ON THE EXISTENCE OF GLOBAL SOLUTION OF THE SYSTEM OF EQUATIONS OF ONE-DIMENSIONAL MOVEMENT OF A VISCOUS LIQUID IN A DEFORMABLE VISCOUS POROUS MEDIUM

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Abstract. The initial-boundary value problem for the system of one-dimensional motion of viscous liquid in deformable viscous porous medium is considered. Local theorem of existence and uniqueness of problem is proved in case compressible liquid. In case of incompressible liquid the theorem of global solvability in time is proved in Holder classes.

Keywords: Darcy's law, poroelasticity, filtration, global solvability, porosity.

1. Introduction

Interest in the mathematical modeling of multiphase flows arises when considering the problems of describing such technogenic systems as modeling processes in oil wells and near-surface formations. In many practical problems, the porosity of the medium is variable, and the porous medium is deformable. At present, rigorous mathematical results in the field of models of filtration in deformable porous media are presented only in a few works and in the case of single-phase filtration.

Mathematical models of fluid filtration in a porous medium apply to a broad range of practical problems. The examples include, but are not limited to filtration near river dams, irrigation and drainage of agricultural fields, dynamics of hydraulic fracturing during oil and gas mining, methane extraction from coal and shale

deposits, flow of magma in the earth's crust, movement of physiological fluids in tissues, tumour growth processes etc. The problems of heat and mass transfer in multiphase media and dynamics of multicomponent media are widely represented in natural processes and human activities. This necessitates the necessity to model the processes of interpenetrating motion of continuous media. The main attention in the paper is paid to investigation of the problem of heat and mass transfer in porous and poroelastic media. The mathematical model considered in this paper takes into account poroelasticity and deformation of the medium, including motion of solid particles of the medium and variable porosity. The resulting mathematical models are usually non-classical and require new approaches, both to investigate their correctness and to numerically simulate them. Mathematical analysis of multiphase and multicomponent systems provides an opportunity to predict the nature of complex flows in various situations that do not have a pre-experimental background. In addition, such analysis serves as a basis for the construction of numerical algorithms, which plays a key role in the development of specific technological processes. Parameters of these models strongly depend on the properties of fluids as well as the porous solid medium. Thus, a vast number of models are currently available (see [1], [2] and references therein). Majority of these models, however, make a simplifying assumption of a static solid porous skeleton, and treat the porosity as a given function. In this work, we try to relax this assumption to account the mobility and poroelastic properties of a solid component. Models with a given porosity function of a solid component are based on a Muskat-Leverett filtering theory. S.N. Antontsev and V.N. Monakhov [3] developed the theory for a special case of two-phase motion of immiscible incompressible liquids in a non-deformable porous medium. A large number of papers are devoted to numerical studies (see, for example, [4]).

The construction of mathematical models of fluid filtration processes in porous media is complicated by the fact that flow is often considered in a mobile inhomogeneous medium, which is characterised by the presence of variable porosity. A special feature of the model considered in this project is the consideration of mobility of solid skeleton and its poroelastic properties. This model is a generalization of the classical Maskett-Leverett model in which porosity is considered as a given function. The consideration of the compressibility of the porous medium is fundamental.

Tertzagi [5] was the first to develop models of poroelastic media that would take into account mobility of the skeleton and its poroelastic properties. He introduced the principle of effective stress, defined as the difference between total stress and the pressure of the liquid phase. This position reflects the fact that the liquid carries a part of the load. The relation between the deformation of the skeleton of the solid matrix and the fluid flow is of key importance here. Bio [6] further developed Tertsagi's theory: he introduced a joint deformation model of a fluid-saturated porous medium and established the theory of poroelasticity. Almost simultaneously and independently, Frenkel [7] developed a similar theory. Later, V.N. Nikolayevsky, P.P. Zolotarev, and Kh.A. Rakhmatullin [8], [9], [10] proposed analogous models in their studies.

O.B. Bocharov [11], allowed the porosity to depend on the pressure, but no deformation of the porous skeleton was considered. V. V. Vedernikov and V. N. Nikolaevskii [12] proposed a two-phase filtration model in a deformable porous medium with a solid skeleton motion being described analogous to the Tertsagi principle and the modified Hooke linear law. There are no results to validate the

model. This was done later in [13]where particular solutions were derived. O.B. Bocharov et all [14] derived the properties of the solutions for a degenerate case. All filtration models are very complex both from a theoretical point of view as well as in their application to specific problems. Only a handful of studies on deformable porous media models published to date included result justification. A mainstream research is based on the classical theory of filtration, and model justification is only examined for a limited number of specific cases. Strict mathematical results are only presented in a few papers exploring the existence and uniqueness of solutions of such problems. For example, A. M. Abourabia et all in [15], [16], [17] reduced the initial system of equations to a single equation of higher order by making a number of simplifying assumptions. M. Simpson et all [17] proved a local solvability of the Cauchy problem in S.L. Sobolev spaces. Y. Geng et all [15], [16], investigated solutions of the "simple wave"type. Numerical studies of such problems were carried out, for example, in [18].

2. Problem Statement

The concepts of volumes of the solid skeleton V_s and pores V_p are introduced for each component of a two-phase medium (the s skeleton and the f liquid phase contained in it). Then the specific pore volume (porosity) can be stated as $\phi = \frac{V_p}{V_t}$, where the total volume is $V_t = V_p + V_s$.

Darcy flow, which describes the fluid velocity relative to the solid velocity, is defined as [19]

$$\vec{q}_D = \phi(\vec{v}_f - \vec{v}_s),$$

where $\vec{v}_f, \vec{v}_s$ are velocities of fluid and porous skeleton respectively.

Mass conservation laws for liquid and solid phases in absence of phase transitions have the form [20]

$$\frac{\partial(\rho_f\phi)}{\partial t} + \bigtriangledown \cdot (\rho_f\phi\vec{v}_f) = 0,$$

$$\frac{\partial(1-\phi)\rho_s}{\partial t} + \bigtriangledown \cdot ((1-\phi)\rho_s\vec{v}_s) = 0,$$

where t is time, ρ_f is density of liquid, ρ_s is density of solid phase, $\bigtriangledown = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ is gradient operator, (x_1, x_2, x_3) are Eulerian coordinates.

Laws of conservation of mass can be wrote in terms of material derivative: ($\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}_s \cdot \bigtriangledown$). Then we obtain

$$\frac{d\rho_f\phi}{dt} = -\bigtriangledown \cdot (\rho_f(\vec{q}_D + \phi\vec{v}_s)),$$

$$\nabla \cdot \vec{v}_s = \frac{1}{1-\phi}\frac{d\phi}{dt}.$$

By motion of fluid in deformable medium it is assumed that [20], [21]:

1. the deviator of stress tensor in liquid phase is neglected ($S_f = 0$), because the fluid viscosity is much lower than the skeleton shear viscosity.

2. the total stress tensor σ is described via stress tensors of solid phase σ_s and liquid phase σ_f by the rule:

$$\sigma = (1-\phi)\sigma_s + \phi\sigma_f = (1-\phi)(S_s - p_sI) - \phi p_fI,$$

and total pressure is described as $p_{tot} = (1-\phi)p_s + \phi p_f$, where σ_s, p_s are stress tensor and pressure of solid phase, $S_s = 2\eta\dot{\varepsilon}_D$ is deviator of stress tensor, $\dot{\varepsilon}_D =$

$\frac{1}{2}\left(\frac{\partial v_s}{\partial \vec{x}} + \left(\frac{\partial v_s}{\partial \vec{x}}\right)^*\right)$ is deviatoric strain rate tensor, η is is the skeleton shear viscosity, σ_f, p_f are stress tensor and pressure of liquid phase.

Terzaghi's principle states that deformation of the solid matrix is determined by an effective stress defined as [5] $\sigma_e = \sigma + p_f I$. Then, for a fully saturated, low-porosity media, the effective dynamic pressure is $p_e = p_{tot} - p_f$ [22].

Notice, that

(1)
$$d\phi = \frac{dV_p}{V_t} - V_p \frac{dV_t}{V_t^2} = \frac{dV_p}{V_t} - \phi \frac{dV_t}{V_t}.$$

If density of solid phase ρ_s is constant, then $dV_s = 0$ and $dV_t = dV_p$. From equation (1) we obtain

(2)
$$d\phi = (1 - \phi)\frac{dV_t}{V_t}.$$

The assumption that porosity is function of effective pressure was used in work [23] $\phi = \phi(p_e)$, in particular: $\phi = \phi_0 \exp\{-bp_e\}$. In approach, used in [23], bulk compressibility of two-phase medium β_t is defined as relativity summary change of volume, responding on changing applied effective dynamic pressure: $\beta_t = -\frac{1}{V_t}(\frac{\partial V_t}{\partial p_e})$.

Equation (2) in this case can be written as

$$d\phi = -(1 - \phi)\beta_t dp_e.$$

Volumetric compressibility is also function of porosity, for example [24]: $\beta_t(\phi) = \phi^b \beta_\phi$, where β_ϕ is the bulk compressibility, b is positive constant

$$\beta_\phi = -1/V_p(\partial V_p/\partial p_e).$$

Thus the temporal variation of the porosity owing to mechanical compaction can be written as [25]:

$$\frac{1}{1 - \phi}\frac{d\phi}{dt} = -\beta_t(\phi)\frac{dp_e}{dt}.$$

The constitutive creep law can be written as [26], [27]

$$\frac{1}{1 - \phi}\frac{d\phi}{dt} = -\frac{p_e}{\xi},$$

where ξ corresponds to a bulk viscosity. This formulation is analogous to a creep-controlled viscous compaction law used in studies dealing with magma transport in the Earth's mantle [28].

The bulk viscosity as rule depends on ϕ, for example: $\xi(\phi) = \frac{\eta}{\phi^m}$, where m is positive constant [29].

Thus the rheological law combining mechanical and viscous compaction can be expressed as [24], [26]

$$\frac{1}{1 - \phi}\frac{d\phi}{dt} = -\beta_t(\phi)\frac{dp_e}{dt} - \frac{p_e}{\xi(\phi)}.$$

The conservation of momentum for the fluid can be stated as Darcy's law [19], [30]

$$\vec{q}_D = -K \bigtriangledown \left(\frac{P_{ex}}{\rho_f g}\right),$$

where K is the hydraulic conductivity, $K = (k'\rho_f g)/\mu$, k', μ are the permeability and the fluid dynamic viscosity, g is the density of the mass forces $(\vec{g} = (0, 0, -g))$,

P_{ex} is the excess fluid pressure, defined as the difference between the fluid pressure and the hydrostatic pressure: $P_{ex} = p_f - p_h$. In this way, we have

$$\vec{q}_D = -\frac{k'}{\mu}(\nabla p_f + \rho_f \vec{g}).$$

In some cases coefficients k', β_t, ξ could be determined in experiment another way. In particular they can be determined as: $\beta_t = \phi^b \beta_\phi, \xi = \eta/\phi^m, k' = k\phi^n$, where k is permability, $b = 1/2, m \in [0, 2], n = 3$ [24].

The laws of momentum conservation for each of phase can be written as [31]

$$\nabla \cdot (\phi \sigma_f) - \rho_f \phi \vec{g} + M = 0,$$

$$\nabla \cdot ((1 - \phi)\sigma_s) - \rho_s(1 - \phi)\vec{g} - M = 0,$$

where M is interphase pulse exchange.

