Characterization of groups $E_6(3)$ and ${}^2E_6(3)$ by Gruenberg–Kegel graph*

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Abstract

The Gruenberg–Kegel graph (or the prime graph) $\Gamma(G)$ of a finite group G is defined as follows. The vertex set of $\Gamma(G)$ is the set of all prime divisors of the order of G. Two distinct primes r and s regarded as vertices are adjacent in $\Gamma(G)$ if and only if there exists an element of order rs in G. Suppose that $L \cong E_6(3)$ or $L \cong {}^2E_6(3)$. We prove that if G is a finite group such that $\Gamma(G) = \Gamma(L)$, then $G \cong L$.

1 Introduction

Given a finite group G, denote by $\omega(G)$ the spectrum of G, that is the set of all its element orders. The set of all prime divisors of the order of G is denoted by $\pi(G)$. *The Gruenberg–Kegel graph* (or *the prime graph*) $\Gamma(G)$ of G is defined as follows. The vertex set is the set $\pi(G)$. Two distinct primes r and s regarded as vertices of $\Gamma(G)$ are adjacent in $\Gamma(G)$ if and only if $rs \in \omega(G)$. A concept of prime graph, which has been later named after them, was introduced by G.K. Gruenberg and O. Kegel. They also gave a characterization of finite groups with disconnected prime graph but did not publish it. This result can be found in [18], where J.S. Williams started the classification of finite simple groups with disconnected Gruenberg–Kegel graph.

Denote the set of orders of maximal abelian subgroups of G by $M(G)$. Note that if $\omega(G) = \omega(H)$ or $M(G) = M(H)$, then $\Gamma(G) = \Gamma(H)$. Consider alternating groups Alt_5 and Alt_6 of degrees 5 and 6, respectively. Then $\Gamma(Alt_5) = \Gamma(Alt_6)$ but $\omega(Alt_5) = \{1, 2, 3, 5\} = \omega(Alt_6) \setminus \{4\}$ and $M(Alt_5) = \{1, 2, 3, 4, 5\} = M(Alt_6) \setminus \{9\}$. We say that a finite group G is *recognizable by* $\Gamma(G)$ ($\omega(G)$ or $M(G)$) if for every finite group H the equality $\Gamma(H) = \Gamma(G)$ ($\omega(H) = \omega(G)$ or $M(H) = M(G)$, respectively) implies that H is isomorphic to G. Clearly, if G is recognizable by $\Gamma(G)$, then it is also recognizable by $\omega(G)$ and $M(G)$. The converse is not true in general: the group Alt_5 is known to be uniquely determined by spectrum [13], while there are infinitely many groups with the same spectrum as Alt_6 [11]. The modern state of the study on characterization of simple groups by Gruenberg–Kegel grpah can be found, for example, in the recent work by P. J. Cameron and the second author [2]. In particular, in [2] the authors have proved that if a finite group L is recognizable by $\Gamma(L)$, then L is almost simple, that is its socle is a nonabelian simple group.

If p is a prime and $q = p^k$ is its power, then by $E_6^+(q)$ and $E_6^-(q)$ we denote the simple exceptional groups $E_6(q)$ and ${}^2E_6(q)$, respectively. Finite groups G such that $\omega(G) = \omega(L)$,

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where $L = E_6^{\pm}(q)$, were described in [8, 7, 21], in particular, if $p \in \{2, 11\}$, then the equality $\omega(G) = \omega(L)$ implies $G \cong L$. In [12] it is proved that if G is a finite group with $M(G) = M(L)$, where $L = E_6(q)$, then G has a unique nonabelian composition factor and this factor is isomorphic to L. Nevertheless, there are few results about groups with Gruenberg–Kegel graph as simple groups $E_6^{\pm}(q)$. In [5] and [9] it is proved that if $L = E_6^{\pm}(2)$ and $\Gamma(G) = \Gamma(L)$, then $G \cong L$. The purpose of this paper is to show that groups $E_6^{+}(3)$ and $E_6^{-}(3)$ are recognizable by their Gruenberg–Kegel graphs.

We prove the following theorem.

Theorem 1.1. *Suppose that $L \cong E_6^{\varepsilon}(3)$ with $\varepsilon \in \{+, -\}$. If G is a finite group such that $\Gamma(G) = \Gamma(L)$, then $G \cong E_6^{\varepsilon}(q)$.*

Remark 1.1. *This result was obtained during The Great Mathematical Workshop* [4].

