmonic vector fields that are constant, $N=1$. There should be three of them: $\ell=-1,0,1$. Then

$$\mathbf{H}_{1,-1}^{(0)} = \nabla Z_{1,-1}^{(1)} = 4\pi\sqrt{\frac{3}{8\pi}}(1,-\mathrm{i},0)^{\perp},\ \mathbf{H}_{10}^{(0)} = \nabla Z_{10}^{(1)} = 4\pi\sqrt{\frac{3}{4\pi}}(0,0,1)^{\perp},$$
$$\mathbf{H}_{11}^{(0)} = \nabla Z_{11}^{(1)} = -4\pi\sqrt{\frac{3}{8\pi}}(1,\mathrm{i},0)^{\perp}.$$

There are only nine vector functions of the first degree

$$\mathbf{A}_{00}^{(1)}(\mathbf{x}) = \int_{\mathbb{S}^2}\mathbf{y}_{00}^{(1)}(\boldsymbol{\eta})C_1^{(3/2)}(\mathbf{x}\cdot\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta} = \frac{3}{\sqrt{4\pi}}\int_{\mathbb{S}^2}\boldsymbol{\eta}(\mathbf{x}\cdot\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta} = \frac{3}{\sqrt{4\pi}}\frac{4\pi}{3}\mathbf{x} = \sqrt{4\pi}\mathbf{x},$$
$$\mathbf{C}_{1\ell}^{(1)}(\mathbf{x}) = \int_{\mathbb{S}^2}\mathbf{y}_{1\ell}^{(3)}(\boldsymbol{\eta})C_1^{(3/2)}(\mathbf{x}\cdot\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta} = 2\pi|\mathbf{x}|\mathbf{y}_{1\,m}^{(3)}(\boldsymbol{\xi})3\int_{-1}^{1}s^2\mathrm{d}\,s = 4\pi|\mathbf{x}|\mathbf{y}_{1\ell}^{(3)}(\boldsymbol{\xi}),$$
$$\mathbf{H}_{2\ell}^{(1)}(\mathbf{x}) = \frac{5}{3}\int_{\mathbb{S}^2}\mathbf{y}_{2\ell}^{(2)}(\boldsymbol{\eta})\underbrace{C_1^{(3/2)}(\mathbf{x}\cdot\boldsymbol{\eta})}_{=3(\mathbf{x}\cdot\boldsymbol{\eta})}\,\mathrm{d}\boldsymbol{\eta} = 4\pi|\mathbf{x}|\left(2\mathbf{y}_{2\ell}^{(1)}(\boldsymbol{\xi}) + \mathbf{y}_{2\ell}^{(2)}(\boldsymbol{\xi})\right).$$

3. Vector potentials for solenoidal basis vector functions

In this section we construct polynomial vector functions, which are vector potentials for solenoidal fields from the basis of the space $\mathbf{L}_2(\mathbb{B}^3)$. Based on these vector functions, a basis for Sobolev space $\mathbf{H}_0^1(\mathbb{B}^3)$ is constructed.

We introduce into consideration polynomial vector fields $\mathbf{U}_{N\ell}^{(N+2k-1)}$, $\mathbf{V}_{N\ell}^{(N+2k)}$ and $\mathbf{W}_{N\ell}^{(N+2k-1)}$, in determining which we use the same approach as in the case of basis vector functions $\mathbf{A}_{N\ell}^{(N+2k-1)}$, $\mathbf{B}_{N\ell}^{(N+2k+1)}$ and $\mathbf{C}_{N\ell}^{(N+2k)}$.

Definition 5.

$$\mathbf{U}_{N\ell}^{(N+2k-1)}(\mathbf{x}) := \int_{\mathbb{S}^2} \mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\eta}) P_{N+2k-1}(\mathbf{x}\,.\,\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta},\ \ N,k\in\mathbb{N}_0, \tag{31}$$

$$\mathbf{V}_{N\ell}^{(N+2k)}(\mathbf{x}) := -\int_{\mathbb{S}^2} \mathbf{y}_{N\ell}^{(3)}(\boldsymbol{\eta}) P_{N+2k}(\mathbf{x}\,.\,\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta},\ \ N\in\mathbb{N}, k\in\mathbb{N}_0, \tag{32}$$

$$\mathbf{W}_{N\ell}^{(N+2k-1)}(\mathbf{x}) := \int_{\mathbb{S}^2} \mathbf{y}_{N\ell}^{(2)}(\boldsymbol{\eta}) P_{N+2k-1}(\mathbf{x}\,.\,\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta},\ \ N\in\mathbb{N}, k\in\mathbb{N}_0, \tag{33}$$

where P_n are Legendre polynomials. For convenience, we also put

$$\mathbf{U}_{00}^{(-1)}(\mathbf{x}) \equiv 0.$$

Note that in this definition, as before, all options are taken into account values for integer variables N and k for which the data integrals exist and are not equal to zero. In this work, the explicit form of polynomial vector fields are not considered and not used, so it will be a separate task.

For example we have

$$\mathbf{V}_{N\ell}^{(N)} = -\frac{1}{2N+1}\int_{\mathbb{S}^2} \mathbf{y}_{N\ell}^{(3)}(\boldsymbol{\xi})(C_N^{(3/2)}(\mathbf{x}.\boldsymbol{\xi}) - C_{N-2}^{(3/2)}(\mathbf{x}.\boldsymbol{\xi}))\,\mathrm{d}\boldsymbol{\xi} = -\frac{1}{2N+1}\mathbf{C}_{N\ell}^{(N)},\ N\in\mathbb{N},$$

$$\mathbf{V}_{1\ell}^{(1)} = -\frac{1}{3}\int_{\mathbb{S}^2} \mathbf{y}_{1\ell}^{(3)}(\boldsymbol{\xi}) C_1^{(3/2)}(\mathbf{x}\,.\,\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = -\frac{1}{3}\mathbf{C}_{1\ell}^{(1)}.$$

Let us formulate the basic properties of vector fields (31), (32) and (33).

