ON SOME SHARP ESTIMATES OF TOEPLITZ OPERATOR IN SOME SPACES OF HARDY-LIZORKIN TYPE OF ANALYTIC FUNCTIONS IN THE POLYDISK

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Abstract. We provide some new sharp assertions on the action of Toeplitz $T\varphi$ operator in new $F^{p,q}_{\alpha}$ type spaces of analytic functions of several complex variables extending previously known assertions proved by various authors.

1. Introduction

Let U^n be the unit polydisk in $\mathbb{C}^n$, $U^n = \{z \in \mathbb{C}^n : \ |z| < 1, \ j = 1, \dots n\}$. Let further $H(U^n)$ be the space of all analytic functions in U^n. Let further also

$$F^{p,q}_{\alpha}(U^n) = \left\{ f \in H(U^n) : \|f\|^p_{F^{p,q}_{\alpha}} = \int_{T^n} \left(\int_{I^n} |f(r\xi)|^q (1-r)^{\alpha q - 1} dr \right)^{\frac{p}{q}} d\xi < \infty \right\},$$

where $0 < p, \ q < \infty$, $\alpha > 0$, $T^n = \{|z_j| = 1, j = 1, \dots n\}$, $I^n = (0,1] \times \cdots \times (0,1]$, $dr = dr_1 \cdots dr_n$, $d\xi = d\xi_1 \cdots d\xi_n$, $(1-r)^{\alpha} = \prod_{k=1}^n (1-r_k)^{\alpha}$, $r_k \in (0,1)$ be the holomorphic Lizorkin-Triebel space, (see, for example, [2], [3], [6], [9], [10]). Note for particular case $p = q$ we have Bergman classical class, for $q = 2$ we have so-called Hardy-Lizorkin space H^p_{β} for some β that is, $H^p_{\beta} = \{f \in H(U^n) : D^{\beta} f \in H^p\}$, $0 < p \leq \infty$, $\beta > 0$, where D^{β} is a fractional derivative of analytic f function in U^n. Note (see definitions bellow) for this particular cases the action of $T\varphi$ classical Toeplitz operator is well-studied in unit disk, unit ball and unit polydisk. We study T_{φ} operators in more general $F^{p,q}_{\alpha}$ type spaces in the polydisk. Our main

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sharp result provide some criteria for symbol of $T\varphi$ to obtain boundednes of T_φ in mentioned type analytic spaces.

We define classical Hardy space $H^p(U^n)$, $0 < p \le \infty$ as follows (see also, for example, [1] and [2]). Let

$$H^p(U^n) = \{f \in H(U^n): \ \|f\|_{H^p} = \sup_{r \in I^n} M_p(f, r) < \infty\},$$

and

$$M_p(f, r) = \left(\int_{T^n} |f(r\xi)|^p dm_n(\xi) \right)^{\frac{1}{p}},$$

where $r\xi = (r_1\xi_1, \ldots, r_n\xi_n)$ and dm_n is a normalized Lebesgues measure on T^n, $r_j \in (0, 1)$, $j = 1, \ldots, n$. Note $M_p(f, r)$ function is growing function by each r_j and we, for $p = \infty$, obtain classical and well-studied class $H^\infty(U^n)$ of all bounded analytic functions in U^n (see for example [8] for this class of functions). Various sharp results on action of Toeplitz operators can bee seen in papers of various authors in various functional spaces in the unit ball and polydisk. We mention, for example, the following papers [4], [5], where such type sharp results can be seen in particular cases of $F^{p,q}_\alpha$ spaces namely in Bergman and in Hardy type spaces in the unit ball and in the unit polydisk. We note similar type results in particular values of parameters is well-known, (see, for example, [1], [2], [4], [5], [9]).

Such type sharp result on boundedness of Toeplitz operators also have various applications (see, for example [1], [2], [5], [9]).

We remind the reader the standard definition of Toeplitz T_h operators in the unit polydisk.

