An inverse problem for pseudoparabolic equation with p-Laplacian and damping term: existence and uniqueness

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Abstract

In this work, we consider the inverse problem of finding a pair of the functions $(u(x,t), f(t))$ which satisfy the nonlinear pseudoparabolic equation with $p-$Laplacian and damping term

$$u_t - \Delta u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \gamma\,|u|^{\sigma-2}\,u + f(t)g(x,t),$$

an initial and Dirichlet boundary conditions, and an integral overdetermintion condition. We study the inverse problem in two cases: the coefficient of the damping term $\gamma\,|u|^{\sigma-2}\,u$ is a positive (nonlinear source term) or negative (an absorption). In this both case, we establish the global and local in time existence and uniqueness of the weak solution to this possed inverse problem under suitable conditions on the exponents $p,\ \sigma$, the dimension d, and the data of the problem.

Keywords: Inverse problem, pseudoparabolic equation, p-Laplacian, damping term, global and local existence, uniqueness.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$, and $Q_T = \{(x,t) :\ x \in \Omega,\ 0 < t \le T\}$ is a cylinder with lateral Γ_T. In this paper, we study the following inverse problem of determinig the pair of the functions $(u(x,t), f(t))$, which satisfy the pseudoparabolic equation with $p-$Laplacian diffusion and damping

$$u_t - \Delta u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \gamma\,|u|^{\sigma-2}\,u + f(t)\cdot g(x,t), \quad \text{in} \ \ Q_T, \tag{1.1}$$

the initial condition

$$u(x,0) = u_0(x) \text{ in } \Omega, \tag{1.2}$$

the Dirihlet boundary condition

$$u(x,t) = 0 \ \text{ on } \ \Gamma_T, \tag{1.3}$$

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and the integral overdetermination condition

$$\int_{\Omega} (u \cdot \omega + \nabla u \cdot \nabla \omega)\, dx = e(t), \ \ t \geq 0. \tag{1.4}$$

Here the coefficient γ might be positive $\gamma > 0$ either negative $\gamma < 0$. The functions $g(x,t)$, $u_0(x)$, $\omega(x)$, and $e(t)$ are given. The exponents p is given positive number and σ is given function, such that

$$1 < p, \ \sigma < \infty. \tag{1.5}$$

Applications of this problem deal with the recovery of unknown coefficient $f(t)$, indicating the intensity of an external source which unknown or impossible to measure during the process. Inverse problems of determining the right-hand side of a differential equation arise in the mathematical modeling of many physical phenomena, when an external source or some of its parameters acting to the motion of the process are unknown or unacceptable for measurement, for example, the source is in a high-temperature environment or underground, etc. Equations like (1.1) with a one time derivative appearing in the highest order term are called pseudo-parabolic or Sobolev equations, and arise in many areas of mathematics and physics. For instance, they have been used, to model thermodynamics processes [24], fluid flow in fissured rock [12], filtration in porous media [13], and nonsteady flow of second order fluids [15], the motion of non-Newtonian fluids [3], [27], in particular, Kelvin-Voigt fluids [6, 7] and many other physical phenomena.

In the case $p = 2$ and $\sigma = 2$, the equation (1.1) becomes the classical pseudoparabolic equation. To our knowledge, the inverse problems for pseudoparabolic equations have not been studied a lot, see for classical pseudoparabolic equations [1], [10], [14], [19], [16], [21], [22, 23], and for pseudoparabolic equations with p-Laplacian and other related equations [4], [2], [8], [17, 18], [26], and references therein.

Recently, Antontsev and et. in [4] have been considered the inverse problem (1.1)-(1.4) with the special right-hand side in the form $F(x,t) = f(t) \cdot (\omega(x) - \Delta\omega(x))$, where $\omega(x)$ is the same function appearing also in the overdetermination condition (1.4). It may restrict the statement of the problem from mathematical and also physical view. Therefore, we consider the inverse problem (1.1)-(1.4) with the right-hand side $F(x,t) = f(t)g(x,t)$, where $g(x,t)$ is an arbitrary function in $L^\infty(0,T; L^2(\Omega))$. They have proved in [4] the local existence of weak solution (without the uniqueness) under the following conditions on the exponents p, σ and the dimension d (see Theorem 3.2, [4]):

$$\sigma > p, \ 2 < p < 4, \ 2 < \sigma < \frac{2d}{d-2}, \text{ and } d \geq 3.$$

The global existence is also established without proof. In this paper, we extend these results, by considering all possible cases of exponents p, σ, and the dimension d. We have also proved the uniqueness of the weak solution in both cases of $\gamma > 0$ either $\gamma < 0$.

