S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 12, стр. 144–144 (2020) УДК 512.545
DOI 10.17377/semi.2015.12.xxx MSC 06F15,20F60

ON CONVEX SUBGROUPS OF CARTESIAN PRODUCT OF m-GROUPS

A.V. ZENKOV

Abstract. We prove that every convex m-subgroup of a Cartesian product of m-groups that admits a faithful m-transitive presentation is an m-subgroup of a suitable projection. As a consequence, we obtain that the Cartesian product of m-groups does not admits a faithful m-transitive presentation.

Keywords: m-group, cartesian product,m-transitive presentation.

1. Introduction

An m-group is an algebraic system G of a signature $m = \langle \cdot, e, ^{-1}, \vee, \wedge, \ _* \rangle$, where $\langle G, \cdot, e, ^{-1}, \vee, \wedge \rangle$ is a lattice-orderd group (for short, ℓ-group) and the unary operation $_*$ can be interpreted as a second-order automorphism of the group $\langle G, \cdot, e, ^{-1} \rangle$ and the antiisomorphism of the lattice $\langle G, \vee, \wedge \rangle$, i.e. for any $x, y \in G$ the folloving relations are true

$$(xy)_* = x_* y_*, (x_*)_* = x, (x \vee y)_* = x_* \wedge y_*, (x \wedge y)_* = x_* \vee y_*.$$

From now one we will denote m-group G with the marked automorphism $*$ as a pair $(G, *)$.

The concept of an m-group as an algebraic system was explicitly formulated by M.Giraudet and J. Rachunek [1]. The introduction of the concept of an m-group as an algebraic system allows us to apply the methods of universal algebra to the study of monotonic permutations groups of totally ordered sets. In particular, it has become possible to write down the properties of such groups in the language of identities, which necessarily leads to the creation of a theory of varieties of m-groups.

Zenkov A.V. On convex subgroups of cartesian product of m-groups.

© 2020 Zenkov A.V..

Поступила 1 октября 2021 г., опубликована 3 февраля 2022 г.

We recall the basic concepts of the representation theory of m-groups by ordered permutations of totally ordered sets. Let Ω be some (infinite) totally ordered set and a be a reversible automorphism of the second-order of Ω. That is, $((\omega)a)a = \omega$ and $\omega < \omega' \Leftrightarrow (\omega)a > (\omega')a$ for all $\omega, \omega' \in \Omega$. Denote by $Aut(\Omega)$ the group (under composition) of all orderd permutations of Ω. It is well known that with respect to the pointwise order $Aut(\Omega)$ is an ℓ-group. Group $Aut(\Omega)$ can be turned into a m-group if the operation $_*$ is defined on it by the rule $g_* = aga$, where $g \in Aut(\Omega)$. By a *faithful* presentation of an m-groups $(G, *)$ by ordered permutations of Ω we mean the m-isomorphism $\nu : G \to Aut(\Omega)$. We write this as (G, Ω, a).

The presentation of (G, Ω, a) is called m - *transitive* if for all $\omega, \omega' \in \Omega$, maybe with the exception of the point o, there is such $x \in G_* = gr.(G, a)$, that $(\omega)x = \omega'$ (here o is a point Ω, fixed relative to the action of a). Importance m-transitive presentations in the study of varieties m-groups are explained by the fact that every subdirect m-irreducible m-group admits an faithful m-transitive presentation [2]. Therefore, every m - variety is generated by its m-transitive groups.

On the other hand, if m-variety is genereted by a certain class of m-groups, then every group of this variety according to Birkhoff's theorem is a homomorphic image of an m-subgroup of the Cartesian product of m-groups of this class. Therefore it is necessary to know the description of those Cartesian product subgroups that admit a faithful m-transitive presentation. We prove that every convex m-subgroup of a Cartesian product of m-groups that admits a faithful m-transitive presentation is an m-subgroup of a suitable projection (Theorm 1). As a consequence, we obtain that the Cartesian product of m-groups does not admits a faithful m-transitive presentation (Corollary 1).

All the concepts of the theory of ℓ-group and terminology used here basically correspond to the book [3].

2. Main result

The subgroup H of an ℓ-group G is its convex ℓ-subgroup if: 1) for any $a, b \in H$ their union $a \vee b \in H$ (H – ℓ-subgroup), 2) H is a convex subset of G, i.e. for any $g \in G$ the inequality $a \leq g \leq b$ implies $g \in H$. So as the set-theoretic intersection of any number of convex ℓ-subgroups is a convex ℓ-subgroup, then the lattice structure is defined in a standard way on the set of all convex ℓ-subgroups of ℓ-group G. This lattice is distributive. By the Birkhoff-Lorentz theorem (see [3, Ch. 3,§1, Theorem 1]), the union of $A \bigvee B$ convex ℓ-subgroups of A and B coincide with the subgroup generated by them.

Let H be a convex ℓ-subgroup of an ℓ-group G. Denote by $R(G : H)$ the set of right cosets. On $R(G : H)$, define $Hx \leq Hy$ if there exsits $h \in H$ such that $hx \leq y$. Then $\geq$ is a partial order of $R(G : H)$. If this order is totally, then H is called *prime*. It is well known (see [3, Ch. 3,§3, Theorem 1]) that the convex ℓ-subgroup H of an ℓ-group G is prime if and only if,for all positive $a, b \in G \setminus H$, $a \wedge b \notin H$ is true. Thus, for any $a \in G \setminus H$, its polar $a^{\perp} = \{g \in G \mid g \perp a\}$ is contained in H ($g \perp a$ means that these elements are orthogonal, i.e. $|a| \wedge |g| = e$).