Theorem 7. *For vector functions $\mathbf{U}_{N\ell}^{(N+2k-1)}$, $\mathbf{V}_{N\ell}^{(N+2k)}$ and $\mathbf{W}_{N\ell}^{(N+2k-1)}$ the following properties are considered:*

- *representation in basis of $\mathbf{L}_2(\mathbb{B}^3)$*
- *values of operators* div, rot
- *boundary values on the sphere*

All of these properties are listed below for each group separately.

$$\begin{cases} \textit{potential field},\ N=0,\ k=1 \\ \mathbf{U}_{00}^{(1)} = \dfrac{\mathbf{A}_{00}^{(1)}}{3} = \dfrac{\sqrt{4\pi}}{3}\mathbf{x} \\ \mathrm{div}\mathbf{U}_{00}^{(1)} = Z_{00}^{(0)} = \sqrt{4\pi} \\ \mathrm{rot}\mathbf{U}_{00}^{(1)} = \mathbf{0} \\ \mathbf{U}_{00}^{(1)}(\mathbf{x})|_{\mathbb{S}^3} = \dfrac{4\pi}{3}\mathbf{y}_{00}^{(1)}(\boldsymbol{\xi}) = \dfrac{\sqrt{4\pi}}{3}\boldsymbol{\xi} \end{cases} \tag{34}$$

$$\left\{\begin{array}{ll} \textit{harmonic vector fields}, \ N\in\mathbb{N}, \ k=0 & \\ \mathbf{U}^{(N-1)}_{N\ell} & = \dfrac{\mathbf{A}^{(N-1)}_{N\ell}}{2N-1} = \dfrac{\mathbf{H}^{(N-1)}_{N\ell}}{(2N-1)(2N+1)} \\ \mathrm{div}\mathbf{U}^{(N-1)}_{N\ell} = 0 & \\ \mathrm{rot}\mathbf{U}^{(N-1)}_{N\ell} = \mathbf{0} & \\ \mathbf{U}^{(N-1)}_{N\ell}|_{\mathbb{S}^3} & = \dfrac{4\pi\Big(N\mathbf{y}^{(1)}_{N\ell}+\mathbf{y}^{(2)}_{N\ell}\Big)}{(2N-1)(2N+1)}, \end{array}\right. \tag{35}$$

$$\left\{\begin{array}{ll} \textit{potential vector fields}, \ N\in\mathbb{N}, \ k=1 & \\ \mathbf{U}^{(N+1)}_{N\ell} & = \dfrac{(2N+1)\mathbf{A}^{(N+1)}_{N\ell}-\mathbf{H}^{(N-1)}_{N\ell}}{(2N+1)(2N+3)} \\ \mathrm{div}\mathbf{U}^{(N+1)}_{N\ell} = Z^{(N)}_{N\ell} & \\ \mathrm{rot}\mathbf{U}^{(N+1)}_{N\ell} = \mathbf{0} & \\ \mathbf{U}^{(N+1)}_{N\ell}|_{\mathbb{S}^3} & = \dfrac{-4\pi\Big(-(N+1)\mathbf{y}^{(1)}_{N\ell}+\mathbf{y}^{(2)}_{N\ell}\Big)}{(2N+1)(2N+3)} \end{array}\right. \tag{36}$$

$$\left\{\begin{array}{ll} \textit{potential vector fields}, \ N\in\mathbb{N}_0, \ k\in\mathbb{N} & \\ \mathbf{U}^{(N+2k-1)}_{N\ell} & = \dfrac{\mathbf{A}^{(N+2k-1)}_{N\ell}-\mathbf{A}^{(N+2k-3)}_{N\ell}}{2N+4k-1} \\ \mathrm{div}\mathbf{U}^{(N+2k-1)}_{N\ell} = Z^{(N+2k-2)}_{N\ell} & \\ \mathrm{rot}\mathbf{U}^{(N+2k-1)}_{N\ell} = \mathbf{0} & \\ \mathbf{U}^{(N+2k-1)}_{N\ell}|_{\mathbb{S}^3} = \mathbf{0} & \\ ||\mathbf{U}^{(N+2k-1)}_{N\ell}||_{\mathbf{H}^1_0} = ||Z^{(N+2k-2)}_{N\ell}||_{L_2} = \dfrac{4\pi}{\sqrt{2N+4k-1}} & \end{array}\right. \tag{37}$$

$$\left\{\begin{array}{l} \textit{solenoidal vector fields}, \ N\in\mathbb{N}, \ k=0 \\ \mathbf{V}^{(N)}_{N\ell} = \dfrac{-\mathbf{C}^{(N)}_{N\ell}}{2N+1} \\ \mathrm{div}\mathbf{V}^{(N)}_{N\ell} = 0 \\ \mathrm{rot}\mathbf{V}^{(N)}_{N\ell} = \mathbf{B}^{(N-1)}_{N\ell} = \dfrac{N+1}{2N+1}\mathbf{H}^{(N-1)}_{N\ell} \\ \mathbf{V}^{(N)}_{N\ell}|_{\mathbb{S}^2} = \dfrac{-4\pi\mathbf{y}^{(3)}_{N\ell}}{2N+1} \\ \Big|\Big|\mathbf{V}^{(N)}_{N\ell}\Big|\Big|^2_{\mathbf{H}^1} = \Big|\Big|\dfrac{\mathbf{C}^{(N)}_{N\ell}}{2N+1}\Big|\Big|^2_{\mathbf{L}_2} + \Big|\Big|\dfrac{N+1}{2N+1}\mathbf{H}^{(N-1)}_{N\ell}\Big|\Big|^2_{\mathbf{L}_2} \sim \Big|\Big|\mathbf{H}^{(N-1)}_{N\ell}\Big|\Big|^2_{\mathbf{L}_2} \end{array}\right. \tag{38}$$

