On Coupon coloring of Cayley graphs of some small Rings

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ABSTRACT - A k- coupon coloring of a graph G without isolated vertices is an assignment of colors from $[k] = \{1, 2, \ldots, k\}$ to the vertices of G such that the neighborhood of every vertex of G contains vertices of all colors from $[k]$. The maximum k for which a k-coupon coloring exists is called the coupon coloring number of G. The Cayley Graph $Cay(G, C)$ of a group G is a graph with vertex set G and edge set $E(Cay(G, C)) = \{gh : hg^{-1} \in C\}$, where C is a subset of G that is closed under taking inverses and does not contain the identity. For a commutative ring R with unity, $Cay(R^+, Z(R)^*)$ is denoted by $\mathbb{C}AY(R)$, where R^+ is the additive group and $Z(R)^*$ is the nonzero zero-divisors of R. In this paper, we have obtained bounds for the coupon coloring number of $\mathbb{C}AY(\mathbb{Z}_n)$ and $\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)$, where $\mathbb{Z}_n$ is the commutative ring of integers modulo n and $\mathbb{Z}_n \times \mathbb{Z}_m$ is the Cartesian product of $\mathbb{Z}_n$ and $\mathbb{Z}_m$. We also found that in some cases these upper bounds are sharp. We have found the coupon coloring number of $Cay(\mathbb{Z}_\mathrm{n}, C)$ when $C = \{1, -1, a\}$, where $a = -a$ and $C = \{1, -1, 2, -2\}$.
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1 Introduction

The concept of coupon coloring number was introduced by Chen et al. in [**?**]. Let G be a graph without isolated vertices. A k - coupon coloring of G is an assignment of colors from $[k] = \{1, 2, \ldots, k\}$ to the vertices of G such that the neighborhood of every vertex of G contains vertices of all colors from $[k]$. The maximum k for which a k-coupon coloring exists is called the coupon coloring number of G and it is denoted by $\chi_c(G)$.

Let $G = (V, E)$ be a graph. $D \subseteq V$ is a dominating set if every vertex in $V \setminus D$ is adjacent to at least one vertex in D. Let $G = (V, E)$ be a graph without isolated vertices. $D' \subseteq V$ is a total dominating set(TDS) if every vertex of G is adjacent to at least one vertex in D'. The minimum cardinality among all the total dominating sets in G is called the total domination number, $\gamma_t(G)$. The coupon coloring number is also referred as the total domatic number introduced in [**?**], which is the maximum number of disjoint total dominating sets. Coupon coloring is studied in [**?**, **?**, **?**]. In [**?**] Y Shi et al. determined coupon coloring number of complete graphs, complete k-partite graphs, wheels, cycles, unicyclic graphs and bicyclic graphs.

Coupon coloring is interesting, not only because of its theoretical value, but for its applications in the network science and some other related fields. Imagine the colors as coupons of different types. Then the requirement of coupon coloring is that every vertex collects coupons of all different types from its neighbors. If we imagine that users $v_1, v_2, \ldots, v_n$ are each assigned a bit from a k-bit message, and that every user has contact with a set of other users, then every user can reconstruct from her contacts the entire message if and only if the graph of contacts has a k-coupon coloring. The task given to a graph of contacts is to determine the coupon coloring number of the graph, to maximize the length of the message that can be transmitted. In addition, results on coupon colorings have concrete applications in network science. One application is to large multi-robot networks [**?**]. One may imagine a network large enough that robots must act based on local information. A graph can be constructed with robots in the network as nodes and there is an edge between nodes if the corresponding robots are able to communicate with each other. An example described in [**?**] is as follows: a group of robots is deployed to monitor an environment. Each robot must monitor many different statistics (e.g. temperature, humidity, etc.), but due to power limitations it is only equipped with a single sensor (thermometer, barometer, etc.). Thus, in order to obtain the remaining data, each robot must communicate with its neighbors. A similar example arises in allocating resources to a network [**?**]. If each vertex of a graph may only use resources available at the vertex or its neighbors, and if some resource (e.g. a printer) must be available to every node in the network, then copies of that resource must be allocated to a dominating set of the network. If every node in the network can accommodate one resource, then finding the coupon coloring number of the network is equivalent to finding the maximum number of resources that can be made available to every node in the network.