The prime ℓ-subgroup H of an m-group $(G, *)$ is called *representing* if it does not contain non-identety m - ideals. The following statement is proved in [4].

Lemma 1. *The m-group $(G, *)$ admits a faithful m-transitive presentation if and only if it contains a representing ℓ-subgroup.*

It is clear that if the representing ℓ-subgroup $H = e$, then itself m-group $(G, *)$ will be totally ordered. In this case, as noted in [5], $(G, *)$ belongs to the variety $\mathcal{I}$, defined by the identity $xx_* = e$.

Let $\{(F_j, \varphi_j) \mid j \in J\}$ be a set of m-groups of the cardinality $|J| > 1$. Denote by $(\overline{F}, \varphi) = \overline{F} = \prod_{j \in J}(F_j, \varphi_j)$ their m-cartesian product. Element $f \in \overline{F}$ will be written as $f = (\dots f_i \dots f_j \dots f_k \dots)$. For $f \in \overline{F}$ and $j \in J$ we denote the j-projection of f as $\widehat{f_j}$. So that $\widehat{f_j} = (\dots e \dots f_j \dots e \dots)$ and then $\overline{f}_j = f\widehat{f_j}^{-1} = (\dots f_i \dots e \dots f_k \dots)$. It follows from definition of above elements that $f = \widehat{f_j}\overline{f}_j$ and $\widehat{f_j} \perp \overline{f}_j$. It is clear that $\widehat{F_j} = \{\widehat{f_j} \mid f \in \overline{F}\}$ and $\overline{F}_j = \{\overline{f}_j \mid f \in \overline{F}\}$ are m-ideals of $\overline{F}$; moreover $\overline{F} = \widehat{F_j} \times \overline{F}_j = \widehat{F_j} \vee \overline{F}_j$. As usual, we identify F_j and $\widehat{F_j}$ everywhere because of their m-isomorphism.

It follows from the above if H is a prime ℓ-subgroup of $(\overline{F}, \varphi)$, then for each $j \in J$ only one of the following conditions is met: a) $\widehat{F_j} \not\subseteq H$ and then $\overline{F}_j \subseteq H$ or b) $\overline{F}_j \not\subseteq H$ and then $\widehat{F_j} \subseteq H$. Obviously that is impossible the following $\overline{F}_j, \widehat{F_j} \subseteq H$.

Let (G, φ) be an arbitrary non-identity convex m-subgroup of $(\overline{F}, \varphi)$. Suppose that (G, φ) admits a faithful m-transitive presentation. By Lemma 1 in (G, φ) there is a representing subgroup V. Then in $(\overline{F}, \varphi)$ there is such a prime ℓ-subgroup H, that

$$V = H \cap G. \qquad (*)$$

The proof of this fact is contained, for example, in [6, Preposition 12.11].

Suppose that $\widehat{F_j} \leq H$ for any $j \in J$. Then $V \cap \widehat{F_j} = H \cap \widehat{F_j} \cap G = \widehat{F_j} \cap G$. Clearly that $\widehat{F_j} \cap G$ is an m-ideal of G wich is contained in V. Therefore $\widehat{F_j} \cap G = e$. In view of this, by the homomorphism theorem, we obtain $G\widehat{F_j}/\widehat{F_j} \cong G/G \cap \widehat{F_j} = G \leq \overline{F}/\widehat{F_j} \cong \overline{F}_j$. Then $G \subseteq \bigcap_{j \in J} \overline{F}_j$. But $\bigcap_{j \in J} \overline{F}_j = e$ and we come to a contradiction.

Therefore, there exsit $j \in J$ such that $\overline{F}_j \leq H$. As above, we can assume $\overline{F}_j \cap G = e$. Hence $G\overline{F}_j/\overline{F}_j \cong G/G \cap \overline{F}_j = G \leq \overline{F}/\overline{F}_j \cong \widehat{F_j}$. Thus we proved.

Theorem 1. *Every convex an m-subgroup (G, φ) of the m-group $(\overline{F}, \varphi)$ that admits a faithful m-transitive presentation is a convex m-subgroup for a suitable j-projection $\widehat{F_j}$.*

The following assertion is true

Corollary 1. *An m-cartesian product $(\overline{F}, \varphi) = \overline{F} = \prod_{j \in J}(F_j, \varphi_j)$, where $|J > 1|$, does not admits a faithful m-transitive presentation.*

Proof. Indeed, if $\overline{F}$ admits a faithful m-transitive presentation, then by Theorem 1 it is isomorphic to its component, which is impossible.□

References

[1] Giraudet M.,Rachunek J., Varieties of half lattice-ordered groups of monotonic permutations of chains,Czech.Math.J.,49:124(1999),743-766.

[2] A.V. Zenkov, On m-transitive groups, Math.Notes,94:1(2013),157-159.

[3] V.M. Kopytov, Lattice-ordered groups,Nauka,Moscow,1984.

[4] S.V. Varaksin, A.V. Zenkov, On presentations of m-groups,Sib.Math.J.,54:2(2013),227-230.

[5] N.V. Bayanova, O.V. Nikonova, Reversing automorphisms of lattice-ordered groups,Sib.Math.J.,36:4(1995),763-768.

[6] Darnel M. Theory of lattice-ordered groups. New York:Marcel Dekker,1995.

Alexey Vladimirovich Zenkov
Altai State Agricultural University,
pr. Krasnoarmeysky, 98,
656049, Barnaul, Russia
E-mail address: alexey_zenkov@yahoo.com