Adding these equations, we obtain the equation of conservation of momentum of the system "solid matrix - pore fluid"[20], [21], [25], namely: the equation of incompressible deformation of the solid skeleton matrix taking into account the effect of the pore fluid pressure:

$$\nabla \cdot \sigma + \rho_{tot}\vec{g} = 0,$$

where $\rho_{tot} = (1 - \phi)\rho_s + \phi\rho_f$ is density of the two-phase medium.

In expanded form, the previous equation can be written as follows [31]

$$\rho_{tot}\vec{g} + div\left((1 - \phi)\eta\left(\frac{\partial \vec{v}_s}{\partial \vec{x}} + (\frac{\partial \vec{v}_s}{\partial \vec{x}})^*\right)\right) - \nabla p_{tot} = 0.$$

In some applications, the force balance equation is written as [20], [31]

$$-\nabla p_{tot} + \rho_{tot}\vec{g} = 0.$$

The equation of energy conservation is taken in the following form [31], [32]

$$(\rho_f c_f \phi + \rho_s c_s(1 - \phi))\frac{\partial \theta}{\partial t} + (\rho_f c_f \phi \vec{v}_f + \rho_s c_s(1 - \phi)\vec{v}_s)\nabla\theta = div(\lambda\nabla\theta),$$

where θ is the temperature of the medium (the same for each phases), c_s and c_f are heat capacities for at constant volume of phases. The thermal conductivity coefficient of the medium $\lambda(\phi)$ is taken in the form $\lambda(\phi) = \lambda_f \phi + \lambda_s(1 - \phi)$, where λ_f, λ_s are the thermal conductivity of liquid and solid phase (averaged thermal conductivity) [31].

Thus, the equations of model in the absence of phase transitions have the form taking into account dependencies coefficients of skeleton viscous and compactness of porosity and temperature [20], [21], [24], [31]:

(3) $\frac{\partial(1-\phi)\rho_s}{\partial t} + div((1-\phi)\rho_s\vec{v}_s) = 0, \quad \frac{\partial(\rho_f\phi)}{\partial t} + div(\rho_f\phi\vec{v}_f) = 0,$

(4) $\phi(\vec{v}_f - \vec{v}_s) = -\frac{K(\phi)}{\mu(\theta)}(\nabla p_f + \rho_f\vec{g}),$

(5) $\nabla \cdot \vec{v}_s = -a_1(\phi)\xi_1(\theta)p_e - a_2(\phi)\xi_2(\theta)(\frac{\partial p_e}{\partial t} + \vec{v}_s \cdot \nabla p_e),$

(6) $\nabla \cdot \sigma + \rho_{tot}\vec{g} = 0, \quad \rho_{tot} = \phi\rho_f + (1 - \phi)\rho_s,$

(7) $(\rho_f c_f \phi + \rho_s c_s (1 - \phi))\dfrac{\partial \theta}{\partial t} + (\rho_f c_f \phi \vec{v}_f + \rho_s c_s (1 - \phi)\vec{v}_s)\nabla\theta = div(\lambda\nabla\theta),$

(8) $p_{tot} = \phi p_f + (1 - \phi)p_s, p_e = (1 - \phi)(p_s - p_f).$

This quasilinear composite type system describes the spatial unsteady isothermal motion of a compressible fluid in a viscoelastic medium. Here $K(\phi), a_1(\phi), a_2(\phi),$ $\xi_1(\theta), \xi_2(\theta)$ are parameters of poroelastic medium.

For the permeability coefficient $K(\phi)$, a well-known dependence of the form is used $K(\phi) = K'\phi^n$, where $K' = const > 0, n = 3,$ [24]. In what follows, the notations are used $k(\phi) = K(\phi)/\mu(\theta), a_1(\phi) = \phi^m, \xi_1(\theta) = 1/\eta(\theta)$, where $\eta(\theta)$ is the coefficient of dynamic viscosity of the skeleton, which characterizes the relationship between the strain rate tensor and the stress tensor and is determined from the experiment under uniaxial compression [33], [34]. The following dependence is taken as a model one: $\eta(\theta) = \eta_r exp(Q_r(1 - \theta/\theta_r)/R\theta), \eta_r, Q_r, \theta_r, R$ are positive constants (analog of the Arrhenius formula for the dependence of the reaction rate on temperature) [24].

Numerical studies of various initial-boundary value problems for the system of equations (2)–(8) were carried out in the works [21], [24], [35]. A numerical analysis of the initial-boundary value problem for the system (3)–(8) is carried out in [36]: difference schemes are constructed and their convergence is established. Questions of justification in these papers were not considered. In some particular cases, the issues of justifying this model are discussed in [37], [38]. The local solvability of the Cauchy problem in Sobolev spaces was established in [17].

The system of equations describing the one-dimensional unsteady motion of a compressible fluid in a viscous porous medium $(a_2(\phi) = 0)$ in the domain $(x, t) \in Q_T = \Omega \times (0, T), \Omega = (0, 1)$, is as follows [21], [31]:

$$\dfrac{\partial(1 - \phi)\rho_s}{\partial t} + \dfrac{\partial}{\partial x}((1 - \phi)\rho_s v_s) = 0,$$

(9)

$$\dfrac{\partial(\rho_f \phi)}{\partial t} + \dfrac{\partial}{\partial x}(\rho_f \phi v_f) = 0,$$

(10) $$\phi(v_f - v_s) = -\dfrac{k(\phi)}{\mu(\theta)}\left(\dfrac{\partial p_f}{\partial x} - \rho_f g\right),$$

(11) $$\dfrac{\partial v_s}{\partial x} = -a_1(\phi)\xi_1(\theta)p_e, \quad p_e = p_{tot} - p_f, p_{tot} = \phi p_f + (1 - \phi)p_s,$$

(12) $$\rho_{tot} g + \dfrac{\partial p_{tot}}{\partial x} = 0,$$

(13) $(\rho_f c_f \phi + \rho_s c_s (1 - \phi))\dfrac{\partial \theta}{\partial t} + (\rho_f c_f \phi v_f + \rho_s c_s (1 - \phi)v_s)\dfrac{\partial \theta}{\partial x} = \dfrac{\partial}{\partial x}(\lambda\dfrac{\partial \theta}{\partial x}).$

The problem is written in the Eulerian coordinates x, t. The real density of the solid particles ρ_s are assumed constant, and Clapeyron dependence is taken as $p_f = R\rho_f, R = const > 0$. At the boundary of the region Ω, the velocities of the phases v_s, v_f are set, and at the initial moment of time the density $\rho_f^0(x)$ and porosity $\phi^0(x)$ are set.

The following conditions are considered for the system (9)–(13):

(14) $v_s|_{x=0,1} = v_f|_{x=0,1} = 0, \rho_f|_{t=0} = \rho^0(x), \phi|_{t=0} = \phi^0(x).$

Following [3], [39], [40] we rewrite the system (9)–(13). Suppose that $\bar{x} = \bar{x}(\tau, x, t)$ is a solution of the Cauchy problem

$$\frac{\partial \bar{x}}{\partial \tau} = v_s(\bar{x}, \tau), \quad \bar{x}\mid_{\tau=t} = x.$$

We set $\hat{x} = \bar{x}(0, x, t)$ and take $\hat{x}$ and t for the new variables. Then $1 - \phi(\hat{x}, t) = (1 - \phi^0(\hat{x}))\hat{J}(\hat{x}, t)$, where $\hat{J}(\hat{x}, t) = \frac{\partial \hat{x}}{\partial x}(\hat{x}, t)$ is the Jacobian of the transformation. Instead of (9)–(13) we have

$$\frac{\partial(1 - \hat{\phi})}{\partial t} + \frac{(1 - \hat{\phi})^2}{1 - \phi^0}\frac{\partial \hat{v}_s}{\partial \hat{x}} = 0, \quad \frac{\partial}{\partial t}(\hat{\rho}_f \hat{\phi}) + \frac{(1 - \hat{\phi})}{1 - \phi^0}\frac{\partial}{\partial \hat{x}}(\hat{\rho}_f \hat{\phi}\hat{v}_f) = \hat{v}_s \frac{(1 - \hat{\phi})}{1 - \phi^0}\frac{\partial}{\partial \hat{x}}(\hat{\rho}_f \hat{\phi}),$$

$$\hat{\phi}(\hat{v}_s - \hat{v}_f) = \frac{k(\hat{\phi})}{\mu(\hat{\theta})}\frac{(1 - \hat{\phi})}{1 - \phi^0}\frac{\partial \hat{p}_f}{\partial \hat{x}}, \quad \frac{(1 - \hat{\phi})}{1 - \phi^0}\frac{\partial \hat{v}_s}{\partial \hat{x}} = -a_1(\hat{\phi})\xi_1(\hat{\theta})\hat{p}_e.$$

Since

$$\hat{v}_s \frac{\partial}{\partial \hat{x}}(\hat{\rho}_f \hat{\phi}) = \frac{\partial}{\partial \hat{x}}(\hat{\rho}_f \hat{\phi}\hat{v}_s) - \hat{\rho}_f \hat{\phi}\frac{\partial \hat{v}_s}{\partial \hat{x}},$$

the continuity equation for the liquid phase can be reduced to the form

$$\frac{1}{(1 - \hat{\phi})}\frac{\partial}{\partial t}(\hat{\rho}_f \hat{\phi}) + \frac{1}{1 - \phi^0}\frac{\partial}{\partial \hat{x}}(\hat{\rho}_f \hat{\phi}(\hat{v}_f - \hat{v}_s)) + \frac{1}{1 - \phi^0}\hat{\rho}_f \hat{\phi}\frac{\partial \hat{v}_s}{\partial \hat{x}} = 0.$$

Using the continuity equation for the solid phase, we have

$$\frac{\partial}{\partial t}(\hat{\rho}_f \frac{\hat{\phi}}{1 - \hat{\phi}}) + \frac{1}{(1 - \phi^0)}\frac{\partial}{\partial \hat{x}}(\hat{\rho}_f \hat{\phi}(\hat{v}_f - \hat{v}_s)) = 0.$$

Finally, passing from $(\hat{x}, t)$ to the mass Lagrangian variables (y, t) by the rule [3] $(1 - \phi^0(\hat{x}))d\hat{x} = dy, \quad y(\hat{x}) = \int\limits_0^{\hat{x}}(1 - \phi^0(\eta))d\eta \in [0, 1]$ and formally replacing y by x, system (9)–(13) can be written in the form

$$(15) \qquad\qquad\qquad \frac{\partial(1 - \phi)}{\partial t} + (1 - \phi)^2\frac{\partial v_s}{\partial x} = 0,$$

$$(16) \qquad\qquad\qquad \frac{\partial}{\partial t}\left(\rho_f \frac{\phi}{1 - \phi}\right) + \frac{\partial}{\partial x}(\rho_f \phi(v_f - v_s)) = 0,$$

$$(17) \qquad\qquad\qquad \phi(v_s - v_f) = \frac{k(\phi)}{\mu(\theta)}((1 - \phi)\frac{\partial p_f}{\partial x} - \rho_f g),$$

$$(18) \qquad\qquad\qquad (1 - \phi)\frac{\partial v_s}{\partial x} = -a_1(\phi)\xi_1(\theta)p_e, \quad p_e = p_{tot} - p_f,$$

$$(19) \qquad\qquad\qquad \rho_{tot} g + (1 - \phi)\frac{\partial p_{tot}}{\partial x} = 0,$$