2 Preliminaries

All graphs considered in this paper are simple, finite and undirected. As usual K_n denote the complete graph with n vertices. For vertices x and y of a graph G, we define degree $d(x, y)$ to be the length of a shortest path from x to y. The minimum and maximum degrees of vertices in a graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. The diameter of a graph G is $diam(G) = \sup\{d(x, y) :$ x and y are vertices of $G\}$. Let G be a graph without isolated vertices. A k-vertex coloring, or simply a k-coloring of G is a mapping c from the vertex set of G to $[k] = \{1, 2, ..., k\}$. A vertex v is said to be a *bad vertex* in a k-coloring c, if its neighborhood does not contain vertices of all colors from $[k]$ and obviously, there is no bad vertices in a coupon coloring. Clearly, coupon coloring is an improper coloring and $\chi_c(G) \leq \delta(G)$.

Let G be a group and let C be a subset of G that is closed under taking inverses and does not contain the identity. Then the Cayley Graph $Cay(G, C)$ is a graph with vertex set G and edge set

$$E(Cay(G, C)) = \{gh : hg^{-1} \in C\}.$$

Let $\mathbb{Z}_\mathrm{n}$ denote the additive group of integers modulo n. If C is a subset of $\mathbb{Z}_\mathrm{n}\backslash\{0\}$, then construct a directed graph $Cay(\mathbb{Z}_\mathrm{n}, C)$ as follows. The vertices of $Cay(\mathbb{Z}_\mathrm{n}, C)$ are elements of $\mathbb{Z}_\mathrm{n}$ and (i, j) is an arc of $Cay(\mathbb{Z}_\mathrm{n}, C)$ if and only if

$j-i \in C$. The graph $Cay(\mathbb{Z}_{\mathrm{n}}, C)$ is called a *Circulant graph* of order n, and C is called its connection set. If the set C is symmetric, that is $C=-C=\{-x : x \in C\}$, then X will be an undirected graph. Let R be a commutative ring with unity. Then the Cayley graph of R with respect to its zero divisors is the graph $Cay(R^{+}, Z(R)^{*})$ denoted by $\mathbb{C}AY(R)$. This is the Cayley graph whose vertices are all elements of the additive group R^{+} and in which two distinct vertices x and y are joined by an edge if and only if $x-y \in Z(R)^{*}$. Here $Z(R)$ is the set of zero divisors of the ring R and $Z(R)^{*}=Z(R) \setminus \{0\}$.

Let $\phi(n)$ denotes the Euler's phi-function. The Cartesian product $R_1 \times R_2$ of two commutative rings R_1 and R_2 is also a commutative ring and is defined as the set consisting of all ordered pairs (a, b) for which $a \in R_1$ and $b \in R_2$.

The following results will be useful for the upcoming sections.

Theorem 2.1. *[?]*

(1) Let G be a complete graph with n vertices. Then $\chi_c(G)=\left\lfloor \frac{n}{2} \right\rfloor$.

(2) Let $G=K_{n_1,n_2,\ldots,n_k}$ be a complete k - partite graph where $k \geq 3$ and $n_1 \leq n_2 \leq, \ldots, \leq n_k$ such that $s=\sum_{i=1}^{k-1} n_i$ and $n=\sum_{i=1}^{k} n_i$. Then

$$\chi_c(G)=\begin{cases} \left\lfloor \frac{n}{2} \right\rfloor & \text{if } s \geq \frac{n}{2}, \\ s & \text{otherwise.} \end{cases}$$

Theorem 2.2. *[?] Let R be a ring. Then the following statements hold:*

1. *$\mathbb{C}AY(R)$ has no edge if and only if R is an integral domain.*
2. *If (R, M) is an Artinian local ring, then $\mathbb{C}AY(R)$ is a disjoint union of $|\frac{R}{M}|$ copies of the complete graph $K_{|M|}$.*
3. *$\mathbb{C}AY(R)$ cannot be a complete graph.*
4. *$\mathbb{C}AY(R)$ is a regular graph of degree $|Z(R)-1|$ with isomorphic components .*

3 Coupon coloring of $\mathbb{C}AY(\mathbb{Z}_n)$

By Theorem **??**, $\mathbb{C}AY(R)$ has no edge if and only if R is an integral domain. So in this section consider only $\mathbb{Z}_n$ with n composite.

Theorem 3.1. *If $Z(\mathbb{Z}_n)$ is an ideal of $\mathbb{Z}_n$, $\chi_c(\mathbb{C}AY(\mathbb{Z}_n))=\left\lfloor \frac{|Z(\mathbb{Z}_n)|}{2} \right\rfloor$.*

