The sum of orders of elements in nonabelian groups of odd order

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Abstract

Denote by $\psi(G)$ the sum of the orders of the elements of a finite group G. We obtain an exact upper bound for $\psi(G)$ on the set of nonabelian groups of given odd order n in terms of the minimal prime divisor of n. We also describe the finite groups on which this bound is achieved.

Keywords: orders of elements, solvable groups.

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1. Introduction

Denote by $\psi(G)$ the sum of the orders of the elements of a finite group G. In [1] the authors show that the maximum of $\psi(G)$ on the set of groups of the given order is attained at a cyclic group. This maximum is strict, that is all other groups of this order have a smaller value of ψ. Using the function $\psi(G)$, it is possible to formulate sufficient conditions for a finite group to be cyclic, abelian, nilpotent, solvable, or supersolvable [2, 3]. The paper [3] gives an upper bound for the values of $\psi(G)$ on the set of nonabelian groups G of even order. The paper [4] contains a similar result for the non-cyclic q^*-groups.

Definition 1. *A finite group G is a q^*-group if q is the smallest prime divisor of the order of G.*

The main result of [4] states that if G is a finite non-cyclic q^*-group of order n and C_n is a cyclic group of order n, then

$$\psi(G) \leqslant \frac{(q^3-q+1)(q+1)}{q^5+1}\psi(C_n).$$

For a positive integer m and a prime q, put

$$M_{q^{m+1}} = \langle a, b | a^{q^m} = b^q = 1, a^b = a^{1+q^{m-1}} \rangle. \tag{1}$$

The main result of this paper is the following statement.

Theorem. *Let q be an odd prime and let G be a nonabelian q^*-group of order n. Then*

$$\psi(G) \leqslant \frac{q^6+q^3-q^2+1}{q^7+1}\psi(C_n)$$

with the equality if and only if G is a direct product of M_{q^3} and a cyclic group of order coprime to q.

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2. Preliminaries

The following lemma contains some useful facts about the function ψ.

Lemma 1. (1) *([5, Lemma 2.2]) If A, B are finite groups of coprime orders, then $\psi(A \times B) = \psi(A)\psi(B)$.*

(2) *([1, Corollary B]) If P is a normal cyclic Sylow subgroup of a finite group G, then $\psi(G) \leqslant \psi(P)\psi(G/P)$, with equality if and only if P is central in G.*

(3) *([5, Lemma 2.9]) If n is a natural number, p and q are the largest and smallest prime divisors of n, then*

$$\psi(C_n) > \frac{q}{p+1}n^2.$$

(4) *([5, Lemma 2.9]) If p is a prime and m is a non-negative integer, then*

$$\psi(C_{p^m}) = \frac{p^{2m+1}+1}{p+1}.$$

(5) *([5, Lemma 2.2]) Let p be a prime. Assume that G is a semidirect product of a normal cyclic p-subgroup C and a nontrivial subgroup F whose order is coprime to p. Put $Z = C_F(C)$. Then*

$$\psi(G) = \psi(C)\psi(Z) + |C|\left(\psi(F) - \psi(Z)\right).$$

Observe that the Item (1) of Lemma 1 is quite obvious and follows for example from [5, Lemma 2.2, Item (3)]. Item (3) is proved in [5, Lemma 2.9] only for $q = 2$, but the proof for the general case is the same (also, though the authors prove strict inequality, the statement of the lemma contains non-strict).

We will need a following generalization of Item (2) of the previous lemma.

Lemma 2. *If H is a normal abelian Hall subgroup of a finite group H, then $\psi(G) \leqslant \psi(H)\psi(G/H)$.*

Proof. By the Schur–Zassenhaus theorem, $G \simeq H \rtimes (G/H)$. It follows from Item (1) of Lemma 1 that it is sufficient to prove that $\psi(H \rtimes (G/H) \leqslant \psi(H \times (G/H))$. Choose $h \in H$ and $g \in G/H$ and denote by gh and $g \cdot h$ their products as elements of semidirect and direct products respectively. Since H is a Hall subgroup, $|g \cdot h| = |g||h|$. Now

$$(gh)^{|g|} = h^{g^{|g|-1}} h^{g^{|g|-2}} \dots h^g h.$$

So $|gh|$ divides $|g \cdot h|$, and the lemma is proved.