$$\left\{\begin{array}{ll} \textit{solenoidalvectorfields}, \ N,k\in\mathbb{N} & \\ \mathbf{V}^{(N+2k)}_{N\ell} & = -\dfrac{\mathbf{C}^{(N+2k)}_{N\ell}-\mathbf{C}^{(N+2k-2)}_{N\ell}}{2N+4k+1} \\ \mathrm{div}\mathbf{V}^{(N+2k)}_{N\ell} = 0 & \\ \mathrm{rot}\mathbf{V}^{(\mathrm{N+2k})}_{\mathrm{N}\ell} & = \mathbf{B}^{(N+2k-1)}_{N\ell} \\ \mathbf{V}^{(N+2k)}_{N\ell}|_{\mathbb{S}^2} & = \mathbf{0} \\ ||\mathbf{V}^{(N+2k)}_{N\ell}||_{\mathbf{H}^1_0} = ||\mathbf{B}^{(N+2k-1)}_{N\ell}||_{\mathbf{L}_2} = 4\pi\sqrt{\dfrac{N(N+1)}{2N+4k+1}} & \end{array}\right. \tag{39}$$

$$(40)\quad \begin{cases} \textit{harmonic vector fields}, N \in \mathbb{N},\ k=0 \\ \mathbf{W}_{N\ell}^{(N-1)} = \dfrac{\mathbf{B}_{N\ell}^{(N-1)}}{2N-1} = \dfrac{(N+1)\mathbf{H}_{N\ell}^{(N-1)}}{(2N-1)(2N+1)} \\ \mathrm{div}\mathbf{W}_{N\ell}^{(N-1)} = 0 \\ \mathrm{rot}\mathbf{W}_{N\ell}^{(N-1)} = \mathbf{0} \\ \mathbf{W}_{N\ell}^{(N-1)}|_{\mathbb{S}^2} = \dfrac{4\pi(N+1)\Big(N\mathbf{y}_{N\ell}^{(1)} + \mathbf{y}_{N\ell}^{(2)}\Big)}{(2N-1)(2N+1)} \end{cases}$$

$$(41)\quad \begin{cases} \textit{solenoidal vector fields},\ N \in \mathbb{N},\ k=1 \\ \mathbf{W}_{N\ell}^{(N+1)} = \dfrac{(2N+1)\mathbf{B}_{N\ell}^{(N+1)} - (N+1)\mathbf{H}_{N\ell}^{(N-1)}}{(2N+1)(2N+3)} \\ \mathrm{div}\mathbf{W}_{N\ell}^{(N+1)} = 0 \\ \mathrm{rot}\mathbf{W}_{N\ell}^{(N+1)} = \mathbf{C}_{N\ell}^{(N)} \\ \mathbf{W}_{N\ell}^{(N+1)}|_{\mathbb{S}^2} = \dfrac{4\pi N\Big(-(N+1)\mathbf{y}_{N\ell}^{(1)} + \mathbf{y}_{N\ell}^{(2)}\Big)}{(2N+1)(2N+3)} \end{cases}$$

$$(42)\quad \begin{cases} \textit{solenoidal vector fields},\ N \in \mathbb{N},\ k \in \mathbb{N}\backslash\{1\} \\ \mathbf{W}_{N\ell}^{(N+2k-1)} = \dfrac{\mathbf{B}_{N\ell}^{(N+2k-1)} - \mathbf{B}_{N\ell}^{(N+2k-3)}}{2N+4k-1} \\ \mathrm{div}\mathbf{W}_{N\ell}^{(N+2k-1)} = 0 \\ \mathrm{rot}\mathbf{W}_{N\ell}^{(N+2k-1)} = \mathbf{C}_{N\ell}^{(N+2k-2)} \\ \mathbf{W}_{N\ell}^{(N+2k-1)}|_{\mathbb{S}^2} = \mathbf{0} \\ ||\mathbf{W}_{N\ell}^{(N+2k-1)}||_{\mathbf{H}_0^1} = ||\mathbf{C}_{N\ell}^{(N+2k-2)}||_{\mathbf{L}_2} = 4\pi\sqrt{\dfrac{N(N+1)}{2N+4k-1}} \end{cases}$$

Proof. Formulas are derived by direct calculations. Let's make a few comments. Formulas for decompositions $\mathbf{U}_{N\ell}^{(N+2k-1)}$, $\mathbf{V}_{N\ell}^{(N+2k)}$ and $\mathbf{W}_{N\ell}^{(N+2k-1)}$ on the basis vector functions of the space $\mathbf{L}_2(\mathbb{B}^3)$ are a consequence of equality $P_{N+1}(t) = \frac{C_{N+1}^{(3/2)}(t) - C_{N-1}^{(3/2)}(t)}{2N+3}$.

Using the obtained formulas and boundary values for basis functions (20), (21), (22) and (23) on the sphere $\mathbb{S}^2$, we can immediately evaluate boundary values for $\mathbf{U}_{N\ell}^{(N+2k-1)}$, $\mathbf{V}_{N\ell}^{(N+2k)}$ and $\mathbf{W}_{N\ell}^{(N+2k-1)}$. Another way might be to directly calculating the integrals (31), (32) and (33) using vector Funk–Hecke formulas, see Theorem 4. Using the properties of vector functions (31), (32) and (31) it is easy to check their orthogonality in the space $\mathbf{H}_0^1(\mathbb{B}^3)$ and calculate the corresponding norms.

Vector functions $\mathbf{U}_{N\ell}^{(N+2k-1)}$ — potential vectr fields, because we can write

$$\mathbf{U}_{N\ell}^{(N+2k-1)}(\mathbf{x}) = \nabla \int_{\mathbb{S}^2} Y_{N\ell}(\boldsymbol{\xi})\mathbb{P}_{N+2k}^{[1]}(\mathbf{x}\cdot\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{S}^2} \mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\eta}P_{N+2k-1}(\mathbf{x}\cdot\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta},$$

where $\mathbb{P}_{N+2k}^{[1]}(t) = \int_1^t P_{N+2k-1}(s)\,\mathrm{d}s$.