Proof . *Let $Z(\mathbb{Z}_n)$ be an ideal of $\mathbb{Z}_n$. Then $n=p^k$ for some prime p and $Z(\mathbb{Z}_n)$ is the maximal ideal of $\mathbb{Z}_n$. By Theorem **??**, $\mathbb{C}AY(\mathbb{Z}_n)$ is the disjoint union of $\left|\frac{\mathbb{Z}_n}{Z(\mathbb{Z}_n)}\right|=p$ copies of the complete graph $K_{|Z(\mathbb{Z}_n)|}=K_{p^{k-1}}$ since, $|Z(\mathbb{Z}_n)|=p^{k-1}$. Therefore, using Theorem **??***

$$\begin{aligned} \chi_c(\mathbb{C}AY(\mathbb{Z}_n)) &= \chi_c(K_{p^{k-1}}) \\ &= \left\lfloor \frac{p^{k-1}}{2} \right\rfloor \\ &= \left\lfloor \frac{|Z(\mathbb{Z}_n)|}{2} \right\rfloor. \end{aligned}$$

The above theorem gives the coupon coloring number of $\mathbb{C}AY(\mathbb{Z}_n)$ when $Z(\mathbb{Z}_n)$ is an ideal of $\mathbb{Z}_n$. Next we consider the case that $Z(\mathbb{Z}_n)$ is an not ideal of $\mathbb{Z}_n$. We found a sharp lower bound of the coupon coloring number of $\mathbb{C}AY(\mathbb{Z}_n)$.

Theorem 3.2. *Suppose that $Z(\mathbb{Z}_n)$ is an not ideal of $\mathbb{Z}_n$. Then $\chi_c(\mathbb{C}AY(\mathbb{Z}_n)) \geq \frac{n}{p}$, where p is the least prime divisor of n.*

Proof . *$Z(\mathbb{Z}_n)$ is an not ideal of $\mathbb{Z}_n$, so there exists at least two prime divisors for n. Let $n = p_1^{r_1} p_2^{r_2} \dots p_m^{r_m}$, where $p_1 < p_2 < \dots < p_m$ and $H = \{0, p_2, 2p_2, \dots, (p_1 - 1)p_2\}$. Define the set $D_i = ip_1 + H, i = 1, 2, \dots, \frac{n}{p_1}$. Clearly, $D_{\frac{n}{p_1}} = H$. D_i's are disjoint, for let $x \in D_i \cap D_j, i \neq j$. Then $x = ip_1 + h_1 = jp_1 + h_2$, $h_1, h_2 \in H$. So, $ip_1 + sp_2 = jp_1 + tp_2$, that is $(i - j)p_1 = (t - s)p_2$ where $i, j \in \{1, 2, \dots, \frac{n}{p_1}\}$ and $s, t \in \{1, 2, \dots, p_1 - 1\}$. Since $i \neq j$, $t \neq s$ and p_1 divides $t - s$, a contradiction. Hence, D_i's are disjoint and so $\left\{D_i : i = 1, 2, \dots, \frac{n}{p_1}\right\}$ is a partition of $\mathbb{Z}_n$. Define the coloring $c : V(\mathbb{C}AY(\mathbb{Z}_n)) \to \left[\frac{n}{p_1}\right]$ by*

$$c(x) = k, \text{ if } x \in D_k.$$

Let $x \in \mathbb{Z}_n$. Then $x \in D_k = kp_1 + H$ for some $k \in \left\{1, 2, \dots, \frac{n}{p_1}\right\}$. So, $x = kp_1 + tp_2, t \in \{0, 1, 2, \dots, p_1 - 1\}$. Then x is adjacent to the vertices $p_1 + tp_2, 2p_1 + tp_2, \dots, (k - 1)p_1 + tp_2, (k + 1)p_1 + tp_2, \dots, \frac{n}{p_1}p_1 + tp_2$ with colors $1, 2, \dots, k - 1, k + 1, \dots, \frac{n}{p_1}$. Also, x is adjacent to $kp_1 + (t + 1)p_2$ with color k. Thus, c is a coupon coloring of $\mathbb{C}AY(\mathbb{Z}_n)$ and so $\mathbb{C}AY(\mathbb{Z}_n) \geq \frac{n}{p}$.

Theorem 3.3. *If n is even, then*

$$\chi_c(\mathbb{C}AY(\mathbb{Z}_n)) = \frac{n}{2}.$$

Proof . *Each color in a coupon coloring must appear at least twice and so there can be at most $\lfloor \frac{n}{2} \rfloor$ colors in a coupon coloring of $\mathbb{C}AY(\mathbb{Z}_n)$. Therefore, $\chi_c(\mathbb{C}AY(\mathbb{Z}_n)) \leq \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$, since n is even.*

*Suppose that $Z(\mathbb{Z}_n)$ is an not ideal of $\mathbb{Z}_n$. Since n is even, 2 is the smallest prime divisor of n. So, by Theorem **??**, $\chi_c(\mathbb{C}AY(\mathbb{Z}_n)) \leq \frac{n}{2}$. Hence, $\chi_c(\mathbb{C}AY(\mathbb{Z}_n)) = \frac{n}{2}$. If $Z(\mathbb{Z}_n)$ is an ideal of $\mathbb{Z}_n$, by Theorem **??**, $\chi_c(\mathbb{C}AY(\mathbb{Z}_n)) = \frac{n}{2}$.*