(20)
$$(\rho_f c_f \phi + \rho_s c_s(1 - \phi))\frac{\partial \theta}{\partial t} = (1 - \phi)\frac{\partial}{\partial x}(\lambda(1 - \phi)\frac{\partial \theta}{\partial x}) + \frac{k(\phi)}{\mu(\theta)}c_f \rho_f((1 - \phi)\frac{\partial p_f}{\partial x} - \rho_f g)\frac{\partial \theta}{\partial x}.$$

Taking into account equation (19), we obtain a representation for p_{tot}:

$$p_{tot} = P^0(t) - \int_0^x \frac{\rho_{tot} g}{1 - \phi}d\xi.$$

We represent the equation (18) as

$$\frac{\partial v_s}{\partial x} = -\frac{a_1(\phi)}{1-\phi}\xi_1(\theta)(p_{tot} - p_f).$$

After integrate this equation by x from 0 to 1 and taking into account the submission for p_{tot} we get

$$P^0 = (\int\limits_0^1 \left(\frac{a_1(\phi)}{1-\phi}\xi_1(\theta)(p_f + \int_0^x \frac{\rho_{tot}g}{1-\phi}d\xi)\right)dx - v_s(1,t) + v_s(0,t))\cdot$$

$$\cdot\left(\int\limits_0^1 \frac{a_1(\phi)}{1-\phi}\xi_1(\theta)dx\right)^{-1}.$$

Converting the original system of equations we get system for finding porosity, pressure of fluid and temperature (20) (using equation (16), taking into account equation (17) and using equation (15), taking into account equation (18)):

(21) $$\frac{\partial}{\partial t}(\frac{p_f}{R\theta}\frac{\phi}{1-\phi}) = \frac{\partial}{\partial x}(\frac{k(\phi)}{\mu(\theta)}\frac{p_f}{R\theta}((1-\phi)\frac{\partial p_f}{\partial x} - \frac{g}{R\theta}p_f),$$

(22) $$\frac{\partial G}{\partial t} = \xi_1(\theta)(p_f - p_{tot}),$$

(23)
$$(\rho_f c_f\phi + \rho_s c_s(1-\phi))\frac{\partial \theta}{\partial t} = (1-\phi)\frac{\partial}{\partial x}(\lambda(1-\phi)\frac{\partial\theta}{\partial x}) + \frac{k(\phi)}{\mu(\theta)}c_f\rho_f((1-\phi)\frac{\partial p_f}{\partial x} - \rho_f g)\frac{\partial\theta}{\partial x},$$

where

$$p_{tot} = \int\limits_0^1 \left(\frac{a_1(\phi)}{1-\phi}\xi_1(\theta)(p_f(\rho_f) + \int_0^x \frac{\rho_{tot}g}{1-\phi}d\xi)\right)dx \left(\int\limits_0^1 \frac{a_1(\phi)}{1-\phi}\xi_1(\theta)dx\right)^{-1} -$$

$$-\int_0^x \frac{\rho_{tot}g}{1-\phi}d\xi,$$

and the function G is defined as follows:
$$\frac{\partial G}{\partial \phi} = \frac{1}{(1-\phi)a_1(\phi)}.$$

3. COMPRESSIBLE FLUID

For system (21)–(23) conditions (14) could be written as

$$((1-\phi)\frac{\partial p_f(\rho_f)}{\partial x} - \rho_f g)\mid_{x=0,x=1} = 0, \quad \frac{\partial\theta}{\partial x}\mid_{x=0,x=1} = 0,$$

(24)

$$p_f\mid_{t=0} = p^0(x), \quad \phi\mid_{t=0} = \phi^0(x), \quad \theta\mid_{t=0} = \theta^0(x),$$

here

$$P^0 = \int\limits_0^1 \left(\frac{a_1(\phi)}{1-\phi}\xi_1(\theta)(p_f + \int_0^x \frac{\rho_{tot}g}{1-\phi}d\xi)\right)dx \left(\int\limits_0^1 \frac{a_1(\phi)}{1-\phi}\xi_1(\theta)dx\right)^{-1} \equiv P^0(\phi,\theta,p_f).$$

In the notation of function spaces we follow [41]: $C^{k+\alpha,m+\beta}(Q_T)$ – Hölder's space, where k, m are natural numbers, $(\alpha, \beta) \in (0, 1]$, with the norm $||f||_{C^{k+\alpha,m+\beta}(Q_T)}$.

Definition 1. *The solution of problem (21)–(24) is the set of functions* $\phi, \rho_f \in C^{2+\alpha,1+\beta}(Q_T)$, *such that* $0 < \phi < 1, \rho_f > 0$. *These functions satisfy the equations (21)–(23) and the initial and boundary conditions (24) and regarded as continuous functions in* Q_T.

Theorem 1. *Suppose that the data of problem (21)–(24) satisfies the following conditions:*
1. the functions $k(\phi), \xi(\phi)$ *and their derivatives up to the second order are continuous for* $\phi \in (0,1), p_f > 0, \theta > 0$, *and satisfy the conditions*

$$k_0^{-1}\phi^{q_1}(1-\phi)^{q_2} \leq k(\phi) \leq k_0\phi^{q_3}(1-\phi)^{q_4}, k_0^{-1}\phi^{q_5}(1-\phi)^{q_6} \leq \lambda(\phi) \leq k_0\phi^{q_7}(1-\phi)^{q_8},$$

$$0 < \xi_1^0 \leq \xi_1(\theta) \leq \xi_1^1 < \infty, \quad 0 < \mu_0 \leq \mu(\theta) \leq \mu_1 < \infty,$$

$$a_1(\phi) = a_0(\phi)\phi^{\alpha_1}(1-\phi)^{\alpha_2-1}, \quad 0 < R_1 \leq a_0(\phi) \leq R_2,$$

where $k_0, \xi_1^0, \xi_1^1, \mu_0, \mu_1\alpha_i, R_i, i = 1,2$ *are positive constants,* $q_1, ..., q_8$ *are fixed real parameters,*
2. the initial functions ϕ^0, p^0, θ^0 *and function* g *satisfy the following smoothness conditions:* $\phi^0 \in C^{2+\alpha}(\overline{\Omega}), p^0 \in C^{2+\alpha}(\overline{\Omega}), \theta^0 \in C^{2+\alpha}(\overline{\Omega}), g \in C^{1+\alpha,1+\alpha/2}(\bar{Q}_T)$, *and the matching conditions*

$$((1-\phi^0)\frac{dp^0}{dx} - \rho(p^0,\theta^0)g(x,0))\mid_{x=0,x=1} = 0,$$

as well as satisfy the inequalities

$$0 < m_0 \leq \phi^0(x) \leq M_0 < 1, 0 < m_1 \leq p^0(x) \leq M_1 < \infty, 0 < m_2 \leq \theta^0(x) \leq M_2 < \infty,$$

$$0 < g(x,t) \leq g_0 < \infty, x \in \overline{\Omega},$$

where m_0, M_0, m_1, M_1, g_0 *are given positive constants.*
Then problem (21)–(24) has a local solution, i.e., there exists a value of $t_0 \in (0,T)$ *such that*

$$(\phi(x,t), p_f(x,t), \theta(x,t)) \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_{t_0}).$$

Moreover, $0 < \phi(x,t) < 1$, $p_f(x,t) > 0$, $\theta(x,t) > 0$ *в* $\overline{Q}_{t_0}$.

Proof. The solvability of problem (21)–(24) is established by using the Tikhonov-Schauder Fixed-Point Theorem [3].

Since the function $\psi = G(\phi)$ is strictly monotone, at $\phi \in (0,1)$, that the inverse function is exist: $\phi = G^{-1}(\psi)$. Assuming that $p(x,t) = p_f(x,t) - p^0(x), \quad \omega(x,t) = G(\phi) - G(\phi^0), \quad \theta_1 = \theta - \theta^0$. We represent the equations (21)–(23) in the form

$$\frac{\partial}{\partial t}\left(\frac{p+p^0}{R(\theta_1+\theta^0)}\frac{\phi(\omega)}{1-\phi(\omega)}\right) =$$

(25)

$$\frac{\partial}{\partial x}\left(\frac{k(\phi(\omega))}{\mu(\theta_1+\theta^0)}\frac{p+p^0}{R(\theta_1+\theta^0)}\left((1-\phi(\omega))\frac{\partial p+p^0}{\partial x} - \frac{g(p+p^0)}{R(\theta_1+\theta^0)}\right)\right),$$

(26)

$$\frac{\partial\omega}{\partial t} = \xi_1(\theta_1+\theta^0)((p+p^0) - \int_0^1\left(\frac{a_1(\phi(\omega))}{1-\phi(\omega)}\xi_1(\theta_1+\theta^0)((p+p^0) + \int_0^x\frac{\rho_{tot}(\omega)g}{1-\phi}d\xi)\right)dx \cdot$$

$$\cdot\left(\int_0^1\frac{a_1(\phi(\omega))}{1-\phi(\omega)}\xi_1(\theta_1+\theta^0)dx\right)^{-1} + \int_0^x(\frac{p+p^0}{R(\theta_1+\theta^0)}\frac{\phi(\omega)}{1-\phi(\omega)} + \rho_s)gd\xi),$$

$$(\frac{c_f(p+p^0)}{R(\theta_1+\theta^0)}\frac{\phi}{1-\phi}+c_s\rho_s)\frac{\partial\theta_1}{\partial t}=\frac{\partial}{\partial x}(\lambda(1-\phi)\frac{\partial(\theta_1+\theta^0)}{\partial x})+$$

(27)

$$+\frac{k(\phi)}{\mu(\theta_1+\theta^0)}\frac{c_f(p+p^0)}{R(\theta_1+\theta^0)}((1-\phi)\frac{\partial(p+p^0)}{\partial x}-\frac{g(p+p^0)}{R(\theta_1+\theta^0)})\frac{\partial(\theta_1+\theta^0)}{\partial x}.$$

For the Banach space, we choose the space $C^{2+\beta,1+\beta/2}(\overline{Q}_{t_0})$, where β is any number from the interval $(0,\alpha)$, $\alpha\in[0,1)$. Let

$$V=\{\bar{p}(x,t),\bar{\omega}(x,t),\bar{\theta}\in C^{2+\alpha,1+\alpha/2}(\overline{Q}_{t_0})|$$

$$\bar{p}\mid_{t=0}=\bar{\omega}\mid_{t=0}=\bar{\theta}\mid_{t=0}=\frac{\partial\bar{\theta}}{\partial x}\mid_{x=0,x=1}=((1-\phi(\bar{\omega}))\frac{\partial(\bar{\rho}+\rho^0)}{\partial x}-(\bar{\rho}+\rho^0)g)\mid_{x=0,x=1}=0,$$

$$m_1^*-p^0(x)\le\bar{p}(x,t)\le M_1^*-p^0(x)<\infty,$$

$$m_1^*=\frac{m_1}{2}(1+\frac{g_0}{Rm_2(1-M_0)})^{-1},M_1^*=2M_1(1+\frac{g_0}{Rm_2(1-M_0)})^{-1},m_2-\theta^0\le\bar{\theta}\le M_2-\theta^0,$$

$$G(\frac{m_0}{2})-G(\phi^0)\le\bar{\omega}(x,t)\le G(\frac{M_0+1}{2})-G(\phi^0)<\infty,\quad(x,t)\in Q_{t_0},$$

$$(|\bar{\omega}|_{1+\alpha,(1+\alpha)/2,Q_{t_0}},|\bar{p}|_{1+\alpha,(1+\alpha)/2,Q_{t_0}},|\bar{\theta}|_{1+\alpha,(1+\alpha)/2,Q_{t_0}})\le K_1,$$

$$(|\bar{\omega}|_{2+\alpha,(2+\alpha)/2,Q_{t_0}},|\bar{p}|_{2+\alpha,(2+\alpha)/2,Q_{t_0}},|\bar{\theta}|_{2+\alpha,(2+\alpha)/2,Q_{t_0}})\le K_1+K_2\},$$

where K_1 is an arbitrary positive constant, while the positive constant K_2 will be given later. We note that on the set V following inequalities hold: $0<m_0/2\le\phi(\bar{\omega})\le(M_0+1)/2<1$.