2 Preliminaries

Recall that a subset of vertices of a graph is called a *coclique*, if every two vertices of this subset are nonadjacent. Suppose that G is a finite group. Denote by $t(G)$ the maximal size of a coclique in $\Gamma(G)$. If $2 \in \pi(G)$, then $t(2, G)$ denotes the maximal size of a coclique containing vertex 2 in $\Gamma(G)$.

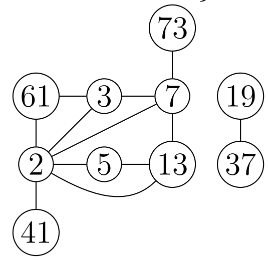
Lemma 2.1 ([15]). *Suppose that G is a finite group with $t(G) \geq 3$ and $t(2, G) \geq 2$. Then the following statements hold.*

1. *There exists a nonabelian simple group S such that $S \trianglelefteq \overline{G} = G/K \leq \mathrm{Aut}(S)$, where K is the solvable radical of G.*
2. *For every cocliue ρ of $\Gamma(G)$ such that $|\rho| \geq 3$, at most one prime of ρ divides $|K| \cdot |\overline{G}/S|$. In particular, $t(S) \geq t(G) - 1$.*
3. *One of the following two conditions holds:*
    - *every prime $p \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$ does not divide $|K| \cdot |\overline{G}/S|$. In particular, $t(2, S) \geq t(2, G)$;*
    - *$S \cong A_7$ or $L_2(q)$ for some odd q, and $t(S) = t(2, S) = 3$.*

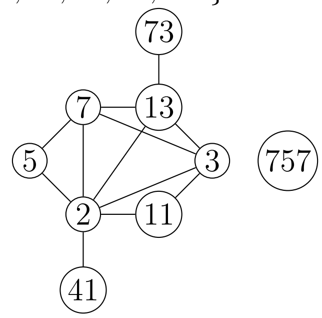
Lemma 2.2. [14, Lemma 1] *Suppose that N is a normal elementary abelian subgroup of a finite group G and $H = G/N$. Define an automorphism $\phi : H \to Aut(N)$ as follows: $n^{\phi(gN)} = n^g$. Then $\Gamma(G) = \Gamma(N \rtimes_{\phi} H)$.*

Lemma 2.3. *Suppose that $L = E_6^{\varepsilon}(q)$, where $\varepsilon \in \{+, -\}$. Then the following statements hold.*

(1) $\pi(E_6^{-}(3)) = \{2, 3, 5, 7, 13, 19, 37, 41, 61, 73\}$ *and* $\Gamma(E_6^{-}(3))$ *is the following*:



(2) $\pi(E_6^+(3)) = \{2,3,5,7,11,13,41,73,757\}$ *and* $\Gamma(E_6^+(3))$ *is the following*:



Proof. Apply a criterion of adjacency of two vertices in $\Gamma(E_6^{\pm}(q))$ [16, Prop 2.5, Prop 3.2, and Prop 4.5]. $\square$

3 Proof of Theorem

In this section we prove Theorem. Consider a finite group G such that $\Gamma(G) = \Gamma(L)$, where $L = E_6^{\varepsilon}(3)$ with $\varepsilon \in \{-,+\}$. Using Lemma 2.3, we find that $t(G) = t(L) = 5$ and $t(2,G) = t(2,L) = 3$. It follows from Lemma 2.1 that $S \lhd \overline{G} = G/K \leq \mathrm{Aut}(S)$, where K is the solvable radical of G and S is a nonabelian simple group. Denote by Ω the set of primes in $\pi(L)$ nonadjacent to 2 in $\Gamma(L)$. Then $\Omega = \{19, 37, 73\}$ if $\varepsilon = -$ and $\Omega = \{73, 757\}$ if $\varepsilon = +$. Lemma 2.1 implies that $t(S) \geq 4$, and primes from Ω belong to $\pi(S)$ and do not divide $|G|/|S|$. We keep these designations and restrictions throughout this section. The proof is split into several lemmas.