Lemma 3. *Let G be a finite group and suppose that there exists $x \in G$ such that $|G : \langle x \rangle| < 2p$, where p is the maximal prime divisor of $|G|$. Then one of the following holds:*

(1) *G has a normal cyclic Sylow p-subgroup;*

(2) *G is solvable and $\langle x \rangle$ is a maximal subgroup of G of index either p or $p + 1$.*

In the following lemma, we calculate $\phi(G)$ for two infinite series of finite groups.

Lemma 4. *If q is a prime and m is a positive integer, then*

$$\psi(M_{q^{m+1}}) = \psi(C_{q^m} \times C_q) = \frac{q^{2m+2} + q^3 - q^2 + 1}{q+1}.$$

In particular,

$$\psi(M_{q^3}) = \frac{q^6 + q^3 - q^2 + 1}{q^7+1}\psi(C_{q^3}).$$

Proof. It is quite easy to see, that

$$\psi(C_{q^m} \times C_q) = (\psi(C_{q^m}) - 1)q + q(q-1) + 1 = q\frac{q^{2m+1} - q}{q+1} + q^2 - q + 1 =$$
$$= \frac{q^{2m+2} + q^3 - q^2 + 1}{q+1}$$

as stated.

Denote $M_{q^{m+1}}$ by G and let a, b be generators of G satisfying the relations (1). An element of G is equal to $b^i a^j$ for some $0 \leqslant i < q$ and $0 \leqslant j < q^m$. Suppose that

$$1 = (b^i a^j)^k = b^{ki}(a^j)^{b^{(k-1)i} + b^{(k-2)i} + \ldots + b^i + 1}$$

for some integer k. Since $a^b = a^{1+q^{m-1}}$, we have

$$(b^i a^j)^k = b^{ki}(a^j)^{(1+q^{m-1})^{(k-1)i} + (1+q^{m-1})^{(k-2)i} + \ldots + (1+q^{m-1})^i + 1}.$$

We get the system of congruences

$$ki \equiv 0 \pmod q \text{ and}$$
$$j((1+q^{m-1})^{(k-1)i} + (1+q^{m-1})^{(k-2)i} + \ldots + (1+q^{m-1})^i + 1) \equiv 0 \pmod{q^m}.$$

By expanding brackets in the second congruence, we get

$$j\left(k + \frac{ki(k-1)q^{m-1}}{2}\right) \equiv 0 \pmod{q^m},$$

or equivalently $jk \equiv 0 \pmod{q^m}$. So the order of $b^i a^j$ is the least common multiple of $q/(q,i)$ and $q^m/(q^m,j)$. If $j \neq 0$, then it is equal to $q^m/(q^m,j)$. Therefore,

$$\psi(G) = \psi(C_q) + q\sum_{j=1}^{j=q^m-1} \frac{q^m}{(q^m,j)}.$$

The number of indices j for which $(q^m,j) = q^{m-l}$ for $0 < l \leqslant m$ is equal to $\varphi(q^l)$. Thus

$$\psi(G) = \frac{q^3+1}{q+1} + q\sum_{l=1}^{l=m}\varphi(q^l)q^l = \frac{q^3+1}{q+1} + (q-1)\sum_{l=1}^{l=m} q^{2l} =$$
$$= \frac{q^3+1}{q+1} + (q-1)q^2\frac{q^{2m}-1}{q^2-1} = \frac{q^{2m+2} + q^3 - q^2 + 1}{q+1}.$$

In particular, if $m = 2$, then

$$\psi(G) = \frac{q^6 + q^3 - q^2 + 1}{q+1}.$$

By Item (4) of Lemma 1

$$\frac{\psi(G)}{\psi(C_{q^3})} = \frac{q^6 + q^3 - q^2 + 1}{q^7+1},$$

and this completes the proof.