Let $h \in L^1(T^n)$. Then we define Toeplitz T_h operator as one integral operator

$$(T_h f)(z) = \frac{1}{(2\pi)^n} \int_{T^n} \frac{f(\xi_1, \ldots, \xi_n) h(\xi_1, \ldots, \xi_n)}{\prod_{k=1}^n (1 - \bar{\xi}_k z_k)} d\xi_1 \cdots d\xi_n,$$

$k = 1, \ldots, n$, $z_k \in U$.

Note that we can easily show $F^{p,q}_{\alpha}$ general mixed norm analytic function spaces in the unit polydisk are Banach spaces for all values of p and q, if $\min(p,q) > 1$ and they are complete metric spaces for all other values of p and q.

We stress that behavior of the operators in the unit polydisk is substantially different from the action of $T\varphi$ operators in the unit ball in $\mathbb{C}^n$ (see [1], [4], [5], [9] for example). Our intention to set criteria for the action of Toeplitz T_φ operators from $F^{p,q}_{\alpha,k}(U^n)$ into Bergman-Sobolev and Hardy-Lizorkin type spaces in the unit polydisk, under the assumption that φ is holomorphic, $\varphi \in H(D^n)$ (some restriction on symbol of Toeplitz operator).

We define some new function spaces in the polydisk for formulation of our main result in the polydisk. Let further dm_{2n} be the normalized Lebesgues measure in U^n, D^s, $0 < s \leq \infty$ be the fractional derivative of holomorphic f function

$$(D^s f)(z) = \sum_{|k|\geq 0} |\frac{\Gamma(s+k+1)\Gamma(s+1)}{\Gamma(k+1)} a_k z^k,$$

$a_k z^k = a_{k_1 \cdots k_m} z_1^{k^1} \cdots z_m^{k_m}$, $f(z) = \sum_{|k|\geq 0} a_k z^k$, $z \in U^n$, $\Gamma(\alpha+1) = \prod_{j=1}^m \Gamma(\alpha_j+1)$, $\alpha_j > -1$, $j = 1, \ldots, m$.

Note if $f \in H(U^n)$ then for any $s \in \mathbb{N}$, $D^s f \in H(U^n)$. Let also $F^{p,q}_{\alpha,k}(U^n) = \left\{ f \in H(U^n) : \|D^k f\|_{F^{p,q}_\alpha} < \infty \right\}$, $0 < p, q, \alpha < \infty$, $k \in \mathbb{N}$.

$$\begin{aligned} &A^s_{\alpha,m}(U^n) = F^{s,s}_{\alpha/s,m} = \\ &= \left\{ f \in H(U^n) : \|f\|^s_{A_{\alpha,m}} = \int_{U^n} |(D^m f)(z)|^s \, (1-|z|)^{\alpha-1} dm_{2n}(z) < \infty \right\}, \end{aligned}$$

$m \in \mathbb{N}$, $0 < s, \alpha < \infty$ (Bergman-Sobolev space). Let further

$$H^s_m(U^n) = \{ f \in H(U^n) : \|D^m f\|_{H^s} < \infty, \; m \in \mathbb{N}, \; 0 < s < \infty \}$$

be analytic Hardy-Lizorkin space in the unit polydisk U^n.

Note it can easily shown that these both scales of analytic function spaces in the unit polydisk are Banach spaces for all values of $s,\ s \geq 1$ and they are complete metric spaces for other values of $s,\ s > 0$.

Throughout the paper, we write C or c (with or without lower indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities), but is independent of the functions or variables being discussed.

2. Main results

We now formulate main results of this note.