Let us construct an operator Λ mapping V in V. Suppose that $\bar{\omega},\bar{p},\bar{\theta}\in V$. Using (26), we define the function ω by the equality

$$\omega=\int_0^t\xi_1(\bar{\theta}+\theta^0)((\bar{p}(x,\tau)+p^0(x))-\int_0^1(\frac{a_1(\phi(\bar{\omega}))}{1-\phi(\bar{\omega})}\xi_1(\bar{\theta}+\theta^0)((\bar{p}+p^0)+$$

(28)
$$+\int_0^x(\frac{\bar{p}+p^0}{R(\bar{\theta}+\theta^0)}\frac{\phi(\bar{\omega})}{1-\phi(\bar{\omega})}+\rho_s)g)d\xi))dx(\int_0^1\frac{a_1(\phi(\omega))}{1-\phi(\omega)}\xi_1(\theta_1+\theta^0)dx)^{-1}+$$

$$+\int_0^x g(\rho_s+\frac{\bar{p}+p^0(\xi)}{R(\bar{\theta}+\theta^0)}\frac{\phi(\bar{\omega})}{1-\phi(\bar{\omega})})d\xi)d\tau.$$

From the representation (28) it follows that smoothness ω is determined by the smoothness of functions $\bar{p},\bar{\omega},\bar{\theta},\bar{\theta}^0,p^0,P^0$ and g. Therefor there exists a value $t_1=t_1(m_0,M_0,m_1,M_1,m_2,M_2)$, such that for all $t_0\le t_1$ the following inequality holds

(29)
$$0<\frac{m_0}{2}\le\phi(x,t)\le\frac{M_0+1}{2},\quad(x,t)\in Q_{t_0}.$$

In particular, we have an estimate

$$|\omega|_{2+\alpha,1+\alpha/2,Q_{t_0}}\le C_0(m_0,M_0,m_1,M_1,K_1,m_2,M_2,T,|g|_{1+\alpha,\Omega},|p^0|_{2+\alpha,\Omega},$$

$$|\theta^0|_{2+\alpha,\Omega},|\phi^0|_{2+\alpha,\Omega}|P^0|_{\alpha/2,[0,T]})(1+t_0(|\bar{\rho}_{xx}|_{\alpha,\alpha/2,\Omega}+|\bar{\theta}_{xx}|_{\alpha,\alpha/2,\Omega})).$$

Taking into account (29) we also have the estimate for function $\omega(x,t)$:

$$G(\frac{m_0}{2})\le\omega(x,t)+G(\phi^0)\le G(\frac{M_0+1}{2}).$$

Using (25), $\bar{p}, \bar{\theta}$ and $\bar{\omega}$, we find the function $p(x,t)$ as a solution of the problem (here and elsewhere, we assume that the initial and boundary conditions are matched):
(30)
$$\frac{\partial}{\partial t}\left(\frac{p+p^0}{R(\bar{\theta}+\theta^0)}\frac{\phi(\bar{\omega})}{1-\phi(\bar{\omega})}\right) = \frac{\partial}{\partial x}\left(\frac{k(\phi(\bar{\omega}))}{\mu(\bar{\theta}+\theta^0)}\frac{\bar{p}+p^0}{R(\bar{\theta}+\theta^0)}\left((1-\phi(\bar{\omega}))\frac{\partial p+p^0}{\partial x}-\frac{g(p+p^0)}{R(\bar{\theta}+\theta^0)}\right)\right),$$

$$p\mid_{t=0}=0, \quad \left((1-\phi(\bar{\omega}))\frac{\partial(p+p^0)}{\partial x}-\frac{g(p+p^0)}{R(\bar{\theta}+\theta^0)}\right)\mid_{x=0,x=1}=0.$$

The equation for $p(x,t)$ is uniformly parabolic. In view of the properties of $\bar{\omega}(x,t), \bar{\theta}(x,t)$ and $p^0(x), \theta^0(x)$ the problem (30) has a classical solution [41]. In addition, we have the following estimate:

$$\left|\frac{1}{a(\bar{\omega})}\frac{\partial a(\bar{\omega})}{\partial t}\right| \leq C_1(m_0, M_0, m_1, M_1, m_2, M_2, \max_{0\leq t\leq T}|p^0(t)|, \max_{0\leq t\leq T}|\theta^0(t)|).$$

Under the additional condition smallness for the value of the time interval the following statement holds.

Lemma 1. *There exists such a t_2, that when $t_0 \leq \min(t_1, t_2)$, the classical solution of problem (25) satisfies the following inequality in Q_{t_0}:*

$$0 < \hat{m}_1 \leq \rho(x,t) + \rho^0(x) \leq \hat{M}_1 < \infty.$$

Proof. Further, setting $U(x,t) = p(x,t) + p^0(x)$, we can express problem (30) in the form

$$\frac{\partial}{\partial t}(a(\bar{\omega}, \bar{\theta})U) = \frac{\partial}{\partial x}\left(K(\bar{\omega})b(\bar{p}, \bar{\theta})\frac{\partial U}{\partial x} - k(\bar{\omega})c(\bar{p}, \bar{\theta})U\right),$$
(31)
$$\left(\frac{\partial U}{\partial x} - \tilde{d}U\right)\mid_{x=0,x=1}=0, \quad U\mid_{t=0}=p^0,$$

Here $a(\omega, \theta) = \frac{\phi(\omega)}{1-\phi(\omega)}\frac{1}{R(\theta+\theta^0)}, b(p,\theta) = \frac{p+p^0}{\mu(\theta+\theta^0)R(\theta+\theta^0)}, c(p,\theta) = \frac{g(p+p^0)}{\mu(\theta+\theta^0)R^2(\theta+\theta^0)^2}$, $\tilde{d}(\omega, \theta) = \frac{g}{(1-\phi(\omega))(\theta+\theta^0)R}, K(\omega) = k(\phi(\omega))(1-\phi(\omega)), \phi(\omega) = G^{-1}(\omega + G(\phi^0))$.

First, we show that $U(x,t) \geq 0, (x,t) \in Q_{t_0}$. In equation (31), let us make the change $U(x,t) = -z(x,t)$. Then

$$z\frac{\partial a}{\partial t} + a\frac{\partial z}{\partial t} = \frac{\partial}{\partial x}\left(Kb\frac{\partial z}{\partial x} - kcz\right).$$

Let

$$z^{(0)}(x,t) = \max\{z, 0\}, \quad z^{(0)}(x,t)\mid_{t=0}=\max\{-\rho^0, 0\} = 0,$$

$$\sigma_\varepsilon(x,t) = z^{(0)}(x,t)(|z^{(0)}(x,t)|^2 + \varepsilon)^{-1/2}, \varepsilon > 0.$$

Let us multiply the equation for the function z by σ_ε and then integrate over Ω. We obtain the equality

$$\frac{d}{dt}\int_0^1 a(|z^{(0)}|^2 + \varepsilon)^{1/2}dx + \int_0^1 \frac{\partial a}{\partial t}(z\sigma_\varepsilon - (|z^{(0)}|^2 + \varepsilon)^{1/2})dx +$$
(32)
$$+\varepsilon\int_0^1\left(Kb\frac{\partial z}{\partial x}\frac{\partial z^{(0)}}{\partial x}(|z^{(0)}|^2 + \varepsilon)^{-3/2} - kcz\frac{\partial z^{(0)}}{\partial x}(|z^{(0)}|^2 + \varepsilon)^{-3/2}\right)dx = 0.$$

Let

$$A^+(t) = \{x \in \Omega \mid z(x,t) > 0\}, \quad A^-(t) = \{x \in \Omega \mid z(x,t) \leq 0\}.$$

Following [42], we get estimates for each term.

$$\int\limits_0^1 \frac{\partial a}{\partial t}(z\sigma_\varepsilon - (|z^{(0)}|^2 + \varepsilon)^{1/2})dx = -\varepsilon \int\limits_{A^+(t)} \frac{\partial a}{\partial t}(|z|^2 + \varepsilon)^{-1/2}dx - \varepsilon^{1/2} \int\limits_{A^-(t)} \frac{\partial a}{\partial t}dx,$$

$$\int\limits_0^1 a(|z^{(0)}|^2 + \varepsilon)^{1/2}dx = \int\limits_{A^+(t)} a(|z|^2 + \varepsilon)^{1/2}dx + \varepsilon^{1/2} \int\limits_{A^-(t)} adx,$$

$$\int\limits_0^1 a(|z^{(0)}|^2 + \varepsilon)^{1/2}|_{t=0}\, dx = \varepsilon^{1/2} \int\limits_0^1 a\,|_{t=0}\, dx, \quad \int\limits_{A^+(t)} a(|z|^2 + \varepsilon)^{1/2}dx \geq \int\limits_{A^+(t)} a|z|dx = \int\limits_0^1 az^{(0)}dx.$$

Last term can be estimated as following:

$$\int\limits_0^1 \varepsilon kcz \frac{\partial z^{(0)}}{\partial x}(|z^{(0)}|^2 + \varepsilon)^{-3/2}dx \leq \frac{\varepsilon}{2} \int\limits_{A^+(t)} (\frac{\partial z^{(0)}}{\partial x})^2(|z^{(0)}|^2 + \varepsilon)^{-3/2}Kbdx+$$

$$+\frac{\varepsilon}{2} \int\limits_{A^+(t)} (\frac{(kcz)^2}{Kb})^2(|z^{(0)}|^2 + \varepsilon)^{-3/2}Kbdx$$

We have

$$\varepsilon z^2(|z^{(0)}|^2 + \varepsilon)^{-3/2} \leq \varepsilon(|z^{(0)}|^2 + \varepsilon)^{-3/2} \leq \sqrt{\varepsilon} \longrightarrow 0, \quad if \quad \varepsilon \to 0.$$

Integrating relation (32) with respect to time, we obtain

$$\int\limits_{A^+(t)} a(|z|^2 + \varepsilon)^{1/2}dx + \varepsilon^{1/2} \int\limits_{A^-(t)} adx + \frac{\varepsilon}{2} \int\limits_0^t \int\limits_{A^+(\tau)} Kb|\frac{\partial z}{\partial x}|^2(z^2 + \varepsilon)^{-3/2}dxd\tau =$$

$$= \varepsilon \int\limits_0^t \int\limits_{A^+(\tau)} \frac{\partial a}{\partial \tau}(|z|^2 + \varepsilon)^{-1/2}dxd\tau + \varepsilon^{1/2} \int\limits_0^t \int\limits_{A^-(\tau)} \frac{\partial a}{\partial \tau}dxd\tau + \varepsilon^{1/2} \int\limits_0^1 a\,|_{t=0}\, dx+$$

$$+\frac{\sqrt{\varepsilon}}{2} \int\limits_0^t \int\limits_{A^+(\tau)} \frac{(kc)^2}{Kb}dx.$$

Following [42], we get estimate passing to the limit as $\varepsilon \to 0$, we find that $z^{(0)} = 0$, i.e. $U \geq 0$.