Here we applied the method that have been proposed by Vasin for Navier-Stokes equation in [25] and used for pseudo-parabolic equation in [4].

The present paper is organized as follows. In Section 2, we introduce the some axillary lemmas that we are used in this work. In Section 3, we prove that the initial inverse problem (1.1)-(1.4) is equivalent to the direct problem (3.6)-(3.9) containing the nonlinear nonlocal operator of u. The global and local in time existence of a weak solution to the direct problem (3.6)-(3.9) with $\sigma = const$ is established in Section 4, when $\gamma > 0$, and is established in Section 5, when $\gamma \leq 0$. For that we construct Galerkin's approximations u^n and derive their first and second estimates. Next using compactness arguments together with the monotony method we realize a passage to the limit as $n \to \infty$. The Section 6 is devoted to the study the uniqueness of the weak solution to the problem (3.6)-(3.9).

2. Preliminaries

In this section, we introduce some auxiliary lemmas and functional spaces that will be used throughout the paper. For the definitions, notations of the function spaces and for their properties, we address the reader also to the monographs [5, 20].

We use the classical and the following nonlinear Gronwall's inequality ([5]) to establish the first and second local estimates.

Lemma 2.1. If $y : \mathbb{R}^+ \longrightarrow [0, \infty)$ is a continuous function such that

$$y(t) \leq C_1 \int_0^t y^\mu(s) ds + C_2, \quad t \in \mathbb{R}^+, \quad \mu > 1$$

for some positive constants C_1 and C_2, then

$$y(t) \leq C_2 \left(1 - (\mu - 1) C_1 C_2^{\mu-1} t\right)^{-\frac{1}{\mu-1}} \quad \text{for} \quad 0 \leq t < t_{\max} := \frac{1}{(\mu - 1) C_1 C_2^{\mu-1}}.$$

The following another very important auxiliary lemma (see [11] or [9], Lemma 2.2., p. 1809.) will be used to prove the uniqueness and passage to the limit in the Galerkin approximation.

Lemma 2.2. For all $p \in (1, \infty)$ and $\delta \geq 0$, there exist constants C_1 and C_2, depending on p and d, such that for all $\xi, \ \eta \in \mathbb{R}^d, \ d \geq 1$, it

$$\left| |\xi|^{p-2}\xi - |\eta|^{p-2}\eta \right| \leq C_1 |\xi - \eta|^{1-\delta} (|\xi| + |\eta|)^{p-2-\delta} \tag{2.1}$$

and

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta \right) \cdot (\xi - \eta) \geq C_2 |\xi - \eta|^{2+\delta} (|\xi| + |\eta|)^{p-2+\delta}. \tag{2.2}$$

3. Weak formulation

Assume that the data of the problem satisfy the following conditions

$$u_0(x) \in W^{1,2}(\Omega) \cap W^{1,p}(\Omega) \cap L^\sigma(\Omega); \tag{3.1}$$

$$|g_0(t)| := \left| \int\limits_\Omega g(x,t)\omega(x)dx \right| \geq l_0 > 0 \text{ for all } \ t \geq 0; \tag{3.2}$$

$$g(x,t) \in L^\infty(0,T;L^2(\Omega)); \tag{3.3}$$

$$e(t) \in W^{1,2}([0,T]), \quad \text{and} \quad \int\limits_\Omega u_0(x) \cdot \omega dx = e(0). \tag{3.4}$$

$$\omega(x) \in W^{1,p}(\Omega) \cap W_0^{1,2}(\Omega) \cap L^\sigma(\Omega); \tag{3.5}$$

