S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

*Том 12, стр. 144–144 (2020)* УДК 512.545
DOI 10.17377/semi.2015.12.xxx MSC 06F15,20F60

# ON CONVEX SUBGROUPS OF CARTESIAN PRODUCT OF $m$-GROUPS

A.V. ZENKOV

Abstract. We prove that every convex $m$-subgroup of a Cartesian product of $m$-groups that admits a faithful transitive presentation is an $m$-subgroup of a suitable projection. As a consequence, we obtain that the Cartesian product of $m$-groups does not admits a faithful transitive presentation.

**Keywords:** $m$-group, cartesian product,transitive presentation of $m$-group.

## 1. Introduction

An $m$-group is an algebraic system $G$ of a signature $m = \langle \cdot, e, ^{-1}, \vee, \wedge, \, _* \rangle$, where $\langle G, \cdot, e, ^{-1}, \vee, \wedge \rangle$ is a lattice-orderd group ($\ell$-group) and the unary operation $_*$ can be interpreted as a second-order automorphism of the group $\langle G, \cdot, e, ^{-1} \rangle$ and the antiisomorphism of the lattice $\langle G, \vee, \wedge \rangle$, i.e. for any $x, y \in G$ the folloving relations are true

$$(xy)_* = x_* y_*, (x_*)_* = x, (x \vee y)_* = x_* \wedge y_*, (x \wedge y)_* = x_* \vee y_*.$$

From now one we will denote $m$-group $G$ with the marked automorphism $\varphi$ as a pair $(G, \varphi)$. Let $(G, \varphi)$ be an $m$-group and $H$ an $\ell$-subgroup of $G$. Then $H$ is called an $m$-subgroup of $(G, \varphi)$ if it stable under $\varphi$. A normal convex $m$-subgroup of $(G, \varphi)$ is called an $m$-ideal of $(G, \varphi)$. The kernels of homomorphisms of $m$-groups are exactly all $m$-ideals. (An $m$-homomorphism is any $\ell$-homomorphism that also respects $\varphi$.)

The concept of an $m$-group as an algebraic system was explicitly formulated by M.Giraudet and J. Rachunek [1]. The introduction of the concept of an $m$-group

Zenkov A.V. On convex subgroups of cartesian product of $m$-groups.



*Поступила 1 октября 2021 г., опубликована 3 февраля 2022 г.*

as an algebraic system allows us to apply the methods of universal algebra to the study of monotonic permutations groups of totally ordered sets. In particular, it has become possible to write down the properties of such groups in the language of identities, which necessarily leads to the creation of a theory of varieties of $m$-groups.

We recall the basic concepts of the representation theory of $m$-groups by order-preserving permutations of totally ordered sets (or chains, for short). Let $\Omega$ be a chain and $a$ be a reversible automorphism of the second-order of $\Omega$. That is, $((\omega)a)a = \omega$ and $\omega < \omega' \Leftrightarrow (\omega)a > (\omega')a$ for all $\omega, \omega' \in \Omega$. Denote by $Aut(\Omega)$ the group (under composition) of all order-preserving permutations of $\Omega$. It is well known that with respect to the pointwise order $Aut(\Omega)$ is an $\ell$-group. The $\ell$-group $Aut(\Omega)$ can be turned into an $m$-group if the operation $_*$ is defined on it by the rule $g_* = aga$, where $g \in Aut(\Omega)$. By a *faithful* presentation of an $m$-group $(G, \varphi)$ by order-preserving permutations of $\Omega$ we mean the $m$-isomorphism $\nu : G \to Aut(\Omega)$. We write this as $(G, \Omega, a)$.

The presentation $(G, \Omega, a)$ of an $m$-group $(G, \varphi)$ is *transitive* if for all $\omega, \omega' \in \Omega$ there is an element $g \in G$ such that $(\omega)g = (\omega')a^{\varepsilon}$, where $\varepsilon = 0$ or $\varepsilon = 1$. Importance transitive presentations of $m$-groups in the study of varieties $m$-groups is explained by the fact that every subdirect $m$-indecomposable $m$-group $(G, \varphi)$ has a faithful transitive presentation [2]. Therefore, every variety of $m$-groups is generated by its groups which admit transitive presentation.