Theorem 1. *Let* $0 < \max(p,q) \leq s,\ 1 < s < \infty,\ m,k \in \mathbb{N},\ k = \alpha + \dfrac{1}{p} - \dfrac{1}{s} + m\left(1 - \dfrac{1}{s}\right)$. *Then* $T_{\bar{\varphi}}$ *operator is bounded operator from* $F^{p,q}_{\alpha,k}(U^n)$ *into* $A^s_{m,m}(U^n)$ *if and only if* $\varphi \in H^\infty(U^n)$ *and* $\|\varphi\|_\infty \leq \|T_{\bar{\varphi}}\|$.

Theorem 2. *Let* $0 < \max(p,q) \leq 1,\ 1 < s < \infty,\ k = \dfrac{1}{p} - \dfrac{1}{s} + \alpha + m,\ k,m \in \mathbb{N}$. *Then* $T_{\bar{\varphi}}$ *operator is bounded from* $F^{p,q}_{\alpha,k}(U^n)$ *into* $H^s_m(U^n)$ *if and only if* $\varphi \in H^\infty(U^n)$ *and* $\|\varphi\|_\infty \leq \|T_{\bar{\varphi}}\|$.

For the proof of our main results we need the following lemmas.

Lemma 1. *Let* $f \in F^{p,q}_{\alpha,k}(U^n),\ s > 1,\ k \in \mathbb{N},\ 0 < p,\ q \leq s$ *and* $k = \dfrac{(\alpha + \frac{1}{p})s - 1}{s} + m(1 - \dfrac{1}{s})$. *Then the following equality is valid*

$$\left(\int_{U^n} |D^k f(w)|^s (1-|w|)^{s(k-1)-(m-1)(1-\frac{1}{s})} dm_{2n}(w)\right)^{\dfrac{1}{s}} \leq$$

$$\leq c\left(\int_{T^n}\left(\int_{I^n} |D^k f(w)|^q (1-|w|)^{\alpha q-1} d|w|\right)^{\dfrac{p}{q}} dm_n(\xi)\right)^{\dfrac{1}{p}}$$

Remark 1. Note similar proof can be provided for a little bit general situation where standard $(1-|w|)^\alpha$ type weights are replaced with $w(r),\ r \in (0,1)$ type weights (see for these weights, for example, [4], [9]).

Lemma 2. *Let $G \in H(U^n)$, $1 < s < \infty$. Then the following estimates are valid*

$$M_s(D^m G, R^2) \leq c(1-R)^{-m} M_s(G,R),\ R \in I^n,\ m \in \mathbb{N}$$

$$M_s(G, R^2) \leq c_1(1-R)^{\frac{1}{s}-1} M_1(G,R),\ R \in I^n.$$

Remark 2. Note it is well known that those estimates are valid for $n = 1$ case (see [2], [5], [6], [9]).

Lemma 3. *(see* [11], [12]*) Let f be analytic in $0 \leq r_j < |z_j| < R_j$, $1 \leq j \leq m$ and $f \in C^s$ continuous in closure of this domain. Then for $0 \leq p \leq q \leq \infty$, $\rho \in (r, R)$*

$$\|f_\rho\|_{H^q} \leq c(m,p,q) \prod_{j=1}^{m} ((\rho_j - r_j),(R_j - \rho_j))^{\frac{1}{q}-\frac{1}{p}} \max_{\substack{V_j = r_j, R_j \\ j = 1, \ldots, m}} \|f_v\|_{H^p}.$$

Lemma 4. *(see* [9], [10]*) Let $0 < \max(p,q) \leq s < \infty$, $\alpha > 0$. Then*

$$\left(\int_{U^n} |f(w)|^s (1-|w|)^{s\left(\alpha + \frac{1}{p}\right) - 2} dm_{2n}(w) \right)^{\dfrac{1}{s}} \leq c \|f\|_{F_\alpha^{p,q}}.$$

Remark 3. It is easy to note that the same approach can be used to get criteria on symbol φ, for which $(T_{\bar{\varphi}})$ operator is bounded from $F_{k,\alpha}^{p,q}$ into X, where X is a different from A_s^m and H^s quazinormed subspace of $H(U^n)$.

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