3. Proof of Theorem

Let G be a nonabelian q^*-group or order n. Suppose that

$$\psi(G) \geqslant \frac{q^6+q^3-q^2+1}{q^7+1}\psi(C_n).$$

Let p be the greatest prime divisor of $|G|$. Since $\psi(C_n) \geqslant \frac{q}{p+1}n^2$ by Item (3) of Lemma 1,

$$\psi(G) > \frac{(q^6+q^3-q^2+1)q}{(q^7+1)(p+1)}n^2.$$

So G contains an element x such that

$$|x| > \frac{(q^6+q^3-q^2+1)q}{(q^7+1)(p+1)}n.$$

Then

$$|G:\langle x\rangle| < \frac{(q^7+1)(p+1)}{(q^6+q^3-q^2+1)q}.$$

We proceed by induction on the number of prime divisors of the order of G. Assume that $p=q$. Then

$$|G:\langle x\rangle| < \frac{(q^7+1)(q+1)}{(q^6+q^3-q^2+1)q}.$$

Since

$$\frac{(q^7+1)(q+1)}{(q^6+q^3-q^2+1)q} < q^2,$$

the index $|G:\langle x\rangle|$ is q. It follows that G is isomorphic to $M_{q^{m+1}}$ (see, for example, [6, Theorem 1.2]). By Lemma 4 and Item (4) of Lemma 1, we have

$$\psi(G) = \frac{q^{2m+2}+q^3-q^2+1}{q+1} \geqslant \frac{q^6+q^3-q^2+1}{q^7+1}\psi(C_n) =$$

$$= \frac{q^6+q^3-q^2+1}{q^7+1}\cdot\frac{q^{2m+1}+1}{q+1},$$

or equivalently

$$-q^{2m+6}+q^{2m+5}-q^{2m+3}+q^{2m+2}+q^{10}-q^9+q^7-q^6 \geqslant 0.$$

By factorizing the left-hand side, we get

$$(-q^{2m+2}+q^6)(q^4-q^3+q^2-1) > 0.$$

This inequality holds only if $m=2$ as stated.

Suppose that $p>q$. Then

$$|G:\langle x\rangle| < \frac{(q^7+1)(p+1)}{(q^6+q^3-q^2+1)q} < p+1.$$

Indeed, the latter inequality is equivalent to

$$q^5+2q^4-3q^3+2q-3>0,$$

which is true for all $q \geqslant 2$. It follows from Lemma 3 that either G contains a normal cyclic Sylow p-subgroup or $|G : \langle x \rangle| = p$.

Assume that G contains a normal cyclic Sylow p-subgroup P. By Item (2) of Lemma 1, we have $\psi(G) \leqslant \psi(P)\psi(G/P)$ with the equality if and only if P is central in G.

If G/P is a nonabelian group, by the inductive hypothesis

$$\psi(G/P) \leqslant \frac{q^6+q^3-q^2+1}{q^7+1}\psi(C_{|G/P|}).$$

And hence

$$\psi(G) \leqslant \frac{q^6+q^3-q^2+1}{q^7+1}\psi(P)\psi(C_{|G/P|}) = \frac{q^6+q^3-q^2+1}{q^7+1}\psi(C_n).$$

So it is possible only if P is central and so G is a direct product of M_{q^3} and a cyclic group of order coprime to q.

Let G/P be abelian. Denote by H a p-complement of G. We write Z for $C_H(P)$. By Item (5) of Lemma 1,

$$\psi(G) < \psi(C_{p^m})\psi(H)\left(\frac{\psi(Z)}{\psi(H)} + \frac{p^m}{\psi(C_{p^m})}\right).$$

If H is cyclic, then

$$\psi(G) < \left(\frac{\psi(Z)}{\psi(H)} + \frac{p^m}{\psi(C_{p^m})}\right)\psi(C_n).$$

Let us bound the right-hand side from above. By Item (4) of Lemma 1

$$\frac{p^m}{\psi(C_{p^m})} = \frac{p^m(p+1)}{p^{2m+1}+1} < \frac{p+1}{p^{m+1}} \leqslant \frac{p+1}{p^2} = \frac{1}{p} + \frac{1}{p^2}. \tag{2}$$