Further we have

$$\begin{aligned}\mathrm{div}\mathbf{U}_{N\ell}^{(N+2k-1)} &= \Delta\int_{\mathbb{S}^2} Y_{N\ell}(\boldsymbol{\xi}))\mathbb{P}_{N+2k}^{[1]}(\mathbf{x}\boldsymbol{.}\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}\\ &= \int_{\mathbb{S}^2} Y_{N\ell}(\boldsymbol{\xi})C_{N+2k-2}^{(3/2)}(\mathbf{x}\boldsymbol{.}\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = Z_{N\ell}^{(N+2k-2)} \in L_2(\mathbb{B}^3).\end{aligned}$$

For $k = 0$ in (31) and (33), we obtain a harmonic field

$$\begin{aligned}\mathbf{U}_{N\ell}^{(N-1)} &= \frac{\nabla Z_{N\ell}^{[1](N)}}{2N-1} = \frac{\nabla Z_{N\ell}^{(N)}}{(2N-1)(2N+1)} = \frac{\mathbf{H}_{N\ell}^{(N-1)}}{(2N-1)(2N+1)}\\ \mathbf{W}_{N\ell}^{(N-1)} &= \frac{\mathbf{B}_{N\ell}^{(N-1)}}{2N-1} = \frac{(N+1)\mathbf{H}_{N\ell}^{(N-1)}}{(2N-1)(2N+1)}.\end{aligned}$$

We already know that vector harmonic polynomials can be obtained also according to different formulas, namely

$$\mathbf{A}_{N\ell}^{(N-1)} = \frac{\mathbf{H}_{N\ell}^{(N-1)}}{2N+1},\ \mathbf{B}_{N\ell}^{(N-1)}(\mathbf{x}) = \frac{N+1}{2N+1}\mathbf{H}_{N\ell}^{(N-1)}(\mathbf{x}).$$

□

To conclude this section, we also define a set of vector fields with zero traces on the boundary.

Definition 6. *Let $N \in \mathbb{N}$ then we define*

(43) $$\overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)}(\mathbf{x}) := N\mathbf{U}_{N\ell}^{(N+1)}(\mathbf{x}) + \mathbf{W}_{N\ell}^{(N+1)}(\mathbf{x}),\ \overset{0}{\mathbf{U}}{}_{00}^{(1)}(\mathbf{x}) := \mathbf{0}.$$

This definition is based on the formulas

$$\begin{aligned}\mathbf{U}_{N\ell}^{(N+1)}(\boldsymbol{\xi}) &= \frac{-4\pi}{(2N+1)(2N+3)}\Big(-(N+1)\mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\xi}) + \mathbf{y}_{N\ell}^{(2)}(\boldsymbol{\xi})\Big),\\ \mathbf{W}_{N\ell}^{(N+1)}(\boldsymbol{\xi}) &= \frac{4\pi N}{(2N+1)(2N+3)}\Big(-(N+1)\mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\xi}) + \mathbf{y}_{N\ell}^{(2)}(\boldsymbol{\xi})\Big).\end{aligned}$$

From these formulas it follows that there is for $\overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)}$

$$\overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)} = \frac{N\mathbf{A}_{N\ell}^{(N+1)} + \mathbf{B}_{N\ell}^{(N+1)} - \mathbf{H}_{N\ell}^{(N-1)}}{2N+3}.$$

As a result, we bring together the properties of $\overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)}$, $N \in \mathbb{N}$

(44) $$\begin{cases}\overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)} = \dfrac{N\mathbf{A}_{N\ell}^{(N+1)} + \mathbf{B}_{N\ell}^{(N+1)} - \mathbf{H}_{N\ell}^{(N-1)}}{2N+3}\\ \mathrm{div}\,\overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)} = NZ_{N\ell}^{(N)}\\ \mathrm{rot}\,\overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)} = \mathbf{C}_{N\ell}^{(N)}\\ \overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)}|_{\mathbb{S}^2} = \mathbf{0}\\ ||\,\overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)}||_{\mathbf{H}_0^1} = \sqrt{N(2N+1)}||Z_{N\ell}^{(N)}||_{L_2} = 4\pi\sqrt{\dfrac{N(2N+1)}{2N+3}}\,.\end{cases}$$

4. Orthogonal decomposition of the space $\mathbf{H}_0^1(\mathbb{B}^3)$

Recall that in a homogeneous space

$$\mathbf{H}_0^1(\mathbb{B}^3) = (H_0^1(\mathbb{B}^3))^3 = \{\mathbf{v} \in \mathbf{H}^1(\mathbb{B}^3) : \mathbf{v}(\boldsymbol{\xi}) = 0,\ \boldsymbol{\xi} \in \mathbb{B}^3\}$$

the inner product $((\cdot,\cdot))_{\mathbf{H}_0^1}$,

$$((\mathbf{u},\mathbf{w}))_{\mathbf{H}_0^1} = \int_{\mathbb{B}^3} \nabla\mathbf{u} : \nabla\mathbf{w}^*\, \mathrm{d}\mathbf{x} = (\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{w})_{\mathbf{L}_2} + (\operatorname{rot}\mathbf{u}, \operatorname{rot}\mathbf{w})_{\mathbf{L}_2}, \tag{45}$$

generates an energy norm in $\mathbf{H}_0^1(\mathbb{B}^3)$, equivalent to the main norm of the space $\mathbf{H}^1(\mathbb{B}^3)$.