On the other hand, if a variety of $m$-groups is genereted by a certain class of $m$-groups, then every group of this variety according to Birkhoff's theorem is a homomorphic image of an $m$-subgroup of the Cartesian product of $m$-groups of this class. Therefore it is necessary to know the description of those Cartesian product subgroups that admit a faithful transitive presentation. We prove that every convex $m$-subgroup of a Cartesian product of $m$-groups that admits a faithful transitive presentation is an $m$-subgroup of a suitable projection (Theorm 1). As a consequence, we obtain that the Cartesian product of $m$-groups does not admits a faithful transitive presentation (Corollary 1).

All the concepts of the theory of $\ell$-group and terminology used here basically correspond to the book [3].

## 2. Main result

The subgroup $H$ of an $\ell$-group $G$ is its convex $\ell$-subgroup if: 1) for any $a, b \in H$ their union $a \vee b \in H$ ($H - \ell$-subgroup), 2) $H$ is a convex subset of $G$, i.e. for any $g \in G$ the inequality $a \leq g \leq b$ implies $g \in H$. So as the set-theoretic intersection of any number of convex $\ell$-subgroups is a convex $\ell$-subgroup, then the lattice structure is defined in a standard way on the set of all convex $\ell$-subgroups of $\ell$-group $G$. This lattice is distributive. By the Birkhoff-Lorentz theorem (see [3, Ch. 3,§1, Theorem 1]), the union of $A \bigvee B$ convex $\ell$-subgroups of $A$ and $B$ coincide with the subgroup generated by them.

Let $H$ be a convex $\ell$-subgroup of an $\ell$-group $G$. Denote by $R(G : H)$ the set of right cosets. On $R(G : H)$, define $Hx \leq Hy$ if there is an element $h \in H$ such that $hx \leq y$. Then $\leq$ is a partial order of $R(G : H)$. If this order is totally, then $H$ is called *prime*. It is well known (see [3, Ch. 3,§3, Theorem 1]) that the convex $\ell$-subgroup $H$ of an $\ell$-group $G$ is prime if and only if,for all positive $a, b \in G \setminus H$,

$a \wedge b \notin H$ is true. Thus, for any $a \in G \setminus H$, its polar $a^{\perp} = \{g \in G \mid g \perp a\}$ is contained in $H$ ($g \perp a$ means that these elements are orthogonal, i.e. $|a| \wedge |g| = e$).

The prime $\ell$-subgroup $H$ of an $m$-group $(G, \varphi)$ is called *representing* if it does not contain non-identety $m$ - ideals. The following statement is proved in [4].

**Lemma 1.** *The $m$-group $(G, \varphi)$ admits a faithful transitive presentation if and only if it contains a representing $\ell$-subgroup.*

It is clear that $H = e$ is the representing $\ell$-subgroup of an $m$-group $(G, \varphi)$, then $G$ is an $o$-group (linearly ordered group). In this case, as noted in [5], $(G, \varphi)$ belongs to the variety $\mathcal{I}$, defined by the identity $xx_* = e$.

Let $\{(F_j, \varphi_j) \mid j \in J\}$ be a collection of $m$-groups of the cardinality $|J| > 1$. We denote by $\overline{F} = \prod_{j\in J} F_j$ the Cartesian product of *$\ell$-groups* $F_j$. Element $f \in \overline{F}$ will be written as $f = (\dots f_i \dots f_j \dots f_k \dots)$. Now, we define the mapping $\varphi : \overline{F} \to \overline{F}$ by the rule $(f)\varphi = (\dots (f_i)\varphi_i \dots (f_j)\varphi_j \dots (f_k)\varphi_k \dots)$. It is not hard to see that $\varphi$ is a second-order automorphism of the group $\overline{F}$ and the antiisomorphism of the lattice $\overline{F}$. Then $(\overline{F}, \varphi) = \prod_{j\in J}(F_j, \varphi_j)$ is an $m$-group wich is called *the Cartesian product* of $m$-groups $(F_j, \varphi_j)$.