Lemma 3.1. *K is nilpotent.*

Proof. Consider the action of G on K by conjugation. Denote $r = 37$ if $\varepsilon = -$ and $r = 757$ if $\varepsilon = +$. Then $r \notin \pi(K)$. Take an element $x \in G$ of order r. Since r is nonadjacent to all the vertices of $\pi(K)$, the action of x on K is fixed-point free. By Thompson's theorem, K is nilpotent. $\square$

Lemma 3.2. *If $\varepsilon = -$, then $S \cong L$.*

Proof. Observe that 73 is the largest prime in $\pi(S)$. According to [19], S is either an alternating group of degree $n \geq 73$ or $L_2(73)$, or one of the groups from [19, Table 1]. If S is an alternating group of degree $n \geq 73$, then $71 \in \pi(S) \setminus \pi(G)$; a contradiction. Note that 19 does not divide $|L_2(73)|$. Inspecting groups from [19, Table 1], we find that $U_4(27)$ and $E_6^-(3)$ are the only groups whose order is divisible by $19 \cdot 37 \cdot 73$ and is not divisible by primes greater than 73. According to [17, Table 2], $t(U_4(27)) = 3$ and hence $S \cong E_6^-(3)$, as claimed. $\square$

Lemma 3.3. *If $\varepsilon = +$, then $S \cong L$.*

Proof. Note that 757 is the largest prime in $\pi(S)$. By [19, Table 3], S is either an alternating group of degree $n \geq 757$, or $L_2(757)$, or a group from the following list:

$$L_3(27), L_4(27), L_2(3^9), G_2(27), L_2(757^2), S_4(757), E_6(3), L_3(3^6), S_6(27), O_7(27), O_8^+(27), U_6(27).$$

If S is an alternating group of degree at least 757, then $17 \in \pi(S) \setminus \pi(G)$. In other cases if $S \not\cong E_6(3)$, then $t(S) \leq 3$ according to [17, Tables 2-4]. Therefore, $S \cong E_6(3)$, as claimed. $\square$

Lemma 3.4. *$G/K \cong L$.*

Proof. Note that $|\operatorname{Aut}(L) : L| = 2$, so either $G/K \cong L$ or $G/K \cong \operatorname{Aut}(L)$. Suppose that $G/K \cong \operatorname{Aut}(L)$. Let γ be a graph automorphism of order 2 of L, By [3, Proposition 4.9.2.], we have $C_L(\gamma) \cong F_4(3)$. Since $73 \in \pi(F_4(3))$ and vertices 2 and 73 are nonadjacent in $\Gamma(G)$, we arrive at a contradiction. □

Lemma 3.5. *If $\varepsilon = -$, then $\pi(K) \subseteq \{3, 7\}$ and if $\varepsilon = +$, then $\pi(K) \subseteq \{3, 13\}$.*

Proof. Take any prime $p \in \pi(K)$. Since K is nilpotent, we can assume that K is a p-group. Factoring G by $\Phi(K)$, we arrive at a situation where K is an elementary abelian p-group. According to [10, Table 5.1], we see that ${}^3D_4(3) \leq G/K$. Consider the action of ${}^3D_4(3)$ on K defined by ϕ as in Lemma 2.2. Take an element $g \in {}^3D_4(3)$ of order 73. If $p \neq 3$, then g fixes an element in K by [20, Proposition 2] and hence 73 and p are adjacent in $\Gamma(G)$. Lemma 2.3 implies that $p \in \{3, 7\}$ if $\varepsilon = -$ and $p \in \{3, 13\}$ if $\varepsilon = +$, as claimed. □

Lemma 3.6. *$\pi(K) \subseteq \{3\}$.*

Proof. Suppose that $7 \in \pi(K)$ or $13 \in \pi(K)$. Factoring G by $\Phi(K)$, we arrive at a situation where K is an elementary abelian group. By Lemma 3.4, we have $G/K \cong L$.

According to [10, Table 5.1], we see that $P\Omega_8^+(3) < L$. Comparing orders of L and $P\Omega_8^+(3)$, we infer that their Sylow 5-subgroups are isomorphic. Therefore, Sylow 5-subgroups of L are non-cyclic [1, Table 3]. Consider a Sylow 5-subgroup P of L. Denote by $\widetilde{P}$ the full preimage of P in G. The conjugation action of 5-elements of $\widetilde{P}$ on K is fixed-point free, so $\widetilde{P}$ is a Frobenious group. Therefore, P is cyclic; a contradiction. □

Lemma 3.7. *$K = 1$.*

Proof. By Lemma 3.6, K is a 3-group. Assume that $K \neq 1$. As above, we can assume that K is elementary abelian. According to [10, Table 5.1], we see that $F_4(3) \leq G/K$. Consider the action of $F_4(3)$ on K as in Lemma 2.2. Since $F_4(3)$ is unisingular [6, Theorem 1.3], any element of order 73 in $F_4(3)$ fixes some non-identity element in K. Therefore, primes 3 and 73 are adjacent in $\Gamma(G)$; a contradiction. □

Lemma 3.7 implies that $G \cong L$. This completes the proof of Theorem.

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