Consider the fraction $\psi(Z)/\psi(H)$. If $Z = H$, then $G = P \times H$ and G is abelian. So Z is a proper subgroup of H. It follows from Item (1) of Lemma 1 that $\psi(H)$ is the product of $\psi(S)$ where S runs over Sylow subgroups of H. Let r be a prime such that the Sylow r-subgroup R does not lie in Z. Thus for $R \in Syl_r(H)$ and we have

$$\frac{\psi(Z)}{\psi(H)} \leqslant \frac{\psi(R \cap Z)}{\psi(R)} \leqslant \frac{r^{2(t-1)+1}+1}{r^{2t+1}+1} \leqslant \frac{1}{r^2-r+1}. \tag{3}$$

Since $G/C_G(P) \leqslant Aut(C_{p^m})$, we have $H/C_H(P) \leqslant Aut(C_{p^m})$. Therefore, $|H/C_H(P)|$ divides $p-1$. In particular, r divides $p-1$ and so $p \geqslant 2r+1$. Since $r \geqslant q$, it is follow that $p \geqslant 2q+1$. Now from the expressions (2) and (3) we obtain the inequality

$$\left(\frac{\psi(Z)}{\psi(H)} + \frac{p^m}{\psi(C_{p^m})}\right)\psi(C_n) \leqslant \left(\frac{1}{(2q+1)^2} + \frac{1}{2q+1} + \frac{1}{q^2-q+1}\right)\psi(C_n).$$

Since

$$\frac{1}{(2q+1)^2} + \frac{1}{2q+1} + \frac{1}{q^2-q+1} \leqslant \frac{q^6+q^3-q^2+1}{q^7+1}$$

holds for all $q > 2$, we have a contradiction.

If H is a non-cyclic subgroup of G, then by Item (5) of Lemma 1, we have

$$\psi(G) = |P|\psi(H) + (\psi(P) - |P|)\psi(Z).$$

Dividing the both sides by $\psi(C_n)$, we get

$$\frac{\psi(G)}{\psi(C_n)} = \frac{|P|}{\psi(P)}\frac{\psi(H)}{\psi(C_{|H|})} + \left(1 - \frac{|P|}{\psi(P)}\right)\frac{\psi(Z)}{\psi(C_{|Z|})}\frac{\psi(C_{|Z|})}{\psi(C_{|H|})}.$$

Since $\psi(Z) \leqslant \psi(C_{|Z|})$,

$$\frac{\psi(G)}{\psi(C_n)} \leqslant \frac{|P|}{\psi(P)}\frac{\psi(H)}{\psi(C_{|H|})} + \left(1 - \frac{|P|}{\psi(P)}\right)\frac{\psi(C_{|Z|})}{\psi(C_{|H|})} =$$

$$= \frac{|P|}{\psi(P)}\left(\frac{\psi(H)}{\psi(C_{|H|})} - \frac{\psi(C_{|Z|})}{\psi(C_{|H|})}\right) + \frac{\psi(C_{|Z|})}{\psi(C_{|H|})}.$$

Since H is non-cyclic q^*-group, according to [4, Theorem 4]

$$\frac{\psi(H)}{\psi(C_{|H|})} \leqslant \frac{(q^3 - q + 1)(q + 1)}{q^5 + 1}.$$

Thus

$$\frac{\psi(G)}{\psi(C_n)} \leqslant \frac{|P|}{\psi(P)}\left(\frac{(q^3 - q + 1)(q + 1)}{q^5 + 1} - \frac{\psi(C_{|Z|})}{\psi(C_{|H|})}\right) + \frac{\psi(C_{|Z|})}{\psi(C_{|H|})}.$$

Since $\psi(C_{|Z|})/\psi(C_{|H|}) > 0$, using inequalities (2) and (3), we get

$$\frac{\psi(G)}{\psi(C_n)} < \frac{(q+1)}{q^2}\frac{(q^3 - q + 1)(q + 1)}{q^5 + 1} + \frac{1}{q^2 - q + 1}.$$

The inequality

$$\frac{(q+1)}{q^2}\frac{(q^3 - q + 1)(q + 1)}{q^5 + 1} + \frac{1}{q^2 - q + 1} < \frac{q^6 + q^3 - q^2 + 1}{q^7 + 1}$$

is equivalent to

$$q^{15} - 3q^{14} + 2q^{12} - 3q^{11} + q^{10} - q^9 - q^7 - 3q^6 + 3q^5 - q^4 - 3q^3 + q^2 - 1 > 0$$

which holds for all $q > 2$; that is a contradiction and we finished the case when G contains a normal cyclic Sylow p-subgroup.