Lemma 3.1. Under the conditions (3.2) and (3.5)-(3.4), the inverse problem (1.1)-(1.4) is equivalent to the following problem for a nonlinear parabolic equation with nonlinear nonlocal operator of the function u

$$u_t - \Delta u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \gamma\,|u|^{\sigma-2}\,u + f(t,u)g(x,t), \quad \text{in} \quad Q_T, \tag{3.6}$$

$$u(x,0) = u_0(x), \;\; x \in \Omega, \tag{3.7}$$

$$u(x,t) = 0, \quad \text{on} \quad \Gamma_T, \tag{3.8}$$

where

$$f(t,u) = \frac{1}{g_0(t)}\left(e'(t) + \int\limits_\Omega |\nabla u|^{p-2}\,\nabla u \cdot \nabla\omega \; dx - \gamma \int\limits_\Omega |u|^{\sigma-2}u \cdot \omega dx\right). \tag{3.9}$$

Proof. 1. Let the pair $(u(x,t), f(t))$ be a solution of the inverse problem (1.1)-(1.4). Multiplying both sides of (1.1) by ω, and integrating by parts, we have

$$\begin{aligned} &\int\limits_\Omega (u_t\omega + \nabla u_t \nabla\omega)\,dx + \int\limits_\Omega |\nabla u|^{p-2}\,\nabla u \cdot \nabla\omega \; dx = \\ &\gamma \int\limits_\Omega |u|^{\sigma-2}u \cdot \omega dx + f(t)\int\limits_\Omega g(x,t)\omega dx. \end{aligned} \tag{3.10}$$

Using

$$\int\limits_\Omega (u_t\omega + \nabla u_t \nabla\omega)\,dx = e'(t)$$

which follows from the overdetermination condition (1.4), and the assumption (3.2), we get from (3.10) the equality (3.9).

2. Let now $u(x,t)$ be a solution to the direct problem (3.6)-(3.8) with (3.9). It means that the pair of functions (u,f) is satisfied the equations (1.1)-(1.3). Thus, the pair (u,f) to be a solution of the inverse problem (1.1)-(1.4) it is sufficient to prove that the function $u(x,t)$

satisfies the overdetermination condition (1.4). Let us assume that for contradiction, i.e. the overdetermination condition (1.4) doesn't hold. Suppose that

$$\int_\Omega (u\omega + \nabla u \nabla \omega)\, dx = e_1(t), \ t \geq 0. \tag{3.11}$$

where $e_1(t) \neq e(t)$ for all $t \geq 0$. Thus, by the conditions (1.4) and (3.4), we have $e_1(t) \in W_2^1([0,T])$ and

$$e_1(0) = \int_\Omega (u_0\omega + \nabla u_0 \nabla \omega)\, dx = e(0)$$

Multiply (3.6) by ω and integrating by parts and using (3.9), we get

$$e_1'(t) + \int_\Omega |\nabla u|^{p-2}\, \nabla u \cdot \nabla \omega \ dx - \gamma \int_\Omega |u|^{\sigma-2} u \cdot \omega dx = f(t,u) g_0(t), \tag{3.12}$$

where $f(t,u)$ is defined in (3.9). Plugging (3.9) into (3.12) we obtain

$$\begin{aligned} & e_1'(t) + \int_\Omega |\nabla u|^{p-2}\, \nabla u \cdot \nabla \omega dx - \gamma \int_\Omega |u|^{\sigma-2} u \cdot \omega dx = \\ & e'(t) + \int_\Omega |\nabla u|^{p-2}\, \nabla u \cdot \nabla \omega dx - \gamma \int_\Omega |u|^{\sigma-2} u \cdot \omega dx. \end{aligned} \tag{3.13}$$

It follows that the following Cauchy problem for $E(t) = e_1(t) - e(t)$:

$$E'(t) = 0, \quad E(0) = e_1(0) - e(0) = 0, \tag{3.14}$$

which yields that $e_1(t) \equiv e(t)$ for all $t > 0$. □

Definition 3.1. A function $u(x,t)$ is a weak solution to the problem (3.6)-(3.9), if:

1. $u \in L^\infty(0,T; W_0^{1,2} \cap W_p^1 \cap L^\sigma) \cap L^p(Q_T) \cap L^\sigma(Q_T), \quad u_t \in L^2(0,T; W_0^{1,2}(\Omega))$;
2. $u(0) = u_0$ a.e. in Ω;
3. For every $\varphi \in W_0^{1,2} \cap W_p^1 \cap L^\sigma(\Omega)$ and for a.a. $t \in (0,T)$ holds

$$\frac{d}{dt} \int_\Omega (u\varphi + \nabla u \cdot \nabla \varphi)\, dx + \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_\Omega \left(\gamma |u|^{\sigma-2} u + f(t,u) g\right) \varphi dx. \tag{3.15}$$

4. Global and local existence in the case: $\gamma > 0$

In this section we consider the problem (3.6)-(3.9) when σ is a constant such that

$$1 < \sigma < \infty$$

and

$$\gamma > 0. \tag{4.1}$$

The following theorems are hold.

Theorem 4.1. [Global existence] Let the conditions (3.1)-(3.4) and (4.1) be fulfilled and assume, that

$$1 < \sigma,\ p \leq 2. \tag{4.2}$$

Then there exists a global weak solution to the problem (3.6)-(3.9) in the sense of Definition 3.1. Moreover, the weak solutions satisfy the following estimates

$$\sup_{t\in[0,T]} \left(\|u(t)\|_{2,\Omega}^2 + \|\nabla u(t)\|_{2,\Omega}^2\right) + \|\nabla u\|_{p,Q_T}^p + \|u\|_{\sigma,Q_T}^\sigma \leq C_1, \tag{4.3}$$

$$\sup_{t\in[0,T]} \left(\|\nabla u(t)\|_{p,\Omega}^p + \|u(t)\|_{\sigma,\Omega}^\sigma\right) + \|u_t\|_{2,Q_T}^2 + \|\nabla u_t\|_{2,Q_T}^2 \leq C_2, . \tag{4.4}$$

where C_1 and C_2 are positive constants depending on data of the problem.

Theorem 4.2. [Local existence]. Let the conditions (3.1)-(3.4) and (4.1) be fulfilled and assume that the condition

$$\sigma \leq 2^*,\ \ 2^* = \frac{2d}{d-2} \text{ if } d > 2; \quad 2^* \in (1, \infty) \text{ if } d = 2 \tag{4.5}$$

holds together with one of the following conditions

$$2 < \sigma \quad \text{or} \quad 2 < p. \tag{4.6}$$

Then there exists a time $T_* \in (0, T)$, such that the problem (3.6)-(3.9) has at least one weak solution $u(x, t)$ in the sense of Definition 3.1, where $T_* := T_2$ and T_2 is defined at (4.44) below. Moreover, these weak solutions satisfy the estimates (4.3)-(4.4) for all $t \in (0, T_*)$ with another positive constants C_1 and C_2 depending on data of the problem.

Remark 4.1. The condition (4.5) assures the passage to the limit as $n \to \infty$ below, see (4.56). We have assumed that the condition (4.6) is fulfilled, because we return to the statement of the Theorem 4.1 in case $p, \sigma \leq 2$.

Proof. The proof of these theorems consists of the steps: construction of Galerkin's approximations; obtain first and second a priory estimates; passage to limit.

4.1. Galerkin's approximations

Let $\{\psi_k\}_{k\in N}$ be an orthonormal family in $L^2(\Omega)$ and a linear combinations are dense in $V := W_0^{1,2} \cap W^{1,p} \cap L^\sigma(\Omega)$. Given $n \in N$, let us consider the n-dimensional space V^n spanned by $\psi_1, \ldots, \psi_n$. For each $n \in N$, we search for approximate solutions

$$u^n(x,t) = \sum_{j=1}^{n} c_j^n(t)\psi_j(x), \quad \psi_j \in V^n, \tag{4.7}$$

where the coefficients $c_1^n(t), \ldots, c_n^n(t)$ are defined as the solutions of the following n ordinary differential equations derived from

$$\begin{aligned} &\int\limits_\Omega (u_t^n\psi_k + \nabla u_t^n \nabla\psi_k)\, dx + \int\limits_\Omega |\nabla u^n|^{p-2}\nabla u^n \cdot \nabla\psi_k\, dx + \\ =&\gamma \int\limits_\Omega |u^n|^{\sigma-2}u^n \cdot \psi_k\, dx + f(t,u^n)\int\limits_\Omega g\psi_k dx, \end{aligned} \tag{4.8}$$

for $k = 1,\ 2,\ \ldots,\ n$.