4 Coupon coloring of $\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)$

If R_1 and R_2 are two finite commutative rings, then $Reg(R_1 \times R_2) = Reg(R_1) \times Reg(R_2)$. So, $(a, b) \in Z(R_1 \times R_2)$ if and only if either $a \in Z(R_1)$ or $b \in Z(R_2)$. Therefore, in $\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)$, $(x, y) \neq (a, b)$, (x, y) is adjacent to (a, b) if and only if either $x - a \in Z(\mathbb{Z}_n)$ or $y - b \in Z(\mathbb{Z}_m)$.

Theorem 4.1. *$\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)$ is a $(nm - \phi(n)\phi(m) - 1)$ - regular graph.*

Proof . *By Theorem **??**, $\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)$ is a $|Z(\mathbb{Z}_n \times \mathbb{Z}_m)| - 1$ regular graph. Here $|Z(\mathbb{Z}_n \times \mathbb{Z}_m)| - 1 = (nm - \phi(n)\phi(m) - 1)$, so $\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)$ is a $(nm - \phi(n)\phi(m) - 1)$ - regular graph.*

Theorem 4.2. *$\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)$ is a connected graph with $diam(\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)) = 2$.*

Proof . *Any $(a,b) \in \mathbb{Z}_n \times \mathbb{Z}_m$ with either $a = 0$ or $b = 0$ is in $Z(\mathbb{Z}_n \times \mathbb{Z}_m)$. Let $(a,b),(x,y) \in \mathbb{Z}_n \times \mathbb{Z}_m = V(\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m))$ such that $(a,b) \neq (x,y)$. If $a = x$ or $b = y$, then (a,b) is adjacent to (x,y). Suppose that neither $a = x$ nor $b = y$, then $(a,b) - (a,y) - (x,y)$ is a path in $\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)$, since*

$$\begin{aligned} (a,b) - (a,y) &= (0, b-y) \in Z(\mathbb{Z}_n \times \mathbb{Z}_m)^* \\ (a,y) - (x,y) &= (a-x, 0) \in Z(\mathbb{Z}_n \times \mathbb{Z}_m)^* \end{aligned}$$

Moreover, there are non-adjacent vertices, $(a,b),(a+1,b+1)$.

Lemma 4.3. *Let $(x,y) \in \mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)$. Then (x,y) is adjacent only to the vertices*

$$\left[\bigcup_{x-i \in Z(\mathbb{Z}_n)} H_i\right] \bigcup \left[\bigcup_{x-i \notin Z(\mathbb{Z}_n)} [(0,y) + J_i]\right] \setminus \{(x,y)\}$$

where $H_i = \{(i,b) : b \in \mathbb{Z}_m\}$, $J_i = \{(i,z) : z \in Z(\mathbb{Z}_m)\}$ and $i \in \mathbb{Z}_n$.

Proof . *Let $(x,y),(a,b) \in \mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)$ and $(a,b) \neq (x,y)$. If $(a,b) \in \bigcup_{x-i \in Z(\mathbb{Z}_n)} H_i$, then $(a,b) \in H_i$ for some i such that $x - i \in Z(\mathbb{Z}_n)$. Therefore, $x - i = z, z \in Z(\mathbb{Z}_n)$ and this implies that $i = z + x$. Thus, $(a,b) \in H_{z+x}$, since $(a,b) \in H_i$. Therefore, $(a,b) = (z+x, s)$ for some $s \in \mathbb{Z}_m, z \in Z(\mathbb{Z}_n)$ and so (a,b) is adjacent to (x,y), since,*

$$(x,y) - (a,b) = (x,y) - (z+x, s) = (-z, y-s) \in Z(\mathbb{Z}_n \times \mathbb{Z}_m).$$

If $(a,b) \in \bigcup_{x-i \notin Z(\mathbb{Z}_n)} [(0,y) + J_i]$, then $(a,b) \in (0,y) + J_i$ for some i such that $x - i \notin Z(\mathbb{Z}_n)$. In this case, $(a,b) = (0,y) + (i,w), w \in Z(\mathbb{Z}_m)$. So,

$$(x,y) - (a,b) = (x,y) - [(0,y) + (i,w)] = (x-i, -w) \in Z(\mathbb{Z}_n \times \mathbb{Z}_m).$$

Therefore, (a,b) is adjacent to (x,y). Thus (x,y) is adjacent to the vertices of $\bigcup_{x-i \in Z(\mathbb{Z}_n)} H_i$ and $\bigcup_{x-i \notin Z(\mathbb{Z}_n)} [(0,y) + J_i]$ except (x,y). Clearly, all these $[m - \phi(m)]n + \phi(m)[n - \phi(n)] - 1 = nm - \phi(n)\phi(m) - 1$ vertices are distinct and by Theorem **??** *these are the only vertices adjacent to (x,y).*

The following theorem gives a sharp lower bound for the coupon coloring number of $\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)$.