Vector field system

$$\left\{\overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)}\right\}_{N\in\mathbb{N}} \bigcup \left\{\mathbf{U}_{N\ell}^{(N+2k+1)}\right\}_{N\in\mathbb{N}_0, k\in\mathbb{N}} \bigcup \left\{\mathbf{V}_{N\ell}^{(N+2k)}, \mathbf{W}_{N\ell}^{(N+2k+1)}\right\}_{N,k\in\mathbb{N}} \tag{46}$$

forms an orthogonal polynomial basis in homogeneous space $\mathbf{H}_0^1(\mathbb{B}^3)$. Orthogonality is understood in the sense of the inner product (45). We have

$$\mathbf{H}_0^1 = \overline{\operatorname{span}}\left\{\overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)}\right\}_{N\in\mathbb{N}}$$
$$\overset{\mathbf{H}_0^1}{\oplus}\ \overline{\operatorname{span}}\left\{\mathbf{U}_{N\ell}^{(N+2k+1)}\right\}_{N\in\mathbb{N}_0, k\in\mathbb{N}} \overset{\mathbf{H}_0^1}{\oplus}\ \overline{\operatorname{span}}\left\{\mathbf{V}_{N\ell}^{(N+2k)}, \mathbf{W}_{N\ell}^{(N+2k+1)}\right\}_{N,k\in\mathbb{N}}.$$

Next, consider the subspace $\mathbf{V} \subset \mathbf{H}_0^1(\mathbb{B}^3)$, containing solenoidal vector fields from $\mathbf{H}_0^1(\mathbb{B}^3)$,

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H}_0^1(\mathbb{B}^3) : \operatorname{div}\,\mathbf{v} = 0\} = \mathbf{H}_0(\operatorname{div} = 0) \cap \mathbf{H}_0^1(\mathbb{B}^3). \tag{47}$$

Then the orthogonal decomposition is valid (see [1],[2], [3], [11])

$$\mathbf{H}_0^1 = \mathbf{V} \overset{\mathbf{H}_0^1}{\oplus} \mathbf{V}^\perp. \tag{48}$$

Here $\mathbf{V}^\perp$ denotes the orthogonal complement of the subspace $\mathbf{V}$ in $\mathbf{H}_0^1$ with respect to the inner product (45).

There is a descriptive characteristic of the subspace (see [2]),

$$\mathbf{V}^\perp = \{(-\Delta_0)^{-1}\nabla q : q \in L_2(\mathbb{B}^3)\}$$

or

$$\mathbf{V}^\perp = \{\mathbf{v} \in \mathbf{H}_0^1 : -\Delta\mathbf{v} = \nabla q,\ q \in L_2(\mathbb{B}^3)/\mathbb{R}\}.$$

Remark. By the Schur complement (Schur operator) of the system (1) –(3) we mean the operator

$$S = \operatorname{div}\Delta_0^{-1}\nabla.$$

The pressure function p in (1) is definied up to a constant and satisaifies the equation $Sp = \operatorname{div}\Delta_0^{-1}\mathbf{f}$. Here we have designated $(-\Delta_0)^{-1}\mathbf{u}$ vector function $\mathbf{v}$ such that $(-\Delta)\mathbf{v} = \mathbf{u}$, $\mathbf{v}(\boldsymbol{\xi}) = \mathbf{0}$ and the divergence operator is an isomorphism of the spaces $\mathbf{V}^\perp$ and $L_2(\mathbb{B}^3)/\mathbb{R}$ (see e.g. [2, Chap. I, Corollary 2.4]), which corresponds to the accuracy of the de Rama complex

$$\mathbb{R} \xrightarrow{\text{id}} H^1(\mathbb{B}^3) \xrightarrow{\nabla} \nabla H^1(\mathbb{B}^3) \xrightarrow{\Delta^{-1}} \mathbf{H}_0^1(\mathbb{B}^3) \xrightarrow{\text{div}} L_2(\mathbb{B}^3)/\mathbb{R}. \tag{49}$$

The de Rama complex forms an exact sequence. This means that the image the left operator is the same as the kernel of the right one.

Theorem 8. *Polynomial basis (46) of the space $\mathbf{H}_0^1(\mathbb{B}^3)$ corresponds to the orthogonal decomposition (48), namely, vector field system*

$$\left\{\mathbf{V}_{N\ell}^{(N+2k)}, \mathbf{W}_{N\ell}^{(N+2k+1)}\right\}_{N,k\in\mathbb{N}}$$

forms an orthogonal polynomial basis in the subspace $\mathbf{V}$, and the system

$$\left\{\overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)}\right\}_{N\in\mathbb{N}} \bigcup \left\{\mathbf{U}_{N\ell}^{(N+2k+1)}\right\}_{N,k\in\mathbb{N}}$$

– orthogonal basis in the subspace $\mathbf{V}^{\perp}$.

Proof. We have by definition,

$$\mathbf{U}_{N\ell}^{(N+2k+1)}(\mathbf{x}) = \int_{\mathbb{S}^2} \mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\eta}) P_{N+2k+1}(\mathbf{x}\cdot\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta},$$

hence

$$\begin{aligned}\Delta\mathbf{U}_{N\ell}^{(N+2k+1)}(\mathbf{x}) &= 3\int_{\mathbb{S}^2} \mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\eta}) C_{N+2k-1}^{(5/2)}(\mathbf{x}\cdot\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta} \\ &= \nabla\int_{\mathbb{S}^2} Y_{N\ell}(\boldsymbol{\eta}) C_{N+2k}^{(3/2)}(\mathbf{x}\cdot\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta} = \nabla Z_{N\ell}^{(N+2k)}.\end{aligned}$$

It is easy to check that for $k \geq 1$ we have
i.e. $\Delta_0^{-1}\nabla Z_{N\ell}^{(N+2k)} = \mathbf{U}_{N\ell}^{(N+2k+1)}$ which means $\mathbf{U}_{N\ell}^{(N+2k+1)} \in \mathbf{V}^{\perp}$.