For $f \in \overline{F}$ and $j \in J$ we denote the $j$-projection of $f$ as $\widehat{f_j}$. So that $\widehat{f_j} = (\dots e \dots f_j \dots e \dots)$ and then $\overline{f}_j = f\widehat{f}_j^{-1} = (\dots f_i \dots e \dots f_k \dots)$. It follows from definition of above elements that $f = \widehat{f_j}\overline{f}_j$ and $\widehat{f_j} \perp \overline{f}_j$. It is clear that $\widehat{F_j} = \{\widehat{f_j} \mid f \in \overline{F}\}$ and $\overline{F}_j = \{\overline{f}_j \mid f \in \overline{F}\}$ are $m$-ideals of $\overline{F}$; moreover $\overline{F} = \widehat{F_j} \times \overline{F}_j = \widehat{F_j} \vee \overline{F}_j$. As usual, we identify $F_j$ and $\widehat{F_j}$ everywhere because of their $m$-isomorphism.

It follows from the above if $H$ is a prime $\ell$-subgroup of $(\overline{F}, \varphi)$, then for each $j \in J$ only one of the following conditions is met: a) $\widehat{F_j} \subsetneqq H$ and then $\overline{F}_j \subseteq H$ or b) $\overline{F}_j \subsetneqq H$ and then $\widehat{F_j} \subseteq H$. Obviously that is impossible the folloving $\overline{F}_j, \widehat{F_j} \subseteqq H$.

Let $(G, \varphi)$ be an arbitrary non-identity convex $m$-subgroup of $(\overline{F}, \varphi)$. Suppose that $(G, \varphi)$ admits a faithful transitive presentation. By Lemma 1 in $(G, \varphi)$ there is a representing $\ell$-subgroup $V$. Then in $(\overline{F}, \varphi)$ there is a prime $\ell$-subgroup $H$ such that

$$V = H \cap G. \qquad (*)$$

The proof of this fact is contained, for example, in [6, Preposition 12.11].

Suppose that $\widehat{F_j} \subseteq H$ for any $j \in J$. Then $V \cap \widehat{F_j} = H \cap \widehat{F_j} \cap G = \widehat{F_j} \cap G$. Clearly that $\widehat{F_j} \cap G$ is an $m$-ideal of $G$ wich is contained in $V$. Therefore $\widehat{F_j} \cap G = e$. In view of this, by the homomorphism theorem, we obtain $G\widehat{F_j} / \widehat{F_j} \cong G / G \cap \widehat{F_j} = G \leq \overline{F} / \widehat{F_j} \cong \overline{F}_j$. Then $G \subseteq \bigcap_{j\in J} \overline{F}_j$. But $\bigcap_{j\in J} \overline{F}_j = e$ and we come to a contradiction.

Therefore, there exsit $j \in J$ such that $\overline{F}_j \subseteq H$. As above, we can assume $\overline{F}_j \cap G = e$. Hence $G\overline{F}_j / \overline{F}_j \cong G / G \cap \overline{F}_j = G \leq \overline{F} / \overline{F}_j \cong \widehat{F_j}$. Thus we proved.

**Theorem 1.** *Every convex an $m$-subgroup $(G, \varphi)$ of the $m$-group $(\overline{F}, \varphi)$ that admits a faithful transitive presentation is a convex $m$-subgroup for a suitable $j$-projection $\widehat{F_j}$.*

The following assertion is true

**Corollary 1.** *An $m$-cartesian product $(\overline{F}, \varphi) = \prod_{j\in J}(F_j, \varphi_j)$, where $|J| > 1$, does not admits a faithful transitive presentation.*

Proof. Indeed, if $(\overline{F}, \varphi)$ admits a faithful transitive presentation, then by Theorem 1 it is isomorphic to its component, which is impossible.□

## References

[1] Giraudet M.,Rachunek J., Varieties of half lattice-ordered groups of monotonic permutations of chains,Czech.Math.J.,49:124(1999),743-766.

[2] A.V. Zenkov, On $m$-transitive groups, Math.Notes,94:1(2013),157-159.

[3] V.M. Kopytov, Lattice-ordered groups,Nauka,Moscow,1984.

[4] S.V. Varaksin, A.V. Zenkov, On presentations of $m$-groups,Sib.Math.J.,54:2(2013),227-230.

[5] N.V. Bayanova, O.V. Nikonova, Reversing automorphisms of lattice-ordered groups,Sib.Math.J.,36:4(1995),763-768.

[6] Darnel M. Theory of lattice-ordered groups. New York:Marcel Dekker,1995.

Alexey Vladimirovich Zenkov
Altai State Agricultural University,
pr. Krasnoarmeysky, 98,
656049, Barnaul, Russia
*E-mail address*: alexey_zenkov@yahoo.com