On some new Uniform estimates and maximal theorems for H^{p} spaces

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We obtain some new uniform estimates and maximal theorems in classical Hardy spaces in the unit disk related with Bergman projection extending some previously well-known inequalities for Hardy spaces.

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1 Introduction

Let further $U = \{z \in C, |z| < 1\}$,be the unit disk on a complex plane C,T be the unit circle on C, let dm_2 be the normalized Lebegues measure on C,let futher U^n be the unit polydisk on $C.. \times ..C$.

And finnaly let H^p be the classical analytic Hardy class in the unit disk for all positive values of p,let $dm(\xi)$ be the normalized Lebegues measure on T. The goal of this short paper is to extend certain classical estimates of complex function theory.

In the unit disk.to several variables.extending certain known one dimensional results to several variables using the so- called expanded Bergman projection.which was recently studied in papers of the author.

The expanded Bergman projection

$$(T_{n,\alpha}f)(w) = C(n,\alpha)\int_U \frac{f(z)(1-|z|)^\alpha}{\prod_{k=1}^{n}(1-\langle \bar{z}, w_k\rangle)^{\frac{\alpha+2}{n}}} dm_2(z), \quad \alpha > -1,$$

where $w = (w_1, \dots, w_n) \in U^n, C(n,\alpha)$ is a Bergman constant from Bergman representation formula, is playing a crucial role during the study of diagonal map (see [2], [3], [5], [7] and references there).

We will now provide new estimates for this operator using, in particular, Steintype maximal functions from [6]. We at the same time extend previously known estimates.

2 Main result

The following theorem is the main result of this note.

Theorem 1. **(a)** *Let* $\Gamma_\gamma(\xi)=\{z\in U\colon\ \left|1-\bar{\xi}z\right|<\gamma(1-|z|)\{, \gamma>1, \xi\in T$. *Let* $\beta\in(0,\frac{1}{2}), \alpha>\beta, n=2$. *Then*

$$\int_T\left(\sup_{z_1\in\Gamma_\gamma(\xi)}\sup_{z_2\in\Gamma_\gamma(\xi)}\left|\mathcal{D}^\alpha_{z_2}T_{n,0}(f)(z_1,z_2)\right|(1-|z_1|)^{\alpha-\beta}(1-|z_2|)^\beta dm(\xi)\right)^2\leqslant C\left\|f\right\|^2_{H^2(U)}.$$

(b) *Let* $p>2, \frac{1}{p}+\frac{1}{q}=1, t\in(-2,-1), \alpha>\max(t+\frac{2}{q},0), n=2$. *Then*

$$\sup_{z_1,z_2\in U}\left|T_{n,\alpha}f(z_1,z_2)\right|(1-|z_1|)^{t+2}(1-|z_2|)^{\frac{\alpha-t}{2}-\frac{1}{q}}$$
$$\leq C\left(\int_T\sup_{z\in\Gamma_\gamma(\xi)}|f(z)|(1-|z|)^{\frac{\alpha}{2}}\right)^p dm(\xi).$$

Remark 1. Putting $n=1,\alpha=0,\beta=0$ in first estimate of Theorem 1 we get the following well-known estimate for H^p classes (see [5], Chapter 1)

$$\int_T\sup_{z\in\Gamma_\gamma(\xi)}|\Phi(z)|^2 dm(\xi)\leq C\left\|\Phi\right\|^2_{H^2(U)}.$$

Putting $n=1,\alpha=0,t=-2$ in the second statement of Theorem 1 we get the well-known estimate (see [1], Theorem 2.5, [4], [5])

$$\sup_{|z|<1}|\Phi(z)|(1-|z|)^{\frac{1}{p}}\leq C\int_T\sup_{w\in\Gamma_\gamma(\xi)}|\Phi(w)|^p dm(\xi)=\left\|\Phi\right\|^p_{H^p}.$$

Proof of Theorem 1. Let $T_{2,0}(f)=\Phi(z_1,z_2)$. Then using Hölder's inequality we obtain

$$|\Phi(z_1,z_2)|\lesssim C\int_U\frac{|f(w)|}{|1-\langle\bar{w},z_1\rangle|\,|1-\langle\bar{w},z_2\rangle|}dm_2(w)$$
$$\leq C\left(\int_U\frac{|f(w)|^2(1-|w|)^{2\beta}}{|1-\langle\bar{w},z_1\rangle|^2}dm_2(w)\right)^{\frac{1}{2}}\left(\int_U\frac{(1-|w|)^{-2\beta}}{|1-\langle\bar{w},z_2\rangle|^2}dm_2(w)\right)^{\frac{1}{2}}.$$