Theorem 4.4. *$\chi_c(\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)) \geq \frac{nm}{p}$, where p is the minimum of the least prime divisors of n and m.*

Proof . *Let $n = p_1^{r_1} p_2^{r_2} \dots p_u^{r_u}$ and $n = q_1^{r_1} q_2^{r_2} \dots p_v^{r_v}$ with $p_1 < p_2 < \dots < p_u$ and $q_1 < q_2 < \dots < q_v$. Without loss of generality, assume that $\min\{p_1, q_1\} = p_1 = p$. Define for all $i = 1, \dots, \frac{n}{p}, j = 0, 1, \dots, m-1$,*

$$D_{i,j} = \left\{ \Big([i-1]p + k, j \Big) : k = 0, 1, \dots, p-1 \right\}$$

Then $\Big\{D_{i,j} : i = 1, \ldots, \frac{n}{p}, j = 0, 1, \ldots, m-1\Big\}$ *is a partition of elements of* $\mathbb{Z}_n \times \mathbb{Z}_m$ *with* $\frac{nm}{p}$ *partite sets. For, let* $(x, y) \in D_{i,j} \cap D_{s,t}$*. Then*

$$(x, y) = \Big([i-1]p + k_1, j\Big) = \Big([s-1]p + k_2, t\Big)$$

So, $[i-1]p+k_1 = [s-1]p+k_2$ *and* $j = t$. $[i-1]p+k_1 = [s-1]p+k_2$ *implies that* $\big([i-1]-[s-1]\big)p = k_2 - k_1$*. Since* $k_1, k_2 \in \{0, 1, \ldots, p-1\}$*,* $k_2 - k_1 \leq p$*. Thus,* $k_2 - k_1 = 0$ *and so* $[i-1] - [s-1] = 0$ *that is* $i = s$*. Therefore,* $D_{i,j} = D_{s,t}$*.*

Define a coloring c *by*

$$c(x, y) = j\frac{n}{p} + i, \text{ for } (x, y) \in D_{i,j}.$$

Let $(x, y) \in D_{s,t}$ *for some* $s \in \Big\{1, \ldots, \frac{n}{p}\Big\}$ *and* $t \in \{0, 1, \ldots, m-1\}$*. Then* $(x, y) = \Big([s-1]p+k, t\Big)$ *for some* $k \in \{0, 1, \ldots, p-1\}$ *and so in* $\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)$*,* (x, y) *is adjacent to the* $\frac{nm}{p} - 1$ *vertices of the set*

$$\Big\{\Big([i-1]p + k, j\Big) : i = 1, \ldots, \frac{n}{p}, j = 0, 1, \ldots, m-1\Big\} \setminus \{(x, y)\},$$

since

$$\begin{aligned}(x, y) - \Big([i-1]p + k, j\Big) &= \Big([s-1]p + k, t\Big) - \Big([i-1]p + k, j\Big) \\ &= \Big(([s-1] - [i-1])p, t - j\Big) \in Z(\mathbb{Z}_n \times \mathbb{Z}_m)^*\end{aligned}$$

So (x, y) *is adjacent to the vertices of colors* $1, 2, \ldots, \frac{nm}{p}$ *except the color* $t\frac{n}{p}+s = c(x, y)$*. But* (x, y) *is also adjacent to the vertex* $\Big([s-1]p + k \pm 1, t\Big)$ *which is distinct from the above vertices and with color* $t\frac{n}{p} + s$*. Thus* c *is a coupon coloring of* $\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)$*. Therefore,* $\chi_c(\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)) \geq \frac{nm}{p}$*.*

Theorem 4.5. *Assume that either* n *or* m *is even. Then*

$$\chi_c(\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)) = \frac{nm}{2}.$$

Proof . *Any color in a coupon coloring must appear at least twice. Here we have* nm *vertices and so there can be at most* $\lfloor\frac{nm}{2}\rfloor$ *colors in a coupon coloring of* $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$*. Therefore,* $\chi_c(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) \leq \lfloor\frac{nm}{2}\rfloor = \frac{nm}{2}$*, since* n *or* m *is even.*

If n *or* m *is even, then* 2 *will be the minimum of the smallest prime divisors of* n *and* m*. So by Theorem* **??***,* $\chi_c(\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)) \geq \frac{nm}{2}$*. Hence,* $\chi_c(\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)) = \frac{nm}{2}$*.*