We can express problem (31) in the form:

$$(33) \qquad U_t - \tilde{a}_{11}U_{xx} + \tilde{a}_1 U_x + \tilde{a}U = 0, \quad (U_x - \tilde{d}U)|_{x=0,1} = 0, \quad U\,|_{t=0} = p^0,$$

where

$$\tilde{a}_{11} = \frac{Kb}{a}, \quad \tilde{a}_1 = \frac{d - (Kb)_x}{a}, \quad \tilde{a} = \frac{a_t + (kc)_x}{a}.$$

Following [41], we move from function $U(x,t)$ to a new function $w(x,t)$ related to it by equality $w(x,t) = e^{-\lambda t}\varphi(x)U(x,t)$, where

$$\varphi = -mx^2 + mx + 1 > 0, m \equiv 2\max_{Q_t}|\tilde{d}| = \frac{4g_0}{(1 - M_0)Rm_2},$$

and number λ will be indicated later.

Function w because of (32) is the solution of the equation

$$w_t - \tilde{a}_{11}w_{xx} + (\tilde{a}_1 + \frac{2\tilde{a}_{11}\varphi_x}{\varphi})w_x + (-2\tilde{a}_{11}\frac{\varphi_x^2}{\varphi^2} + \tilde{a}_{11}\frac{\varphi_{xx}}{\varphi} - \tilde{a}_1\frac{\varphi_x}{\varphi} + \tilde{a} + \lambda)w = 0,$$

$$w_x|_{x=0,1} = ((\frac{\varphi_x}{\varphi} + \tilde{d})w)|_{x=0,1}, \quad w|_{t=0} = \varphi(x)p^0,$$

where $w_x|_{x=0} \geq 0$, $w_x|_{x=1} \leq 0$, because $\varphi|_{x=0,1} = 1$, $\varphi_x|_{x=0} = m > 0$, $\varphi_x|_{x=1} = -m \equiv -2\max\tilde{d} < 0$. Choosing

$$\lambda > \max_{Q_t}[2\tilde{a}_{11}\frac{\varphi_x^2}{\varphi^2} - \tilde{a}_{11}\frac{\varphi_{xx}}{\varphi} + \tilde{a}_1\frac{\varphi_x}{\varphi} - \tilde{a}],$$

the function w reaches a positive maximum at $t = 0$, such

$$U(x,t)e^{-\lambda t}\varphi(x) \leq \max_{Q_t}(U(x,t)e^{-\lambda t}\varphi(x)) = \max_{Q_t} w(x,t) \leq$$

$$\leq \max_x w|_{t=0} = \max_x(U(x,t)e^{-\lambda t}\varphi(x))|_{t=0}.$$

Therefore, we obtain upper bound for U:

$$U \leq e^{\lambda t}M_1(1 + \frac{g_0}{(1-M_0)Rm_2}).$$

Then there is a value $\tilde{t}_2 = \ln 2^{1/\lambda}$, that for all $t \leq \tilde{t}_2$ we have the estimate for ρ from above from Lemma 1.

To obtain a lower estimate we represent equation (31) in the form ($z(x,t) = 1/U(x,t)$)

$$z_t - \tilde{a}_{11}z_{xx} + \frac{2\tilde{a}_{11}}{z}(z_x^2) + \tilde{a}_1 z_x - \tilde{a}z = 0.$$

We move from function $z(x,t)$ to a new function $w_1(x,t)$ related to it by equality $w_1(x,t) = e^{-\lambda_1 t}\varphi(x)z(x,t)$, where φ defined as before for the upper estimate and number λ_1 will be indicated later.

Function w_1 is the solution of the problem

$$w_{1t} - \tilde{a}_{11}w_{1xx} + (\tilde{a}_1 + \frac{2\tilde{a}_{11}\varphi_x}{\varphi} - 4\tilde{a}_{11}\frac{\varphi_x}{\varphi^2})w_{1x} + (-2\tilde{a}_{11}\frac{\varphi_x^2}{\varphi^2} + \tilde{a}_{11}\frac{\varphi_{xx}}{\varphi} - \tilde{a}_1\frac{\varphi_x}{\varphi} +$$

$$+2\tilde{a}_{11}\frac{\varphi_x^2}{\varphi^3} - \tilde{a} + \lambda_1)w_1 + 2\tilde{a}_{11}\frac{w_{1x}^2}{\varphi^2 w} = 0,$$

$$w_{1x}|_{x=0,1} = ((\frac{\varphi_x}{\varphi} - \tilde{d})w_1)|_{x=0,1}, \quad w_1|_{t=0} = \varphi(x)\frac{1}{p^0},$$

where $w_{1x}|_{x=0} \geq 0$, $w_{1x}|_{x=1} \leq 0$, because $\varphi|_{x=0,1} = 1$, $\varphi_x|_{x=0} = m > 0$, $\varphi_x|_{x=1} = -m \equiv -2\max\tilde{d} < 0$. Choosing

$$\lambda_1 > \max_{Q_t}[2\tilde{a}_{11}\frac{\varphi_x^2}{\varphi^2} - \tilde{a}_{11}\frac{\varphi_{xx}}{\varphi} + \tilde{a}_1\frac{\varphi_x}{\varphi} - 2\tilde{a}_{11}\frac{\varphi_x^2}{\varphi^3} + \tilde{a}],$$

the function w_1 reaches a positive maximum at $t = 0$, such

$$\max_{Q_t}(\frac{1}{U(x,t)}e^{-\lambda_1 t}\varphi(x)) = \max_{Q_t} w_1(x,t) \leq \max_x w_1|_{t=0} = \max_x(\frac{1}{U(x,t)}e^{-\lambda_1 t}\varphi(x))|_{t=0}.$$

Accordingly, we obtain

$$\frac{1}{U(x,t)} \leq \frac{1}{m_1}e^{\lambda_1 t}(1 + \frac{g_0}{R(1-M_0)m_2}).$$

Therefore
$$U(x,t) \geq m_1 e^{-\lambda_1 t}(1 + \frac{g_0}{R(1 - M_0)m_2})^{-1}.$$

Then there exists a value $\tilde{t}_3 = \ln 2^{1/\lambda_1}$, that for all $t \leq \tilde{t}_3$ we have the estimate for ρ from above from Lemma 1. □

Choosing $t \leq t_2 = \min\{\tilde{t}_2, \tilde{t}_3\}$, we proof Lemma1.

Using (27), $\bar{p}, \bar{\theta}$ and $\bar{\omega}$, we find the function $\theta_1(x,t)$ as a solution of the problem (here and elsewhere, we assume that the initial and boundary conditions are matched):

(34)
$$(\frac{c_f(\bar{p}+p^0)}{R(\bar{\theta}+\theta^0)} \frac{\phi(\bar{\omega})}{1-\phi(\bar{\omega})} + c_s \rho_s)\frac{\partial \theta_1}{\partial t} = \frac{\partial}{\partial x}(\lambda(1 - \phi(\bar{\omega}))\frac{\partial(\theta_1+\theta^0)}{\partial x})+$$

$$+k(\phi(\bar{\omega}))\frac{c_f(\bar{p}+p^0)}{R(\bar{\theta}+\theta^0)}((1 - \phi(\bar{\omega}))\frac{\partial(\bar{p}+p^0)}{\partial x} - \frac{g(\bar{p}+p^0)}{R(\bar{\theta}+\theta^0)})\frac{\partial(\theta_1+\theta^0)}{\partial x},$$

$$\theta_1|_{t=0} = 0, \frac{\partial \theta_1}{\partial x}|_{x=0,x=1} = 0.$$

Further, setting $\tilde{U} = \theta_1 + \theta^0$ in (34), we have classical maximum principle for function $\tilde{U}$: [41] $m_2 = \min_{x\in[0,1]} \theta^0 \leq \tilde{U} \leq \max_{x\in[0,1]} \theta^0 = M_2$. Therefore $m_2 - \theta^0 \leq \theta_1 \leq M_2 - \theta^0$, and we have following estimates for function θ_1 :

$$|\theta_1|_{\alpha,\alpha/2,Q_{t_0}} \leq C_2(m_0, M_0, m_1, M_1, m_2, M_2, K_1),$$

$$|\theta_1|_{2+\alpha,1+\alpha/2,Q_{t_0}} \leq C_3(m_0, M_0, m_1, M_1, m_2, M_2, K_1, |\theta^0|_{2+\alpha,\omega}, |\phi^0|_{2+\alpha,\omega}, |\rho^0|_{2+\alpha,\omega}) \cdot$$
$$\cdot(1 + |\theta_{1x}|_{\alpha,\alpha/2,Q_{t_0}}).$$

Using compact attachments $C^{2+\alpha,1+\alpha/2}(Q_{t_0})$ in $C^{1+\alpha,\alpha/2}(Q_{t_0})$, we get

$$|\theta_{1x}|_{\alpha,\alpha/2,Q_{t_0}} \leq \varepsilon|\theta_1|_{2+\alpha,1+\alpha/2,Q_{t_0}} + C_\varepsilon|\theta_1|_{\alpha,\alpha/2,Q_{t_0}}.$$

Then
$$|\theta_1|_{2+\alpha,1+\alpha/2,Q_{t_0}} \leq C_3(1 + \varepsilon|\theta_1|_{2+\alpha,1+\alpha/2,Q_{t_0}} + C_\varepsilon C_5).$$

By choosing ε we get
$$|\theta_1|_{2+\alpha,1+\alpha/2,Q_{t_0}} \leq C_4(m_0, M_0, m_1, M_1, m_2, M_2, K_1).$$

In view of Lemma 1 and the properties of $\bar{\omega}$, we have the following estimates [41]:
$$|\rho|_{\alpha,\alpha/2,Q_{t_0}} \leq C_5,$$

$$|\rho|_{2+\alpha,1+\alpha/2,Q_{t_0}} \leq C_6\left(1 + |\rho^0|_{2+\alpha,\Omega} + |\bar{\rho}_x|_{\alpha,\alpha/2,Q_{t_0}} + |\bar{\omega}_t|_{\alpha,\alpha/2,Q_{t_0}} + |\bar{\omega}_x|_{\alpha,\alpha/2,Q_{t_0}}\right),$$

in which the constants C_5, C_6 depend on $K_1, m_0, m_1, m_2, M_0, M_1, M_2$. Therefore
$$|\rho|_{2+\alpha,1+\alpha/2,Q_{t_0}} \leq C_7(K_1, m_0, m_1, m_2, M_0, M_1, M_2).$$