Further, by definition, we have $\overset{0}{\mathbf{U}}{}_{00}^{(1)} = \mathbf{0}$, and for $N \geq 1$ we have

$$\begin{aligned}&\Delta\,\overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)} = (\nabla\,\mathrm{div} - \mathrm{rot}\,\mathrm{rot})\left(\mathbf{W}_{N\ell}^{(N+1)}(\mathbf{x}) + N\mathbf{U}_{N\ell}^{(N+1)}(\mathbf{x})\right) \\ &= -\,\mathrm{rot}\,\mathrm{rot}\,\mathbf{W}_{N\ell}^{(N+1)}(\mathbf{x}) + N\nabla\,\mathrm{div}\,\mathbf{U}_{N\ell}^{(N+1)}(\mathbf{x}) = -\,\mathrm{rot}\,\mathbf{C}_{N\ell}^{(N)}(\mathbf{x}) + N\nabla Z_{N\ell}^{(N)}(\mathbf{x}).\end{aligned}$$

Since $\mathbf{C}_{N\ell}^{(N)}(\mathbf{x}) = -\mathrm{rot}(\mathbf{x}Z_{N\ell}^{(N)}(\mathbf{x})) = \mathbf{x}\times\nabla Z_{N\ell}^{(N)}(\mathbf{x})$, then

$$\begin{aligned}&-\,\mathrm{rot}\mathbf{C}_{N\ell}^{(N)}(\mathbf{x}) = \mathrm{rot}\,\mathrm{rot}\left(\mathbf{x}Z_{N\ell}^{(N)}(\mathbf{x})\right) = (2N+1)\mathrm{rot}\,\mathrm{rot}\left(\mathbf{x}Z_{N\ell}^{[1](N)}(\mathbf{x})\right) \\ &= (2N+1)\mathbf{B}_{N\ell}^{(N-1)}(\mathbf{x}) = (2N+1)\frac{N+1}{2N+1}\nabla Z_{N\ell}^{(N)}(\mathbf{x}) = (N+1)\nabla Z_{N\ell}^{(N)}(\mathbf{x}).\end{aligned}$$

As a result, we have $\Delta\,\overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)} = (2N+1)\nabla Z_{N\ell}^{(N)}$, $Z_{N\ell}^{(N)} \in L_2(\mathbb{B}^3)/\mathbb{R}$, because $N \geq 1$. Hence $\overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)} \in \mathbf{V}^{\perp}$. □

3. Stokes system

Consider the boundary value problem (1)– (3) in a weak (variational) formulation (see e.g. [1, p. 26]): find a pair of functions $\mathbf{u} \in \mathbf{H}_0^1(\mathbb{B}^3)$ and $p \in L_2(\mathbb{B}^3)/\mathbb{R}$,

$$\int_{\mathbb{B}^3} \nabla\mathbf{u} : \nabla\mathbf{v}\mathrm{d}\mathbf{x} - \int_{\mathbb{B}^3} p(\mathbf{x})\mathrm{div}\,\mathbf{v}\,\mathrm{d}\mathbf{x} = \int_{\mathbb{B}^3} \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}\ \forall\ \mathbf{v} \in \mathbf{H}_0^1(\mathbb{B}^3)$$

or

$$((\mathbf{u},\mathbf{v}))_{\mathbf{H}_0^1} - (p, \mathrm{div}\ \mathbf{v})_{\mathbf{L}_2} = (\mathbf{f},\mathbf{v})_{\mathbf{L}_2}\ \forall\ \mathbf{v} \in \mathbf{H}_0^1(\mathbb{B}^3). \tag{50}$$

Theorem 9. *Assume that the right–hand side* $\mathbf{f} \in \mathbf{L}_2(\mathbb{B}^3)$ *and has series an expansion (30) in basis vector functions. Then the weak solution of the Stokes problem, the velocity vector* $\mathbf{u}$ *and the pressure* p*, have the forms*

$$\mathbf{u} = \sum_{N=1}^{\infty}\sum_{k=1}^{\infty} u_{N\ell}^{(N+2k)V}\mathbf{V}_{N\ell}^{(N+2k)} + \sum_{N=1}^{\infty}\sum_{k=1}^{\infty} u_{N\ell}^{(N+2k+1)W}\mathbf{W}_{N\ell}^{(N+2k+1)} \in \mathbf{H}_0^1(\mathbb{B}^3), \tag{51}$$

$$p = \sum_{N=1}^{\infty} p_{N\ell}^{(N)} Z_{N\ell}^{(N)} + \sum_{N=0}^{\infty}\sum_{k=1}^{\infty} p_{N\ell}^{(N+2k)} Z_{N\ell}^{(N+2k)} \in L_2(\mathbb{B}^3)/\mathbb{R}, \tag{52}$$

where the coefficients are calculated by the formulas

$$u_{N\ell}^{(N+2k)V} = \frac{\left(\mathbf{f}, \mathbf{V}_{N\ell}^{(N+2k)}\right)_{\mathbf{L}_2}}{||\mathbf{V}_{N\ell}^{(N+2k)}||^2_{\mathbf{H}_0^1}}, \quad u_{N\ell}^{(N+2k+1)W} = \frac{\left(\mathbf{f}, \mathbf{W}_{N\ell}^{(N+2k+1)}\right)_{\mathbf{L}_2}}{||\mathbf{W}_{N\ell}^{(N+2k+1)}||^2_{\mathbf{H}_0^1}},$$

$$p_{N\ell}^{(N)} = \frac{-\left(\mathbf{f}, \overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)}\right)_{\mathbf{L}_2}}{N||Z_{N\ell}^{(N)}||^2_{L_2}}, \quad p_{N\ell}^{(N+2k)} = \frac{-\left(\mathbf{f}, \mathbf{U}_{N\ell}^{(N+2k+1)}\right)_{\mathbf{L}_2}}{||Z_{N\ell}^{(N+2k)}||^2_{L_2}}.$$

Proof. For solutions problem in a weak setting, it is suffices to require the equality (50) only on the basic functions of the space $\mathbf{H}_0^1(\mathbb{B}^3)$.