Lemma 4.6. *Suppose that* p *and* q *are prime numbers with* $p \leq q$*. Then a dominating set of* $\mathbb{C}AY(\mathbb{Z}_p \times \mathbb{Z}_q)$ *contains at least* p *vertices.*

Proof . *Suppose that D is a dominating set of $\mathbb{C}AY(\mathbb{Z}_p \times \mathbb{Z}_q)$ with $p-1$ vertices. Define $H_i = \{(i,b) : b \in \mathbb{Z}_q\}$ for all $i \in \mathbb{Z}_p$. Then $\{H_i : i \in \mathbb{Z}_p\}$ is a partition of the vertices of $\mathbb{C}AY(\mathbb{Z}_p \times \mathbb{Z}_q)$. Since D contains only $p-1$ vertices, there must exist at least one $s \in \mathbb{Z}_p$ such that no vertex of H_s is in D. Consider the vertices of $H_s = \{(s,0),(s,1),\ldots,(s,q-1)\}$. By Lemma **??**, for all $j \in \mathbb{Z}_q$, (s,j) is adjacent only to the vertices*

$$\begin{aligned} & \left[\bigcup_{s-i\in Z(\mathbb{Z}_p)} H_i\right] \bigcup \left[\bigcup_{s-i\notin Z(\mathbb{Z}_p)} [(0,j)+\{(i,0)\}]\right] \setminus \{(s,j)\} \\ = & \left[\bigcup_{s-i=0} H_i\right] \bigcup \left[\bigcup_{s-i\in\{1,2,\ldots,p-1\}} \{(i,j)\}\right] \setminus \{(s,j)\} \\ = & \ H_s \setminus \{(s,j)\} \cup \{(s-1,j),(s-2,j),\ldots(s-(p-1),j)\} \\ = & \ H_s \setminus \{(s,j)\} \cup \big\{(s-k,j) : k \in \{1,2,\ldots,p-1\}\big\} \end{aligned}$$

Since D is a dominating set and no vertex of H_s is in D, there should exist $k \in \{1,2,\ldots,p-1\}$ such that $(s-k,j) \in D$ for all $j \in \mathbb{Z}_q$. That is

$$\{(s-k_0,0),(s-k_1,1),\ldots,(s-k_{q-1},q-1)\} \subseteq D$$

where, $k_t \in \{1,2,\ldots,p-1\}, t \in \mathbb{Z}_q$. Then D should contain at least q vertices. But this is not possible, since $p < q$ and D contains only $p-1$ vertices.

Theorem 4.7. *Suppose that p and q are prime numbers. Then*

$$\chi_c(\mathbb{C}AY(\mathbb{Z}_p \times \mathbb{Z}_q)) = \max\{p,q\}.$$

Proof . *Without loss of generality, assume that $p \le q$. By Theorem **??**, $\chi_c(\mathbb{C}AY(\mathbb{Z}_p \times \mathbb{Z}_q)) \ge q$. From Lemma **??**, a dominating set of $\mathbb{C}AY(\mathbb{Z}_p \times \mathbb{Z}_q)$ should contain at least p vertices. So any total dominating set of $\mathbb{C}AY(\mathbb{Z}_p \times \mathbb{Z}_q)$ contains at least p vertices. Hence, there can be at most q disjoint total dominating sets. that is $\chi_c(\mathbb{C}AY(\mathbb{Z}_p \times \mathbb{Z}_q)) \le q$.*

Corollary 4.8. *Suppose that p and q are distinct prime numbers and $n = pq$. Then $\chi_c(\mathbb{C}AY(\mathbb{Z}_n)) = max\{p,q\}$*

The Corollary is immediate from the theorem, since $\mathbb{Z}_p \times \mathbb{Z}_q$ is isomorphic to $\mathbb{Z}_{\mathrm{pq}}$.

Theorem **??** says that $\chi_c(\mathbb{C}AY(\mathbb{Z}_n \times \mathbb{Z}_m)) \ge \frac{nm}{p}$, where p is the minimum of the least prime divisors of n and m. By Theorem **??**, this lower bound is sharp when either n or m even and by Theorem **??**, this lower bound is sharp when n and m are primes.