Let $C_8 = max\{C_0, C_4, C_7\}$. Choose K_2 so that $C_8 \leq (K_1 + K_2)/2$. Then, for $t_0 < \min(t_1, t_2, (K_1 + K_2)^{-1})$ we obtain

$$|\rho|_{2+\alpha,1+\alpha/2,Q_{t_0}} \leq K_1 + K_2, \quad |\omega|_{2+\alpha,1+\alpha/2,Q_{t_0}} \leq K_1 + K_2.$$

It remains to verify conditions

$$|\rho|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \leq K_1, \quad |\omega|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \leq K_1, |\theta|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \leq K_1,.$$

Integrating equation (30), (33) with respect to time, we obtain $|\rho|_{0,Q_{t_0}} \leq C_6 t_0, |\theta|_{0,Q_{t_0}} \leq C_6 t_0$. From the equation (28) we obtain $|\omega|_{0,Q_{t_0}} \leq C_7 t_0$. Further, using for ρ, ω, θ an inequality of the form [3]

$$|u|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \leq C|u|_{2+\alpha,1+\alpha/2,Q_{t_0}}^c |u|_{0,Q_{t_0}}^{1-c}, \quad c = (1+\alpha)(2+\alpha)^{-1},$$

we find that there exists a sufficiently small value of t_0, depending on K_1 and K_2, such that the required estimates hold: $|\rho|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \leq K_1, |\omega|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \leq K_1, |\theta|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \leq K_1$.

Thus, the operator Λ maps the set V into itself for sufficiently small values of t_0. Using the estimates obtained above, we can easily show the continuity of the operator Λ in the norm of the space $C^{2+\beta,1+\beta/2}(\overline{Q}_{t_0})$. By the Tikhonov-Schauder theorem, there exists a fixed point $(\rho, \omega) \in V$ of the operator Λ.

Uniqueness is established in the standard way [43].

Theorem 1 is proved. □

4. The Case of Incompressible Medium

Definition 2. *By a solution of problem (9)–(14) we mean the set of functions $\phi, \phi_t, \theta, v_s, v_f \in C^{2+\alpha,1+\beta}(Q_T), p_f, p_s \in C^{1+\alpha,1+\beta}(Q_T)$, such that $0 < \phi < 1$, $0 < \theta < \infty$. These functions satisfy the equations (9)–(13) and the initial and boundary conditions (14) and regarded as continuous functions in Q_T.*

Theorem 2. *Suppose that $\rho_f = const > 0$ and the data of problem (9)–(14) satisfies the following conditions: 1) the functions $k(\phi), a_1(\phi), \lambda(\phi), \xi_1(\theta)$ and their derivatives up to the second order are continuous for $\phi \in (0,1)$, $\theta \in (0,\infty)$ and satisfy the conditions*

$$k_0^{-1}\phi^{q_1}(1-\phi)^{q_2} \leq k(\phi) \leq k_0\phi^{q_3}(1-\phi)^{q_4},$$

$$k_0^{-1}\phi^{q_5}(1-\phi)^{q_6} \leq \lambda(\phi) \leq k_0\phi^{q_7}(1-\phi)^{q_8}, \xi_1(\theta) > 0, \theta \in (0,\infty),$$

$$a_1(\phi) = a_0(\phi)\phi^{\alpha_1}(1-\phi)^{\alpha_2-1}, \quad 0 < R_1 \leq a_0(\phi) \leq R_2 < \infty,$$

where $k_0, \alpha_i, R_i, i = 1,2$ are positive constants, $q_1,...,q_8$ are fixed real numbers, $\mu(\theta) = const$. 2) the function g, the initial functions ϕ^0 and θ^0 satisfy the following smoothness conditions:

$$g \in C^{1+\alpha,1+\beta}(\bar{Q}_T), \quad \theta^0, \phi^0 \in C^{2+\alpha}(\bar{\Omega}),$$

and the inequalities

$$0 < m_0 \leq \phi^0(x) \leq M_0 < 1, \quad 0 < m \leq \theta^0(x) \leq M < \infty, \quad |g(x,t)| \leq g_0 < \infty,$$

$$x \in \bar{\Omega}, \quad t \in (0,T),$$

where m_0, M_0, m, M, g_0 are given positive constants. Then problem (9)–(14) has a local solution, i.e., there exists a value of t_0 such that

$$\phi(x,t), \phi_t(x,t), \theta(x,t) \in C^{2+\alpha,1+\beta}(\bar{Q}_{t_0}), (v_s(x,t), v_f(x,t)) \in C^{2+\alpha,\beta}(\bar{Q}_{t_0}),$$

$$(p_f(x,t), p_s(x,t)) \in C^{1+\alpha,\beta}(\bar{Q}_{t_0}).$$

Moreover, $0 < \phi(x,t) < 1$, $0 < \theta(x,t) < \infty$ in $\bar{Q}_{t_0}$.

Theorem 3. *Let, in addition to the conditions of Theorem 2, the functions $k(\phi), \xi(\phi, \theta)$ satisfy the conditions*

$$k(\phi) = \frac{K}{\mu}\phi, \quad a_1(\phi) = \phi^4,$$

where K, μ are positive constants. Then for all $t \in [0, T], T < \infty$ uniqueness solution of problem (9)–(14) exists, and there are numbers $0 < m_1 < M_1 < 1, 0 < m_2 < M_2$ such that $m_1 \leq \phi(x, t) \leq M_1, m_2 \leq \theta(x, t) \leq M_2, (x, t) \in Q_T$.

2. LOCAL SOLVABILITY

We first prove the local theorem.

Proof. When proving Theorems 2 and 3, it is convenient to use the Lagrange variables [39]. Suppose that $\bar{x} = \bar{x}(\tau, x, t)$ is a solution of the Cauchy problem

$$\frac{\partial \bar{x}}{\partial \tau} = v_s(\bar{x}, \tau), \quad \bar{x}\mid_{\tau=t} = x.$$

We set $\hat{x} = \bar{x}(0, x, t)$ and take $\hat{x}$ and t for the new variables. Then $\hat{J}(\hat{x}, t) = \frac{\partial \hat{x}}{\partial x}(x, t) = (1 - \phi(\hat{x}, t))/(1 - \phi^0(\hat{x}))$ is the Jacobian of the transformation. Following [43], we rewrite the system (9)–(13):

$$(35) \quad \frac{\partial}{\partial t}\left(\frac{\phi}{1-\phi}\right) = \frac{\partial}{\partial x}\left(k(\phi)(1-\phi)\frac{\partial}{\partial x}\left(\frac{1}{\xi_1(\theta)}\frac{\partial G(\phi)}{\partial t}\right) - k(\phi)g(\rho_{tot} + \rho_f)\right),$$

$$(36) \quad \left((1-\phi)\frac{\partial}{\partial x}\left(\frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t}\right) - g(\rho_{tot} + \rho_f)\right)\mid_{x=0, x=1} = 0, \quad \phi\mid_{t=0} = \phi^0(x),$$

$$(37) \quad \left(c_s\rho_s + c_f\rho_f\frac{\phi}{1-\phi}\right)\frac{\partial \theta}{\partial t} + c_f\rho_f\phi(v_f - v_s)\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\left(\lambda(1-\phi)\frac{\partial \theta}{\partial x}\right),$$

$$(38) \quad \frac{\partial \theta}{\partial x}\mid_{x=0, x=1} = 0, \quad \theta\mid_{t=0} = \theta^0(x),$$

$$(39) \quad \frac{\partial G(\phi)}{\partial t} = \xi_1(\theta)p_e, \quad \frac{dG}{d\phi} = \frac{1}{a_1(\phi)(1-\phi)}.$$

In the system (35) - (39), the basic equations are (35) and (37) for the required functions ϕ and θ.

We substitute in the coefficients of the equation (35) and the boundary condition (36) instead of $\theta(x, t)$ an arbitrary smooth function $\theta_0(x, t) \in C^{2+\alpha_1, 1+\beta_1}(Q_T)$, which satisfies the inequalities $0 < m \leq \theta^0(x) \leq M < \infty$. We retain the previous notation ϕ for solving the arising problem and the latter is called Problem I.

Lemma 2. *Let the data of problem I satisfy the conditions of the theorem. Then problem I has a unique local solution, i.e., there exists a value of t_0 such that*

$$(\phi, \phi_t) \in C^{2+\alpha, 1+\beta}(Q_{t_0}), \phi \in (0, 1).$$

Proof. Suppose that $z = \frac{1}{\xi_1(\theta_0)}\frac{\partial G}{\partial t}$, we arrive at the following problem for G, z :

$$(40) \qquad z = \frac{1}{\xi_1(\theta_0)}\frac{\partial G}{\partial t}, \quad G\mid_{t=0} = G(\phi^0) = G^0(x),$$

$$(41) \quad \frac{z}{d(G,\theta_0)} - \frac{\partial}{\partial x}\left(a(G)\frac{\partial z}{\partial x} - b(G)\right) = 0, \quad \left(a(G)\frac{\partial z}{\partial x} - b(G)\right)\mid_{x=0,x=1} = 0,$$

where

$$d(G,\theta_0))) = \frac{1-\phi(G)}{a_1(\phi(G))\xi_1(\theta_0)}, \quad a(G) = k(\phi(G))(1-\phi(G)), \quad b(G) = k(\phi(G))g(\rho_{tot}+\rho_f).$$

Since $0 < m_0 \le \phi^0(x) \le M_0 < 1$ and the function $G(\phi)$ is monotone, then $G(m_0) \le G^0(x) \le G(M_0)$. From (40) when the inequality $\max_{(x,t)}|\xi_1(\theta)z(x,t)| \le c_0$ we have that there is a value t_0, such that for all $t \le t_0$ the estimates take place

$$G_1(m_0) = G(m_0) - c_0 t_0 \le G(x,t) \le G(M_0) + c_0 t_0 = G_2(M_0),$$
(42)
$$0 \le G^{-1}(G_1(m_0)) \le \phi(x,t) \le G^{-1}(G_2(M_0)) < 1.$$

Let $G_0(x,t)$ be a function continuous in x and t, satisfying inequalities (42) and having a continuous derivative $\partial G_0/\partial x$ with respect to x, t. Substituting $G_0(x,t)$ instead of $G(x,t)$ into the coefficients of the equation (41) and the boundary conditions, we arrive at a linear problem for z, in which $a > 0, b > 0$ and $d > 0$. The solution to this problem is unique. Existence follows, for example, from Hilbert's theorem [44] for ordinary linear equations of the second order. The t variable plays the role of a parameter. Thus, $(z, z_x, z_{xx}) \in C(Q_{t_0})$. After finding $z(x,t)$, we can find a new value $G(x,t)$ from the equation (40). This value will satisfy the condition (42).