Due to the orthogonal decomposition $\mathbf{H}_0^1(\mathbb{B}^3) = \mathbf{V} \oplus \mathbf{V}^{\perp}$ we can first put $\mathbf{v} \in \mathbf{V}$, that is,

$$\mathbf{v} \in \left\{\mathbf{V}_{N\ell}^{(N+2k)}, \mathbf{W}_{N\ell}^{(N+2k+1)}\right\}_{N,k\in\mathbb{N}}.$$

Then we obtain the equalities $((\mathbf{u},\mathbf{v}))_{\mathbf{H}_0^1(\mathbb{B}^3)} = (\mathbf{f},\mathbf{v})_{\mathbf{L}_2(\mathbb{B}^3)}$ and, therefore, the solution will have the form (51). To find the second function, consider the rest of the basis $\mathbf{H}_0^1(\mathbb{B}^3)$, we can assume that

$$\mathbf{v} \in \left\{\overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)}, \mathbf{U}_{N\ell}^{(N+2k+1)}\right\}_{N\in\mathbb{N}_0,k\in\mathbb{N}}.$$

In this case, from (50) we have the equalities $(p, \operatorname{div}\, \mathbf{v})_{L_2(\mathbb{B}^3)} = -(\mathbf{f},\mathbf{v})_{\mathbf{L}_2(\mathbb{B}^3)}$, which determine the pressure p by the formula (52) and this completes the proof. □

In proving above theorem, we saw that the use of the orthogonal basis when solving the Stokes problem in a weak setting, it leads to a separate determination of the velocity and pressure. Another approach for excluding the pressure field from the system may consist of applying the operator rotor.

Below we will show that the problem (1) — (3) can be split into two independent subproblems

1) $\nabla p = \mathbf{f}_1$ in $\mathbb{B}^3$,

2) $\operatorname{rot}\operatorname{rot}\mathbf{u} = -\Delta\mathbf{u} = \mathbf{f}_2$ in $\mathbb{B}^3$, $\mathbf{u} = 0$ on $\mathbb{S}^2 = \partial\mathbb{B}^3$,

где $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$ — a special decomposition of the right–hand side in (1).

Hint: since $\operatorname{div}\mathbf{u} = 0$, then $-\Delta\mathbf{u} = \operatorname{rot}\operatorname{rot}\mathbf{u}$.

For this purpose, we consider the influence of the solenoidal part of $\mathbf{f}$, i.e. $\overset{\circ}{\mathbf{J}}$, to determine the pressure function p. It is clear that this influence will be felt only in the coefficients $p_{N\ell}^{(N)}$. We have

$$p_{N\ell}^{(N)} = \frac{-\left(\mathbf{f}, \overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)}\right)_{\mathbf{L}_2}}{N||Z_{N\ell}^{(N)}||_{L_2}^2} = \frac{-\left(\overset{\circ}{\mathbf{J}}, \mathbf{B}_{N\ell}^{(N+1)}\right)_{\mathbf{L}_2}}{16\pi^2 N} + \ ... \ = \frac{-||\mathbf{B}_{N\ell}^{(N+1)}||_{\mathbf{L}_2}^2}{16\pi^2 N} f_{N\ell}^{(N+1)B} + ...$$
$$= -\frac{N+1}{2N+5} f_{N\ell}^{(N+1)B} + ... \, .$$

Here we used the formula (44)

$$\overset{0}{\mathbf{U}}{}_{N\ell}^{(N+1)} = \frac{N\mathbf{A}_{N\ell}^{(N+1)} + \mathbf{B}_{N\ell}^{(N+1)} - \mathbf{H}_{N\ell}^{(N-1)}}{2N+3}$$

and norms $||Z_{N\ell}^{(N)}||_{L_2}^2 = \dfrac{16\pi^2}{2N+3}$, $||\mathbf{B}_{N\ell}^{(N+1)}||_{\mathbf{L}_2}^2 = \dfrac{16\pi^2 N(N+1)}{2N+5}$.

Then the expansion for p has the form

$$p(\mathbf{x}) = -\sum_{N=1}^{\infty} \frac{N+1}{2N+5} f_{N\ell}^{(N+1)B} Z_{N\ell}^{(N)}(\mathbf{x})$$
$$+ \Big(\text{this part is determined only by the potential part of } \ \mathbf{f}\Big).$$

Thus, there is an equality for the pressure gradient

$$\nabla p(\mathbf{x}) = \mathbf{f}_1 := -\sum_{N=1}^{\infty} \frac{N+1}{2N+5} f_{N\ell}^{(N+1)B} \nabla Z_{N\ell}^{(N)}(\mathbf{x}) + \underbrace{\sum_{N=0}^{\infty}\sum_{k=0}^{\infty} f_{N\ell}^{(N+2k-1)A} \mathbf{A}_{N\ell}^{(N+2k-1)}}_{\text{potential part of } \mathbf{f}}$$
$$= -\sum_{N=1}^{\infty} \frac{2N+1}{2N+5} f_{N\ell}^{(N+1)B} \mathbf{B}_{N\ell}^{(N-1)}(\mathbf{x}) + \sum_{N=0}^{\infty}\sum_{k=0}^{\infty} f_{N\ell}^{(N+2k-1)A} \mathbf{A}_{N\ell}^{(N+2k-1)}.$$

As a result, we obtain a complete decomposition of pressure

$$p(\mathbf{x}) = -\sum_{N=1}^{\infty} \frac{2N+1}{2N+5} f_{N\ell}^{(N+1)B} Z_{N\ell}^{(N)}(\mathbf{x}) + \sum_{N=0}^{\infty}\sum_{k=0}^{\infty} f_{N\ell}^{(N+2k-1)A} Z_{N\ell}^{[1](N+2k)}. \tag{53}$$