5 Coupon coloring of some circulant graphs

Theorem 5.1. *Let $C = \{1,-1,a\}$, where $a = -a$. Then*

$$\chi_c(Cay(\mathbb{Z}_n, C)) = \begin{cases} 3, & \text{if } n \equiv 0 (mod\ 3) \\ 2, & \text{otherwise.} \end{cases}$$

Proof . *Clearly $Cay(\mathbb{Z}_n, C)$ is a 3-regular graph. Let $a \in \mathbb{Z}_n$ and let $a = -a$. Then n must be even, since $2a = a + a = 0 = n$.*

Case 1: $n \equiv 0(mod\ 3)$

Define $c : V(Cay(\mathbb{Z}_n, C)) \to [3]$ by

$$c(i) = \begin{cases} 1, & \text{if } i \equiv 0(mod\ 3) \\ 2, & \text{if } i \equiv 1(mod\ 3) \\ 3, & \text{if } i \equiv 2(mod\ 3) \end{cases}$$

Then c is a coupon coloring on $Cay(\mathbb{Z}_n, C)$. For let $i \in V(Cay(\mathbb{Z}_n, C))$ such that $i \equiv 0(mod\ 3)$. Then $c(i) = 1$ and the neighbors of i are $i + a, i - 1$ and $i + 1$. Since $i \equiv 0(mod\ 3)$, $i + a = i + \frac{n}{2} \equiv 0(mod\ 3)$, $i - 1 \equiv -1 \equiv 2(mod\ 3)$ and $i + 1 \equiv 1(mod\ 3)$. So $c(i + a) = 1$, $c(i - 1) = 3$ and $c(i + 1) = 2$. All other possibilities can be proved similarly.

Case 2: $n \equiv 1(mod\ 3)$

Suppose that X has a 3-coupon coloring. Then at least one color should be given to at most $\lfloor \frac{n}{3} \rfloor$ vertices. The vertices of this color class D must be a total dominating set.

If $\lfloor \frac{n}{3} \rfloor$ is even, then $\lfloor \frac{n}{3} \rfloor = \frac{n-1}{3}$ vertices can dominate at most $\frac{(n-1)/3}{2} \times 6 = n - 1$, since vertices of P_2's can totally dominate maximum number of vertices. So in this case D cannot be a TDS.

If $\lfloor \frac{n}{3} \rfloor$ is odd, then $\lfloor \frac{n}{3} \rfloor$ vertices can dominate at most $\frac{\frac{(n-1)}{3} - 3}{2} \times 6 + (3 + 2 + 2 + 1) = n - 2$, since $\lfloor \frac{n}{3} \rfloor$ is odd, we have to choose one P_3 and maximum number of P_2's. Therefore in this case also D cannot be a TDS. So $\chi_c(Cay(\mathbb{Z}_n, C)) \leq 2$.

Define $c : V(Cay(\mathbb{Z}_n, C)) \to [2]$ by

$$c(i) = \begin{cases} 1, & \text{if } i \equiv 0(mod\ 2) \\ 2, & \text{if } i \equiv 1(mod\ 2) \end{cases}$$

Clearly, c is a 2-coupon coloring of $Cay(\mathbb{Z}_n, C)$, and so $\chi_c(Cay(\mathbb{Z}_n, C)) \geq 2$. Hence $\chi_c(Cay(\mathbb{Z}_n, C)) = 2$.

Case 3: $n \equiv 2(mod\ 3)$

Proof is similar to the case 2.

Theorem 5.2. *Let $Cay(\mathbb{Z}_n, C)$ be the circulant graph with $C = \{1, -1, 2, -2\}$, $2 \neq -2$. Then $\chi_c(Cay(\mathbb{Z}_n, C)) \leq 3$. In particular, $\chi_c(Cay(\mathbb{Z}_n, C)) = 3$ if $n \equiv 0(mod\ 6)$ and $n \equiv 0(mod\ 7)$.*

Proof . *Since $2 \neq -2$, $Cay(\mathbb{Z}_n, C)$ is a 4-regular graph. Let a be any vertex of $Cay(\mathbb{Z}_n, C)$. Then neighbors of a are $a - 1, a - 2, a + 1$ and $a + 2$. Since $(a - 1)^{-1}(a + 1) = 2 \in C$, there is an edge between $a - 1$ and $a + 1$.*

If c is a 4-coupon coloring of $Cay(\mathbb{Z}_n, C)$, then without loss of generality we may assume that $c(a - 1) = 1$, $c(a + 1) = 2$, $c(a - 2) = 3$ and $c(a + 2) = 4$. Then $c(a)$ cannot be 1,2,3 or 4.

(i) If $c(a) = 1$, then the vertex $a + 1$ have two neighbors with color 1. Since $\Delta(Cay(\mathbb{Z}_n, C)) = 4$, this will make the vertex $a + 1$, a bad vertex.

(ii) If $c(a) = 2$, then the vertex $a - 1$ have two neighbors with color 2 and so $a - 2$ is a bad vertex.