The system (4.8) of ODEs is supplemented with the following Cauchy data

$$u^n(0) = u_0^n \quad \text{in } \Omega. \tag{4.9}$$

and assume that

$$u_0^n \to u_0(x) \text{ as } n \to \infty \ \text{ in } \ W_0^{1,2} \cap W^{1,p} \cap L^\sigma(\Omega). \tag{4.10}$$

According to the general theory of nonlinear ODE, the problem (4.8)-(4.9) has a solution $c_j^n(t)$ in $[0, t_0]$, where $t_0 \in (0,T]$. The solution can be extended to $[0,T]$ by a priory estimate which we shall obtain below.

4.2. First and second a priory estimates

Let us consider the case (4.1). In this case we obtain the global a priori estimates.

Lemma 4.1. Assume that

$$u_0(x) \in W_0^{1,2}(\Omega)$$

and the conditions (3.2)-(3.4), (4.1), (4.9)-(4.10) are fulfilled and

$$1 < \sigma, p \leq 2 \tag{4.11}$$

holds. Then the following a priori estimate is valid for all $t \in (0,T]$

$$\sup_{t\in[0,T]} \left(\|u^n\|_{2,\Omega}^2 + \|\nabla u^n\|_{2,\Omega}^2\right) + \|\nabla u^n\|_{L^p(Q_T)}^p + \|u^n\|_{L^\sigma(Q_T)}^\sigma \leq K_0 < \infty. \tag{4.12}$$

If in addition to above conditions

$$u_0(x) \in W_0^{1,2} \cap W^{1,p}(\Omega) \cap L^\sigma(\Omega)$$

holds, then the second estimate is valid for all $t \in (0,T]$

$$\sup_{t\in[0,T]} \left(\|\nabla u^n\|^p_{L^p(\Omega)} + \|u^n\|^\sigma_{L^\sigma(\Omega)} \right) + \|u^n_t\|^2_{L^2\left(0,T;W^{1,2}_0(\Omega)\right)} \le K_1 < \infty. \tag{4.13}$$

Proof. Multiplying both sides of (4.8) by $c^n_k(t)$ and summing on k, and adding $\gamma \|u^n\|^\sigma_{L^\sigma(\Omega)}$ on both side of result, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u^n\|^2_{2,\Omega} + \|\nabla u^n\|^2_{2,\Omega} \right) + \|\nabla u^n\|^p_{L^p(\Omega)} + \gamma \|u^n\|^\sigma_{L^\sigma(\Omega)} = 2\gamma \|u^n\|^\sigma_{L^\sigma(\Omega)} + I_1, \tag{4.14}$$

where

$$\begin{aligned} I_1 =& \frac{1}{g_0(t)} \left(e'(t) + \int_\Omega |\nabla u^n|^{p-2} \nabla u^n \cdot \nabla \omega \; dx - \right. \\ & \left. - \gamma \int_\Omega |u^n|^{\sigma-2} u^n \cdot \omega dx \right) \int_\Omega g(x,t) u^n(x) dx = \sum_{i=1}^{3} I_{1i}. \end{aligned} \tag{4.15}$$

Using the Holder's and Youn's inequalities and the assumptions in (4.11), we estimate each term on the right hand side of (4.14)

$$2\gamma \|u^n\|^\sigma_{L^\sigma(\Omega)} \le C(|\gamma|,\Omega,\sigma) \|u^n\|^\sigma_{2,\Omega} \le \frac{1}{8}\|u^n\|^2_{2,\Omega} + C^{\frac{2}{2-\sigma}}(\gamma,\Omega,\sigma), \tag{4.16}$$