Hence since $\beta\in(0,\frac{1}{2})$ and $\alpha>\beta$,

$$\left|\mathcal{D}^\alpha_{z_2}\Phi(z_1,z_2)\right|\lesssim C\left(\int_U\frac{|f(w)|^2(1-|w|)^{-2\beta}}{|1-\langle\bar{w},z_1\rangle|^2}dm_2(w)\right)^{\frac{1}{2}}\left(\int_U\frac{(1-|w|)^{2\beta}}{|1-\langle\bar{w},z_2\rangle|^{2+2\alpha}}dm_2(w)\right)^{\frac{1}{2}},$$

and we have

$$\sup_{z_1,z_2\in\Gamma_\gamma(\xi)}\left|\mathcal{D}^\alpha_{z_2}\Phi(z_1,z_2)\right|(1-|z_2|^{\alpha-\beta})(1-|z_1|^\beta)$$
$$\leq C\sup_{z_1\in\Gamma_\gamma(\xi)}\left(\int_U\frac{|f(w)|^2(1-|w|)^{-2\beta}}{|1-\langle\bar{w},z_1\rangle|^2}dm_2(w)(1-|z_1|)^{2\beta}\right)^{\frac{1}{2}}=G_1(f),\ \text{(see[4])}.$$

Note that

$$(\left|1-\left\langle|\lambda|\,\bar{\xi},z\right\rangle\right|)\asymp(\left|1-\left\langle\bar{\lambda},z\right\rangle\right|),\quad z\in U,\lambda\in\Gamma_\beta(\xi).$$

Hence

$$G_1(f)(\xi) \lesssim C \sup_{0<r<1} \left(\int_U \frac{(1-|z|)^{-2\beta} |f(z)|^2 (1-r)^{2\beta}}{\left|1-\langle r\bar{\xi}, z\rangle\right|^2} dm_2(z) \right)^{\frac{1}{2}} = \widetilde{G_1}(f,\xi,\beta).$$

Obviously for $\gamma \in (1, 2-2\beta), \beta \in (0, \frac{1}{2})$,

$$\widetilde{G_1}(f,\xi,\beta) \lesssim C \sup_{0<r<1} \left(\int_T \frac{(1-r)^{\gamma-1} |f(r\xi)|^2}{|1-\langle r\xi, \varphi\rangle|^\gamma} dm(\xi) \right)^{\frac{1}{2}}.$$

Hence it is enough to use estimates for Stein-type maximal functions [6]

$$\left\| \sup_{0<r<1} \left(\int_T \frac{(1-r)^{\alpha-1} |f(r\varphi)|^p}{|1-\langle r\bar{\varphi}, \xi\rangle|^\alpha} d(\varphi) \right)^{\frac{1}{p}} \right\|_{L^p} \leq C \|f\|_{H^p},$$

$f \in H^p, p > 1, \beta \in (0, \frac{1}{p}), \alpha \in (1, 2-\beta p)$ to get what we need. So the proof of first estimate is complete.

Let us prove the second estimate. First, we have the following chain of known estimates (see for example [4]).

$$\int_U d\mu(z) \lesssim C \int_T \int_{\Gamma_t(\xi)} \frac{d\mu(z)}{1-|z|} dm(\xi), \tag{1}$$

$$\int_T |M_{H-L} f(\xi)|^p \, d\xi \leq C \int_T |f(\xi)|^p \, d\xi, \; p > 1, \tag{2}$$

where M_{H-L} is a classical maximal Hardy-Littlewood operator.

$$\int_U |f(z)|^{\widetilde{p}} \, dm_2(z) \lesssim C \int_T \left(\sup_{z \in \Gamma_\gamma(\xi)} |f(z)| \right)^{\widetilde{p}} C(\mu)(\xi) d\xi, \tag{3}$$

where μ is a positive Borel measure, $0 < \widetilde{p} < \infty$, f is measurable in U and as usual $C(\mu)(\xi) = \sup\limits_{\xi \in I} \frac{1}{|I|} \int_{\triangle I} d\mu(\xi), \triangle I = \{z = r\xi, \xi \in I, 1-|z| < r < 1\}, I \subset T$. Using (1) we have

$$|\Phi(z_1, z_2)| \lesssim C(\alpha) \int_U \frac{|f(w)| (1-|w|^\alpha)}{(1-\langle z_1, \bar{w}\rangle)^{\frac{\alpha+2}{2}} (1-\langle z_2, \bar{w}\rangle)^{\frac{\alpha+2}{2}}} dm_2(w),$$

where $z_1, z_2 \in U$ and $C(\alpha)$ is a Bergman projection constant. Further using (1) and applying twice Hölder's inequality we get

$$|\Phi(z_1, z_2)| \lesssim C \left(\int_T \left(\int_{\Gamma_\alpha(\xi)} \frac{|f(w)|^2 (1-|w|)^{2\alpha - t} dm_2(w)}{(1-|w|)^2 |1-\langle z_1, \bar{w}\rangle|^{\alpha+2}} \right)^{\frac{p}{2}} d\xi \right)^{\frac{1}{p}} \times$$

$$\times \left(\int_T \left(\int_{\Gamma_\alpha(\xi)} \frac{(1-|w|)^t dm_2(w)}{|1-\langle z_2, \bar{w}\rangle|^{\alpha+2}} \right)^{\frac{q}{2}} dm(\xi) \right)^{\frac{1}{q}} \lesssim B(f)(1-|z_2|)^{-\left(\frac{\alpha-t}{2} - \frac{1}{q}\right)},$$