(iii) If $c(a) = 3$*, then the vertex* $a-1$ *have two neighbors with color 3 and* $a-1$ *will be a bad vertex.*

(iv) If $c(a) = 4$*, then the vertex* $a+2$ *have two neighbors with color 4 and so* $a+2$ *is a bad vertex.*

Hence 4-coupon coloring is not possible and so $\chi_c(Cay(\mathbb{Z}_n, C)) \leq 3$.
Claim 1 *:* $\chi_c(Cay(\mathbb{Z}_n, C)) = 3$ *if* $n \equiv 0(mod\ 6)$
Define $c : V(Cay(\mathbb{Z}_n, C)) \rightarrow [3]$ *by*

$$c(i) = \begin{cases} 1, & if\ i \equiv 0, 1(mod\ 6) \\ 2, & if\ i \equiv 2, 3(mod\ 6) \\ 3, & if\ i \equiv 4, 5(mod\ 6) \end{cases}$$

Then c *is a coupon coloring of* $Cay(\mathbb{Z}_n, C)$*. For let* $i \in V(Cay(\mathbb{Z}_n, C))$ *such that* $i \equiv 0(mod\ 6)$*. Then neighbors of* i *are* $i-1, i-2, i+1$ *and* $i+2$*.*

$c(i-1) = 3$*, since* $i-1 \equiv -1 \equiv 5(mod\ 6)$*. Similarly,* $c(i-2) = 3$*, since* $i-2 \equiv -2 \equiv 4(mod\ 6)$*,* $c(i+1) = 1$*, since* $i+1 \equiv 1(mod\ 6)$ *and* $c(i+2) = 2$*, since* $i+2 \equiv 2(mod\ 6)$*. Therefore, the four neighbors of* i *colored with all the three colors. Other cases can be proved similarly. Hence the claim 1.*
Claim 2 *:* $\chi_c(Cay(\mathbb{Z}_n, C)) = 3$ *if* $n \equiv 0(mod\ 7)$
Define $c : V(Cay(\mathbb{Z}_n, C)) \rightarrow [3]$ *by*

$$c(i) = \begin{cases} 1, & if\ i \equiv 0, 1, 2(mod\ 7) \\ 2, & if\ i \equiv 4, 6(mod\ 7) \\ 3, & if\ i \equiv 3, 5(mod\ 7) \end{cases}$$

Then c *is a coupon coloring of* $Cay(\mathbb{Z}_n, C)$*. For, let* $i \in V(Cay(\mathbb{Z}_n, C))$ *such that* $i \equiv 0(mod\ 7)$*. Then neighbors of* i *are* $i-1, i-2, i+1$ *and* $i+2$*.*

$c(i-1) = 2$*, since* $i-1 \equiv -1 \equiv 6(mod\ 7)$*, so . Similarly,* $c(i-2) = 3$*, since* $i-2 \equiv -2 \equiv 5(mod\ 7)$*,* $c(i+1) = 1$*, since* $i+1 \equiv 1(mod\ 7)$ *and* $c(i+2) = 1$*, since* $i+2 \equiv 2(mod\ 6)$*. Therefore, the four neighbors of* i *colored with all the three colors. Other cases can be proved similarly.*

References

[1] Bob Chen, Jeong Han Kim, Michael Tait, and Jacques Verstraete. On coupon colorings of graphs. **Discrete Applied Math.**, 193:94-101, 2015.

[2] F Bullo, J Cortes, and S Martinez. Distributed control of robotic networks: a mathematical approach to motion coordination algorithms, volume 27. Princeton University Press, 2009.

[3] G. Aalipour and S. Akbari, Some properties of a Cayley graph of a commutative ring, **Comm. Algebra**, vol. 42, pp. 1582-1593, 2014.

[4] G Aalipour and S Akbari. On the cayley graph of a commutative ring with respect to its zero-divisors. **Comm. Algebra**, 44(4):1443-1459, 2016.

[5] H Aram, S Sheikholeslami, L Volkmann, On the total domatic number of regular graphs, **Trans. Comb.** 1 (1) (2012) 45-51.

[6] He Chen and Zemin Jin. Coupon coloring of cographs. **Appl. Math. Comput.**, 308:90-95, 2017.

[7] W Abbas, M Egerstedt, C-H Liu, R Thomas, P Whalen, Deploying Robots with two sensors in $K_{1,6}$-free graphs. Manuscript available online at http://arxiv.org/abs/1308.5450.

[8] Y Shi, M Wei, J Yue, and Y Zhao. Coupon coloring of some special graphs. **J. Comb. Optim.**, 33(1):156-164, 2015.

[9] Z L Nagy. Coupon-coloring and total domination in hamiltonian planar triangulations. **Graphs Comb.**, 34(6):1385-1394, 2018.