To prove the solvability of problem I, we use the method of successive approximations. Let $z^i(x,t)$ and $G^i(x,t)$ be a solution to the problem

$$\frac{\partial G^{i+1}}{\partial t} = \xi_1(\theta_0)z^{i+1}, \quad G^{i+1}(x,0) = G^0(x),$$

$$\frac{z^{i+1}}{d(G^i)} - \frac{\partial}{\partial x}\left(a(G^i)\frac{\partial z^{i+1}}{\partial x} - b(G^i)\right) = 0,$$

$$\left(a(G^i)\frac{\partial z^{i+1}}{\partial x} - b(G^i)\right)\mid_{x=0,x=1} = 0,$$

where $i = 0, 1, 2, ...$ Substituting $G^0(x)$ into the equation for z at the first step, we find $z^1(x,t)$. After that, from the equation for G we find $G^1(x,t)$, etc. For each i there is a unique solution $z^i(x,t)$ and $G^i(x,t)$, satisfying (42). It is checked in a standard way that for a small value of t_0 the solutions $z^i(x,t)$, $G^i(x,t)$ and their derivatives up to the second order inclusive are bounded uniformly in i.

We put $y^{i+1} = z^{i+1} - z^i, \omega^{i+1} = G^{i+1} - G^i$. We have

$$\frac{\partial \omega^{i+1}}{\partial t} = \xi_1(\theta_0)y^{i+1}, \quad \omega^{i+1}\mid_{t=0} = 0,$$

$$\frac{y^{i+1}}{d(G^i)} + A_1\omega^i - \frac{\partial}{\partial x}\left(ay_x^{i+1} + A_2\omega^i\right) = 0,$$

$$(ay_x^{i+1} + A_2\omega^i)|_{x=0,x=1} = 0,$$

where the coefficients A_1, A_2 are easily recovered and are limited. We have following inequalities

$$\int_0^1 (|y^{i+1}|^2 + |y_x^{i+1}|^2)dx \le c_1 \int_0^1 |\omega^i|^2 dx \le c_1 \max_x |\omega^i|^2,$$

$$\max_x |\omega^{i+1}| \le c_1 \int_0^t \max_x |y^{i+1}|d\tau,$$

where the constant c_1 does not depend on i. Taking into account the last inequality for the function $v^i(t) = \max_x |y^i(x,t)|^2$ we get $v^{i+1}(t) \le c_2 \int_0^t v^i(\tau)d\tau$ and therefore [41], $v^i(t) \le (c_2 T)^i v^0/i! \to 0$ for $i \to \infty$. After that it is easy to establish that the sequences z^i, G^i are fundamental in $C(Q_{t_0})$ and have limits $z(x,t) \in C(Q_{t_0})$ and $G(x,t) \in C(Q_{t_0})$. The sequences z_x^i, z_{xx}^i, G_t^i are also fundamental. Passing to the limit as $i \to \infty$, we obtain that the limit functions satisfy the problem (40), (41). The uniqueness of the solution is proved similarly to [45]. Increasing the smoothness of the initial data to those specified in the conditions of Theorem 2 allows us to obtain that $\phi(x,t), \phi_t(x,t) \in C^{2+\alpha,1+\beta}(\bar{Q}_{t_0})$.

Lemma 2 is proved.   □

Substituting $\theta_0(x,t)$ and the solution to Problem I into the coefficients of equation (37), we arrive at a linear problem for $\theta(x,t)$ of the form

$$Q\frac{\partial\theta}{\partial t} + V\frac{\partial\theta}{\partial x} = \frac{\partial}{\partial x}\left(\lambda(1-\phi)\frac{\partial\theta}{\partial x}\right),$$

$$\frac{\partial\theta}{\partial x}|_{x=0,x=1} = 0, \quad \theta|_{t=0} = \theta^0(x),$$

where

$$Q = \rho_s c_s + \rho_f c_f \frac{\phi}{1-\phi}, \quad V = c_f\rho_f\phi(v_f - v_s) = \rho_f c_f k(\phi)\left((1-\phi)\frac{\partial z}{\partial x} + g(\rho_{tot} + \rho_f)\right).$$

The unique solvability of this problem in Holder classes follows from [41], and the solution satisfies the estimate

$$0 < \underline{\theta} = \min_x \theta^0(x) \le \theta(x,t) \le \max_x \theta^0(x) = \bar{\theta} < \infty.$$

After these remarks, the local solvability of the problem (35) - (38) can easily be obtained using the Schauder theorem according to the scheme used in [45].

After finding ϕ, θ, the remaining functions from the system (9)–(13) can be defined as follows. We find the phase velocities from (9)

$$v_f(x,t) = -\frac{1}{\phi}\int_0^x \frac{\partial\phi}{\partial t}d\xi \in C^{2+\alpha,\beta}(Q_{t_0}),$$

$$v_s(x,t) = -\frac{1}{1-\phi}\int_0^x \frac{\partial(1-\phi)}{\partial t}d\xi \in C^{2+\alpha,\beta}(Q_{t_0}).$$

From (12) we find $p_{tot}(x,t) = p^0(t) - \int\limits_0^x \rho_{tot} g d\xi \in C^{3+\alpha,1+\beta}(Q_{t_0})$.

From (11) we have $p_e(x,t) = -\frac{\partial v_s}{\partial x}\xi(\phi,\theta) \in C^{1+\alpha,\beta}(Q_{t_0})$, then

$$p_f(x,t) = p_{tot} - p_e \in C^{1+\alpha,\beta}(Q_{t_0}), \quad p_s(x,t) = \frac{p_{tot}}{1-\phi} - \frac{\phi}{1-\phi}p_f \in C^{1+\alpha,\beta}(Q_{t_0}).$$

Theorem 2 is proved.

$\square$

3. GLOBAL SOLVABILITY

Now we will prove the Theorem 3.

Proof. By Theorem 2, we will assume that on the interval $[0,t_0]$ there exists a solution to the problem (9)–(14), and $0 < \phi(x,t) < 1$, $0 < \theta(x,t) < \infty$, $x \in \Omega$, $t \in [0, t_0]$. After obtaining the necessary a priori estimates that do not depend on the value of t_0, the local solution can be continued to the entire segment $[0, T]$.

Lemma 3. *Under the conditions of Theorem 3, for all $t \in [0,T]$ the following relations hold:*

$$(43) \qquad \int_0^1 s(x,t)dx = \int_0^1 s^0(x)dx, \quad s = \frac{\phi}{1-\phi}, \quad s^0 = s(x,0),$$

$$(44) \qquad 0 < \underline{\theta} \equiv \min_{x \in [0,1]} \theta^0(x) \leq \theta(x,t) \leq \max_{x \in [0,1]} \theta^0(x) \equiv \overline{\theta} < \infty,$$

$$\int_0^1 \frac{1}{\xi_1(\theta)}\frac{a_1}{1-\phi}\left(\frac{\partial G}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^1 k(\phi)(1-\phi)\left|\frac{\partial}{\partial x}\left(\frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t}\right)\right|^2 dx \leq$$

$$(45) \qquad\qquad\qquad \leq \frac{1}{2}\int_0^1 \frac{k(\phi)}{1-\phi}g^2(\rho_{tot} + \rho_f)^2 dx \leq N.$$

Hereinafter, N denotes a constant that depends only on the data of the problem (9)–(14) and does not depend on t_0.

Proof. Let us integrate the equation (35) over x from 0 to 1 and take into account the boundary condition (36). After integration over time from 0 to the current value of t, we arrive at the equality (43).

The equation (37) is written in a divergent form:

$$\frac{\partial}{\partial t}\left(\theta(c_s\rho_s + c_f\rho_f\frac{\phi}{1-\phi})\right) + \frac{\partial}{\partial x}\left(\theta c_f\rho_f\phi(v_f - v_s) - \lambda(1-\phi)\frac{\partial\theta}{\partial x}\right) =$$

$$(46) \qquad\qquad = \theta\left[\frac{\partial}{\partial t}\left(c_s\rho_s + c_f\rho_f\frac{\phi}{1-\phi}\right) + \frac{\partial}{\partial x}(c_f\rho_f\phi(v_f - v_s))\right].$$

The right-hand side of this equality is equal to zero, since the second equation from (9) in Lagrange variables becomes [43]

$$\frac{\partial}{\partial t}\left(\frac{\phi}{1-\phi}\right) + \frac{\partial}{\partial x}(\phi(v_f - v_s)) = 0.$$

In particular, from (46) we have

$$\int_0^1 \left(c_f \rho_f \frac{\phi}{1-\phi} + c_s \rho_s \right) \theta dx = \int_0^1 \left(c_f \rho_f \frac{\phi^0}{1-\phi^0} + c_s \rho_s \right) \theta^0 dx,$$

and therefore $\theta(x,t) \in L_1[0,1]$ for all $t \in [0,T]$.

Let the smooth function $\kappa(\theta)$ satisfy the condition $\kappa''(\theta) = d^2\kappa/d\theta^2 \geq 0$. Multiplying the equation (37) by $\kappa'(\theta) \equiv d\kappa/d\theta$, and following the equality (46) we reduce the resulting equality to the form

$$\frac{\partial}{\partial t}\left(\left(c_s \rho_s + c_f \rho_f \frac{\phi}{1-\phi} \right) \kappa(\theta) \right) + \frac{\partial}{\partial x}(c_f \rho_f \phi (v_f - v_s)\kappa(\theta)) =$$

(47)
$$= \frac{\partial}{\partial x}\left(\lambda(1-\phi)\frac{\partial\kappa(\theta)}{\partial x} \right) - \kappa''(\theta)\left(\frac{\partial\theta}{\partial x} \right)^2 \lambda(1-\phi).$$

In the case $\kappa(\theta) = \theta^p, p > 1$, from (47) we deduce

$$\int_0^1 \theta^p(x,t)dx \leq \max_{x\in[0,1]} \left(\frac{c_f\rho_f}{c_s\rho_s} \frac{\phi^0(x)}{1-\phi^0(x)} + 1 \right) \int_0^1 |\theta^0(x)|^p dx.$$

Whence, in the standard way, we get that $\theta(x,t) \leq \max_{x\in[0,1]} \theta^0(x)$ for all $t \in [0,T], x \in [0,1]$. Put $\theta_1 = 1/\theta$ and the equation (35) can be represented as

$$\left(c_s \rho_s + c_f \rho_f \frac{\phi}{1-\phi} \right) \frac{\partial\theta_1}{\partial t} + c_f \rho_f (v_f - v_s)\frac{\partial\theta_1}{\partial x} =$$

$$= \frac{\partial}{\partial x}\left(\lambda(1-\phi)\frac{\partial\theta_1}{\partial x} \right) - 2\lambda(1-\phi)\left(\frac{\partial\theta_1}{\partial x} \right)^2 \theta.$$

Multiplying (37) by $\kappa_1'(\theta_1) = d\kappa_1/d\theta_1$, $\kappa_1 = \theta_1^p$, and integrating over x, we arrive at a relation of the form (43) for $\theta_1(x,t)$. Therefore $\theta(x,t) \geq \min_{x\in[0,1]} \theta^0(x)$ for all $t \in [0,T], x \in [0,1]$.