Let $|G : \langle x\rangle| = p$, $P \in Syl_p(G)$ and $|P| = p^{m+1}$. There are three possible cases: P is cyclic, P is isomorphic to $C_{p^m} \times C_p$, and $P \simeq M_{p^{m+1}}$.

Let us show that P is normal in G. If $Q \in Syl_q(G)$ then Q lies in $\langle x\rangle$. So Q is cyclic and G contains a normal q-complement N (see, for example, [7, Theorem 5.14]). By the inductive hypothesis P is normal in N and therefore in G. So $G = P \rtimes H$ where H is a cyclic subgroup of G. The case of cyclic P has been considered.

If P is isomorphic to $C_{p^m} \times C_p$, then $\psi(G) \leqslant \psi(P)\psi(H)$ by Lemma 2. It follows from Lemma 4 that

$$\frac{\psi(G)}{\psi(C_n)} \leqslant \frac{\psi(P)}{\psi(C_{p^{m+1}})} = \frac{p^{2m+2} + p^3 - p^2 + 1}{p^{2m+3} + 1} \leqslant \frac{q^6 + q^3 - q^2 + 1}{q^7 + 1}.$$

The last inequality holds for all $m \geqslant 2$. If $m = 0$, then P is cyclic. Assume that $m = 1$. Since H acts nontrivially on the cyclic group $P/(P \cap H)$ of order p, its order has a non-identity common divisor with $p - 1$. So $p \geqslant 2q + 1$. The inequality

$$\frac{(2q+1)^4 + (2q+1)^3 - (2q+1)^2 + 1}{(2q+1)^5 + 1} \leqslant \frac{q^6 + q^3 - q^2 + 1}{q^7 + 1}$$

is equivalent to

$$8q^9 + 20q^8 + 24q^7 + 31q^6 + 28q^5 + q^4 - 4q^3 + 17q^2 + 16q + 3 \geqslant 0,$$

which is satisfied for all $q \geqslant 1$; that is a contradiction.

Finally, consider the case $P \simeq M_{p^{m+1}}$. Let h be an arbitrary element of the group H and φ be the automorphism of P induced by conjugation by h. Let a, b be generators for P such that relations

$$a^{p^m} = b^p = 1, a^b = a^{1+p^{m-1}}$$

are satisfied. Obviously, $\varphi(g) = g$ for every $g \in \langle a \rangle$. If $\varphi(b) = b^\gamma a^\alpha$, then

$$\varphi(a^b) = a^{\varphi(b)} = a^{b^\gamma a^\alpha} = a^{b^\gamma} = a^{(1+p^{m-1})^\gamma} = a^{(1+p^{m-1})}.$$

Hence

$$(1 + p^{m-1})^\gamma \equiv 1 + p^{m-1} \pmod{p^m},$$

or equivalently

$$(1 + p^{m-1})((1 + p^{m-1})^{\gamma-1} - 1) \equiv 0 \pmod{p^m}.$$

So $\gamma - 1 \equiv 0 \pmod{p}$, i.e. $\gamma = 1$.

Since the element h was chosen arbitrary, we have that subgroup H centralizes the normal series

$$1 \leqslant \langle a \rangle \leqslant P.$$

Since H is p'-group, we have $G = P \times H$.

By Item (1) of Lemma 1

$$\frac{\psi(G)}{\psi(C_n)} = \frac{\psi(P)\psi(H)}{\psi(C_{|P|})\psi(C_{|H|})}.$$

Recall that H is cyclic subgroup, and therefore

$$\frac{\psi(G)}{\psi(C_n)} = \frac{\psi(P)}{\psi(C_{|P|})} \leqslant \frac{p^6 + p^3 - p^2 + 1}{p^7 + 1} \leqslant \frac{q^6 + q^3 - q^2 + 1}{q^7 + 1};$$

that is a contradiction, and the proof is complete.

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