$$|I_{11}| = \left| \frac{1}{g_0(t)} \int_\Omega g u^n e'(t) dx \right| \le \frac{1}{8}\|u^n\|^2_{2,\Omega} + \frac{1}{2l_0^2}\|g(t)\|^2_{2,\Omega} |e'(t)|^2, \tag{4.17}$$

$$\begin{aligned} &|I_{12}| = \left| \frac{1}{g_0(t)} \int_\Omega g u^n dx \cdot \int_\Omega |\nabla u^n|^{p-2} \nabla u^n \cdot \nabla \omega \; dx \right| \le \\ &\frac{1}{l_0}\|g(t)\|_{2,\Omega} \|u^n\|_{2,\Omega} \|\nabla \omega\|_{p,\Omega} \|\nabla u^n\|^{p-1}_{p,\Omega} \le \\ &\frac{1}{2}\|\nabla u^n\|^p_{p,\Omega} + C(p)\left(\frac{1}{l_0}\|g(t)\|_{2,\Omega} \|\nabla\omega\|_{p,\Omega} \|u^n\|_{2,\Omega} \right)^p \le \\ &\frac{1}{2}\|\nabla u^n\|^p_{p,\Omega} + \frac{1}{8}\|u^n\|^2_{2,\Omega} + C_0'(t) \end{aligned} \tag{4.18}$$

where $C_0'(t) = \left(C(p)\frac{1}{l_0} \max_{t\in[0,T]} \|g(t)\|_{2,\Omega} \|\nabla\omega\|_{p,\Omega} \right)^{\frac{2}{2-p}}$

$$\begin{aligned} &|I_{13}| = \left| -\frac{1}{g_0(t)}\gamma \int_\Omega g u^n dx \cdot \int_\Omega |u^n|^{\sigma-2} u^n \omega \; dx \right| \le \\ &\frac{1}{l_0}|\gamma| \|g\|_{2,\Omega} \|u^n\|_{2,\Omega} \|\omega\|_{\sigma,\Omega} \|u^n\|^{\sigma-1}_{\sigma,\Omega} \le \\ &C(\sigma,\Omega)\frac{1}{l_0}|\gamma| \|g\|_{2,\Omega} \|\omega\|_{\sigma,\Omega} \|u^n\|^\sigma_{2,\Omega} \le \frac{1}{8}\|u^n\|^2_{2,\Omega} + C_0''(t). \end{aligned} \tag{4.19}$$

where $C_0''(t) = \left(C(\sigma,\Omega) \frac{1}{l_0} |\gamma| \, \|g(t)\|_{2,\Omega} \, \|\omega\|_{\sigma,\Omega} \right)^{\frac{2}{2-\sigma}}$

Plugging the inequalities (4.16)-(4.19) into (4.14), we get

$$\begin{aligned} &\frac{d}{dt}\left(\|u^n\|_{2,\Omega}^2 + \|\nabla u^n\|_{2,\Omega}^2\right) + \|\nabla u^n\|_{L^p(\Omega)}^p + \gamma \|u^n\|_{L^\sigma(\Omega)}^\sigma \le \\ &\|u^n\|_{2,\Omega}^2 + C_0(t) \le \|u^n\|_{2,\Omega}^2 + \|\nabla u^n\|_{2,\Omega}^2 + C_0(t), \end{aligned} \tag{4.20}$$

where $C_0(t) = C^{\frac{2}{2-\sigma}}(|\gamma|,\Omega,\sigma) + \frac{2}{l_0^2}\|g(t)\|_{2,\Omega}^2 |e'(t)|^2 + C_0'(t) + C_0''(t)$, and due to the conditions (3.3)-(3.4), $C_1(t) \in L^1([0,T])$.

Omitting the second and third terms on left hand side and applying Grönwall's lemma for $y(t) := \|u^n\|_{2,\Omega}^2 + \|\nabla u^n\|_{2,\Omega}^2$ we get from (4.20)

$$\begin{aligned} &\|u^n\|_{2,\Omega}^2 + \|\nabla u^n\|_{2,\Omega}^2 \le e^t \left(\|u_0^n\|_{2,\Omega}^2 + \|\nabla u_0^n\|_{2,\Omega}^2 + \int_0^t C_0(\tau) e^{-\tau} d\tau \right) \\ &e^T \left(\|u_0\|_{2,\Omega}^2 + \|\nabla u_0\|_{2,\Omega}^2 + \int_0^T C_0(\tau) d\tau \right) := K_0' < \infty. \end{aligned} \tag{4.21}$$

Now integrate (4.20) by τ from 0 to t and take supremum by t. Then after using (4.21), we obtain the estimate (4.12).