For the velocity vector $\mathbf{u}$, Laplace equation will be fulfilled

$$-\Delta \mathbf{u} = \mathbf{f}_2 := \sum_{N=1}^{\infty} \frac{2N+1}{2N+5} f_{N\ell}^{(N+1)B} \mathbf{B}_{N\ell}^{(N-1)}(\mathbf{x}) + \overset{\circ}{\mathbf{J}}$$
$$= \sum_{N=1}^{\infty} f_{N\ell}^{(N+1)B} \left(\mathbf{B}_{N\ell}^{(N+1)}(\mathbf{x}) + \frac{2N+1}{2N+5} \mathbf{B}_{N\ell}^{(N-1)}(\mathbf{x})\right)$$
$$+ \sum_{N=1}^{\infty} \sum_{\boxed{k=1}}^{\infty} f_{N\ell}^{(N+2k+1)B} \mathbf{B}_{N\ell}^{(N+2k+1)} + \sum_{N=1}^{\infty}\sum_{k=0}^{\infty} f_{N\ell}^{(N+2k)C} \mathbf{C}_{N\ell}^{(N+2k)},$$

and we have a formal solution

$$\mathbf{u} = -\sum_{N=1}^{\infty} f_{N\ell}^{(N+1)B} \frac{\mathbf{W}_{N\ell}^{(N+3)}}{2N+5} - \sum_{N=1}^{\infty}\sum_{k=1}^{\infty} f_{N\ell}^{(N+2k+1)B} \frac{\mathbf{W}_{N\ell}^{(N+2k+3)} - \mathbf{W}_{N\ell}^{(N+2k+1)}}{2N+4k+5}$$

$$+ \sum_{N=1}^{\infty} f_{N\ell}^{(N)C} \frac{\mathbf{V}_{N\ell}^{(N+2)}}{2N+3} + \sum_{N=1}^{\infty}\sum_{k=1}^{\infty} f_{N\ell}^{(N+2k)C} \frac{\mathbf{V}_{N\ell}^{(N+2k+2)} - \mathbf{V}_{N\ell}^{(N+2k)}}{2N+4k+3}. \tag{54}$$

We verify this by direct calculations,

$$\begin{aligned}
\Delta\mathbf{u} &= \operatorname{rot}\operatorname{rot} \sum_{N=1}^{\infty} f_{N\ell}^{(N+1)B} \frac{\mathbf{W}_{N\ell}^{(N+3)}}{2N+5} \\
&\quad + \operatorname{rot}\operatorname{rot} \sum_{N=1}^{\infty}\sum_{k=1}^{\infty} f_{N\ell}^{(N+2k+1)B} \frac{\mathbf{W}_{N\ell}^{(N+2k+3)} - \mathbf{W}_{N\ell}^{(N+2k+1)}}{2N+4k+5} \\
&\quad - \operatorname{rot}\operatorname{rot} \sum_{N=1}^{\infty} f_{N\ell}^{(N)C} \frac{\mathbf{V}_{N\ell}^{(N+2)}}{2N+3} - \operatorname{rot}\operatorname{rot} \sum_{N=1}^{\infty}\sum_{k=1}^{\infty} f_{N\ell}^{(N+2k)C} \frac{\mathbf{V}_{N\ell}^{(N+2k+2)} - \mathbf{V}_{N\ell}^{(N+2k)}}{2N+4k+3}. \\
&= \operatorname{rot} \sum_{N=1}^{\infty} f_{N\ell}^{(N+1)B} \frac{\mathbf{C}_{N\ell}^{(N+2)}}{2N+5} + \operatorname{rot} \sum_{N=1}^{\infty}\sum_{k=1}^{\infty} f_{N\ell}^{(N+2k+1)B} \frac{\mathbf{C}_{N\ell}^{(N+2k+2)} - \mathbf{C}_{N\ell}^{(N+2k)}}{2N+4k+5} \\
&\quad - \operatorname{rot} \sum_{N=1}^{\infty} f_{N\ell}^{(N)C} \frac{\mathbf{B}_{N\ell}^{(N+1)}}{2N+3} - \operatorname{rot} \sum_{N=1}^{\infty}\sum_{k=1}^{\infty} f_{N\ell}^{(N+2k)C} \frac{\mathbf{B}_{N\ell}^{(N+2k+1)} - \mathbf{B}_{N\ell}^{(N+2k-2)}}{2N+4k+3} \\
&= -\sum_{N=1}^{\infty} f_{N\ell}^{(N+1)B} \frac{(2N+1)\mathbf{B}_{N\ell}^{(N-1)} + (2N+5)\mathbf{B}_{N\ell}^{(N+1)}}{2N+5} \\
&\quad - \operatorname{rot} \sum_{N=1}^{\infty}\sum_{k=1}^{\infty} f_{N\ell}^{(N+2k+1)B} \mathbf{V}_{N\ell}^{(N+2k+2)} \\
&\quad - \sum_{N=1}^{\infty} f_{N\ell}^{(N)C} \mathbf{C}_{N\ell}^{(N)} - \operatorname{rot} \sum_{N=1}^{\infty}\sum_{k=1}^{\infty} f_{N\ell}^{(N+2k)C} \mathbf{W}_{N\ell}^{(N+2k+1)} \\
&= -\sum_{N=1}^{\infty} f_{N\ell}^{(N+1)B} \frac{(2N+1)\mathbf{B}_{N\ell}^{(N-1)}}{2N+5} - \sum_{N=1}^{\infty}\sum_{k=0}^{\infty} f_{N\ell}^{(N+2k+1)B} \mathbf{B}_{N\ell}^{(N+2k+1)} \\
&\quad - \sum_{N=1}^{\infty}\sum_{k=0}^{\infty} f_{N\ell}^{(N+2k)C} \mathbf{C}_{N\ell}^{(N+2k)} = -\mathbf{f}_2.
\end{aligned}$$

In the above formal calculations were used the properties of vector polynomial functions form Theorm 7. Thus, knowing the decomposition of the right–hand side $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$, Stokes problem can be split into two independent subproblems for which we have given formal solution (53) and (54).

3. Conclusion

It is constructed polynomial vector function which form an orthogonal basis in $\mathbf{H}_0^1(\mathbb{B}^3)$. After the result is established, we solve system of Stokes equations in the ball. The constructed polynomial vector functions also can be used when solving some problems of vector analysis related with operators ∇, **div** and **rot**.

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