$p > 2, \alpha > t + \frac{2}{q}, t \in (-2, -1), \alpha > \frac{t}{2}, \frac{1}{p} + \frac{1}{q} = 1.$

Using Fubini's theorem, duality argument

$$B(f) = \sup_{\|\varphi\|_{L(\frac{p}{2})'}} \int_T \int_{\Gamma_\alpha(\xi)} \frac{|f(w)|^2 (1-|w|)^{2\alpha-t} dm_2(w)}{(1-|w|)^2 |1-\langle z_1, \bar{w}\rangle|^{\alpha+2}} |\psi(\xi)| \, dm(\xi)$$

$$= \sup_{\|\varphi\|_{L(\frac{p}{2})'}} \int_U \frac{|f(w)|^2 (1-|w|)^{2\alpha-t}}{|1-\langle z_1, \bar{w}\rangle|^{\alpha+2}} \int_T |\psi(\xi)| \, \chi_{\Gamma_\alpha(\xi)}(z) d\xi \frac{dm_2(w)}{(1-|w|)^2}.$$

Hence using (3) and the estimate

$$\sup_{z\in\Gamma_\eta} \frac{1}{1-|z|} \int_T |\psi(\xi)| \, \chi_{\Gamma_\tau(\xi)}(z) dm(\xi) \leq C M_{H-L}(\varphi)(\xi), \text{ (see[4])},$$

we have ($\widetilde{f} = f(1-|w|)^{\frac{\alpha}{2}}$)

$$B(f) \lesssim \sup_\varphi \int_T (A_\infty(\widetilde{f})(\xi))^2 M_{H-L}(\varphi)(\xi) C \left(\frac{(1-|w|)^{\alpha-t-1}}{|1-\langle z, w\rangle|^{\alpha+2}} \right)(\xi) d\xi$$

$$\lesssim \sup_\varphi \int_T (A_\infty(\widetilde{f})(\xi))^2 M_{H-L}(\varphi)(\xi) d\xi \sup_{\widetilde{w}\in U} \int_U \frac{(1-|w|)^{\alpha-t-1}(1-|\widetilde{w}|)^N dm_2(w)}{|1-\langle z_1, \widetilde{w}\rangle|^{\alpha+2} |1-\langle \widetilde{w}, w\rangle|^{N+1}},$$

where M_{H-L} is a maximal Hardy-Littlewood function. We used the fact that

$$\|C(F)\|_{L^\infty} = \sup_{\widetilde{w}\in U} \int_U \frac{|F(z)| \, dm_2(z)}{|1-\langle \widetilde{w}, z\rangle|^N} (1-|\widetilde{w}|)^{N-1}, \ N > 1.$$

From last estimate, Hölder's inequality and (2) we fimally get

$$|\Phi(z_1, z_2)| \, (1-|z_1|)^{t+2} (1-|z_2|)^{\frac{\alpha-t}{2}-\frac{1}{q}} \leq C \left\| \widetilde{f} \right\|_{L^p},$$

$t \in (-2, -1), p > 2$. The proof of Theorem 1 is complete. □

References

[1] A. V. Aleksandrov: Essay on non locally convex Hardy classes. Lecture notes in Mathematics, Springer-Verlag, Complex Analysis and spectral theory. edited by V. V. Havin and N. K. Nikolski (1981), 1-99.

[2] E. Amar, C. Menini: A counterexample to the corona theorem for operators on $H^2(D^n)$. Pacific Journal of Mathematics, v. 206, no 2, 2002.

[3] D. Clark: Restrictions of H^p functions in the polydisk. American Journal of Mathematics, (110)(1988), 1119-1152.

[4] R. Coifman, Y. Meyer, E. Stein: Some new functionl classes and their applications to harmonic analysis. Journal of Functional Analisys, (62)(1985), 304-335.

[5] A. E. Djrbashian, F. A. Shamoyan: Topics in the Theory of A^p_α Spaces. Leipzig, Teubner, 1988.

[6] I. Verbitsky: Multipliers in spaces with fractional norms and inner functions, Siberian Math. Journal, (Russian), no 2, 1985, 49-72.

[7] G. Ren, J. Shi: The diagonal mapoing in mixed norm spaces, Studia Math., 163, 103-117, 2004.