Multiplying the equation (35) by $\frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t}$ and integrating over x we arrive at the relation

$$\int_0^1 \frac{1}{\xi_1(\theta)}\frac{a_1(\phi)}{1-\phi}\left(\frac{\partial G}{\partial t} \right)^2 dx + \int_0^1 k(\phi)(1-\phi)\left| \frac{\partial}{\partial x}\left(\frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t} \right) \right|^2 dx =$$

$$= \int_0^1 k(\phi)g(\rho_{tot} + \rho_f)\frac{\partial}{\partial x}\left(\frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t} \right) dx \leq$$

$$\leq \frac{1}{2}\int_0^1 k(\phi)(1-\phi)\left| \frac{\partial}{\partial x}\left(\frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t} \right) \right|^2 dx + \frac{1}{2}\int_0^1 \frac{k(\phi)}{1-\phi}g^2(\rho_{tot} + \rho_f)^2 dx.$$

The last term on the right-hand side is bounded uniformly in t_0, since $\phi < 1$ and, therefore, $\rho_{tot} \leq \max(\rho_f, \rho_s)$. Finally, due to (43) we have

(48)
$$\int_0^1 \frac{dx}{1-\phi} = 1 + \int_0^1 s^0(x)dx.$$

Lemma 3 is proved. □

Lemma 4. *Under the conditions of Theorem 3, for all $t \in [0,T], x \in [0,1]$ the estimate takes place*

(49)
$$0 < m \leq \phi(x,t) \leq M < 1.$$

Proof. Under the conditions of Theorem 3, the function G has the form

$$G = ln|\phi| - ln|\phi - 1| - \frac{1}{\phi} - \frac{1}{2\phi^2} - \frac{1}{3\phi^3}.$$

It is monotonically increasing. Therefore, if we prove the boundedness of G, we come to the estimates of (49). To prove the boundedness of the function G, we first establish an embedding of the form

$$\frac{1}{\phi} \in L_1(0, 1).$$

By virtue of the properties of the function $\xi_1(\theta)$ and (45) we have

$$\int_0^1 \frac{a_1(\phi)}{1 - \phi} \left(\frac{\partial G}{\partial t}\right)^2 dx = \int_0^1 \frac{1}{\phi^4(1 - \phi)^3} \left(\frac{\partial \phi}{\partial t}\right)^2 dx \le N \max_{x,t} \xi_1(\theta) \equiv N_1.$$

From the last inequality and (48) we have

$$\int_0^1 \frac{1}{\phi^2(1 - \phi)^2} \left|\frac{\partial \phi}{\partial t}\right| dx \le \left(\int_0^1 \frac{dx}{1 - \phi}\right)^{1/2} \left(\int_0^1 \frac{1}{\phi^4(1 - \phi)^3} \left|\frac{\partial \phi}{\partial t}\right|^2 dx\right)^{1/2} \le$$

$$\le (1 + \int_0^1 s^0(x)dx)^{1/2} N_1^{1/2} \equiv N_2.$$

Therefore

$$\left|\frac{d}{dt} \int_0^1 r(\phi)dx\right| \le \int_0^1 \frac{1}{\phi^2(1 - \phi)^2} \left|\frac{\partial \phi}{\partial t}\right| dx \le N_2,$$

where

$$r(\phi) = -\frac{1}{\phi} + \frac{1}{1 - \phi} + 2ln\frac{\phi}{1 - \phi}.$$

By integrating the last inequality over t from 0 to the current value, we obtain

$$-N_2 t + \int_0^1 r(\phi^0)dx \le \int_0^1 r(\phi)dx \le N_2 t + \int_0^1 r(\phi^0)dx.$$

From which it follows

$$-\frac{N_2 t}{3} - \frac{1}{3} \int_0^1 r(\phi^0)dx + \frac{1}{3} \int_0^1 s^0(x)dx + \frac{4}{3} \le \int_0^1 \frac{1}{\phi}dx \le -1 + 3\int_0^1 s^0(x)dx + N_2 t -$$

$$- \int_0^1 r(\phi^0)dx,$$

and

$$\left|\int_0^1 \frac{1}{\phi}dx\right| \le N_3,$$

where

$$N_3 = \max\{\frac{N_2 T}{3} + \frac{1}{3} \int_0^1 r(\phi^0)dx - \frac{1}{3} \int_0^1 s^0(x)dx - \frac{4}{3}, -1 + 3\int_0^1 s^0(x)dx + N_2 T - \int_0^1 r(\phi^0)dx\}.$$

Finally, returning to (45), we have

$$\int_0^1 \left|\frac{\partial}{\partial x}\left(\frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t}\right)\right| dx \le \left(\int_0^1 k(\phi)(1 - \phi)\left|\frac{\partial}{\partial x}\left(\frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t}\right)\right|^2 dx\right)^{1/2} \left(\int_0^1 \frac{1}{k(\phi)(1 - \phi)}dx\right)^{1/2} \le$$

$$\le (2N\frac{\mu}{K})^{1/2} \left(\int_0^1 \frac{1}{\phi(1 - \phi)}dx\right)^{1/2} \le (2N\frac{\mu}{K})^{1/2} \left(N_3^{1/2} + (1 + \int_0^1 s^0(x)dx)^{1/2}\right) \equiv N_4,$$

and there is a point $x_0(t)$ at which $\frac{\partial G}{\partial t}(x_0(t), t) = 0$, therefore

$$\min_{x \in (0,1)} \left| \frac{1}{\xi_1(\theta)} \right| \left| \frac{\partial G}{\partial t} \right| \leq \left| \frac{1}{\xi_1(\theta)} \frac{\partial G}{\partial t} \right| \leq \int_0^1 \left| \frac{\partial}{\partial x} \left(\frac{1}{\xi_1(\theta)} \frac{\partial G}{\partial t} \right) \right| dx \leq N_4.$$

Taking into account (44) and the conditions of Theorem 3, from the last inequality we have

$$\left| lns - \frac{1+s}{s} - \frac{1}{2} \frac{(1+s)^2}{s^2} - \frac{1}{3} \frac{(1+s)^3}{s^3} \right| = |G(x,t)| \leq |G^0(x)| + T N_4 \max_{x,t} \xi_1(\theta) \equiv N_5.$$

The constants m and M are restored from here as follows. If $s > 1$, then from last inequality follows

$$lns \leq N_5 + \frac{1}{s} + 1 + \frac{1}{2} \left(\frac{1}{s^2} + \frac{2}{s} + 1 \right) + \frac{1}{3} \left(\frac{1}{s^3} + \frac{3}{s} + \frac{3}{s^2} + 1 \right) \leq N_5 + 7.$$

Then we have

$$1 < s \leq e^{N_5 + 7}.$$

If $0 < s < 1$, then $ln\frac{1}{s} \leq N_5$ and we have the estimate $e^{-N_5} \leq s$. Therefore

$$m = \frac{1}{1 + e^{N_5}}, \quad M = \frac{1}{1 + e^{-N_5 - 7}}.$$

Thus we arrive at (49). □

Let $z = \frac{1}{\xi_1(\theta)} \frac{\partial G}{\partial t}$. The problem (35), (36) takes the form

$$\frac{a_1(\phi)\xi_1(\theta)z}{(1 - \phi)} = \frac{\partial}{\partial x} \left(k(\phi)(1 - \phi)\frac{\partial z}{\partial x} - k(\phi)g(\rho_{tot} + \rho_f) \right),$$

$$\left(k(\phi)(1 - \phi)\frac{\partial z}{\partial x} - k(\phi)g(\rho_{tot} + \rho_f) \right) |_{x=0, x=1} = 0.$$

By Lemmas 3 and 4, we have

$$\int_0^t \int_0^1 \theta_x^2 dx d\tau + \int_0^1 (z^2 + z_x^2) dx \leq N_6,$$

where N_6 is a positive constant depending on the initial data, parameters and problem constants, but does not depend on t_0.

Using the representation

$$G(\phi) = \int_0^t \xi_1(\theta)z d\tau + G(\phi^0),$$

we get

$$G'(\phi)\phi_x = \int_0^t (z_x \xi_1(\theta) + z\xi_1'\theta_x)d\tau + G_x(\phi^0).$$

Therefore

$$\int_0^1 \phi_x^2 dx \leq N_7.$$

The equation for function $z(x, t)$ takes form

$$a_0(\phi, \theta)z = a_1(\phi)z_{xx} + a_1'(\phi)\phi_x z_x + a_2'(\phi)\phi_x.$$

The coefficients $a_0(\phi, \theta) > 0$, $a_1(\phi) > 0$, $a_2(\phi)$ are limited and easy to calculate.
We have

$$\int_0^1 z_{xx}^2 dx \le C_1 \left(\int_0^1 (z^2 + \phi_x^2) dx + \int_0^1 |z_{xx} z_x \phi_x| dx \right),$$

where

$$I_1 = \int_0^1 |z_{xx}||z_x \phi_x| dx \le \max |z_x| \left(\int_0^1 z_{xx}^2 dx \right)^{1/2} \left(\int_0^1 \phi_x^2 dx \right)^{1/2} \le$$

$$\le C_1 \left(\left(\int_0^1 z_{xx}^2 dx \right)^{1/2} \left(\int_0^1 \phi_x dx \right)^{1/2} + \left(\int_0^1 z_{xx}^2 dx \right)^{3/4} \left(\int_0^1 \phi_x dx \right)^{1/2} \right).$$

The constant C_1 is not depend on t_0.

Therefore

$$\max_x |z_x| + \int_0^1 z_{xx}^2 dx \le N_8.$$

The equation for the function $\theta(x, t)$ has the form

$$\theta_t + a_3(\phi, z_x)\theta_x = a_4(\phi)\theta_{xx} + a_5(\phi)\phi_x\theta_x,$$

where the coefficients $a_4(\phi) > 0$, $a_3(\phi, z_x)$, $a_5(\phi)$ are limited and easy to calculate.

Since

$$\int_0^1 |\theta_x \theta_{xx} \phi_x| dx \le \max_x |\theta_x| \left(\int_0^1 \theta_{xx}^2 dx \right)^{1/2} \left(\int_0^1 \phi_x^2 dx \right)^{1/2} \le$$

$$\le c \left(\int_0^1 \theta_{xx}^2 dx \right)^{3/4} \left(\int_0^1 \phi_x^2 dx \right)^{1/2} \left(\int_0^1 \theta_x^2 dx \right)^{1/4},$$

then from the equation for θ we have

$$\int_0^1 \theta_x^2 dx + \int_0^t \int_0^1 (\theta_t^2 + \theta_{xx}^2) dx d\tau \le N_9.$$

To complete the proof of Theorem 3, it is necessary to obtain the Holder continuity in x, t of the functions ϕ_x and z_x included in the coefficients of the equations for z and θ. From the embedding $z_{xx} \in L_2[0, 1]$ and the representation for ϕ we have $\phi_{xx} \in L_2[0, 1]$. Then for $w = \theta_x$ we get

$$\int_0^1 (\theta_t^2 + w_x^2) dx + \int_0^t \int_0^1 (w_t^2 + w_{xx}^2) dx d\tau \le N_{10}.$$

After that we deduce that $|\phi_{xt}| \le N_{11}$. Finally, following [45] for the function $\sigma = z_t$ we get $\sigma_x \in L_2[0, 1]$.

Theorem 3 is proved. □

CONCLUSION

The local solvability in the Holder classes of the initial - boundary value problem of one-dimensional fluid motion in a nonisothermal viscous porous medium is proved. An example of decidability is given at any finite time interval.

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