In fact, multiplying both sides of (4.8) by $\frac{dc_k^n}{dt}$, summing on k and integrating the results with respect to τ from 0 to $t \in [0,T]$, we obtain

$$\begin{aligned} &\int_0^t \left(\|u_t^n(\tau)\|_{2,\Omega}^2 + \|\nabla u_t^n(\tau)\|_{2,\Omega}^2\right) d\tau + \frac{1}{p}\|\nabla u^n\|_{L^p(\Omega)}^p = \\ &\frac{\gamma}{\sigma}\|u^n\|_{L^\sigma(\Omega)}^\sigma + \frac{1}{p}\|\nabla u^n(0)\|_{L^p(\Omega)}^p - \frac{\gamma}{\sigma}\|u^n(0)\|_{L^\sigma(\Omega)}^\sigma + I_2, \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} I_2 = \int_0^t \frac{1}{g_0(\tau)} \Bigg(&e'(\tau) + \int_\Omega |\nabla u^n|^{p-2} \nabla u^n \cdot \nabla \omega \, dx \\ &- \gamma \int_\Omega |u^n|^{\sigma-2} u^n \cdot \omega dx \Bigg) \int_\Omega g(x,\tau) u_t^n(x,\tau) dx d\tau. \end{aligned} \tag{4.23}$$

Add the term $\frac{\gamma}{\sigma}\|u^n\|_{\sigma,\Omega}^\sigma$ on both sides of (4.22), and using

$$\|u^n\|_{L^\sigma(\Omega)} \le C(\sigma,\Omega)\|u^n\|_{L^2(\Omega)} \le C(\sigma,\Omega) K_0^{\frac{1}{2}}, \ \forall t \in [0,T], \tag{4.24}$$

which yeilds from (4.11) and (4.21), and applying the Hölder inequality together with (4.10) and (4.12), we obtain the relation

$$\begin{aligned}
&\int_0^t \left(\|u_t^n(\tau)\|_{2,\Omega}^2 + \|\nabla u_t^n(\tau)\|_{2,\Omega}^2\right) d\tau + \frac{1}{p}\|\nabla u^n\|_{L^p(\Omega)}^p + \frac{\gamma}{\sigma}\|u^n\|_{L^\sigma(\Omega)}^\sigma = \\
&\frac{2\gamma}{\sigma}\|u^n\|_{L^\sigma(\Omega)}^\sigma + \frac{1}{p}\|\nabla u^n(0)\|_{L^p(\Omega)}^p - \frac{\gamma}{\sigma}\|u^n(0)\|_{L^\sigma(\Omega)}^\sigma + I_2 \leq \\
&\frac{2|\gamma|}{\sigma}C(\sigma,\Omega)K_0^{\frac{\sigma}{2}} + \frac{1}{p}\|\nabla u_0\|_{L^p(\Omega)}^p + \frac{|\gamma|}{\sigma}\|u_0\|_{L^\sigma(\Omega)}^\sigma + |I_2|.
\end{aligned} \tag{4.25}$$

Next, applying Holder and Young's inequalities and (4.12), we estimate I_2 as follow

$$\begin{aligned}
&|I_2| \leq \frac{1}{l_0}\int_0^t \|u_t^n(\tau)\|_{2,\Omega}\|g\|_{2,\Omega}\left(|e'(\tau)| + \|\nabla\omega\|_{p,\Omega}\|\nabla u^n\|_{p,\Omega}^{p-1} + \right. \\
&\left.|\gamma|\,\|\omega\|_{\sigma,\Omega}\|u^n\|_{\sigma,\Omega}^{\sigma-1}\right) d\tau \leq \frac{1}{2}\int_0^t \|u_t^n(\tau)\|_{2,\Omega}^2\, d\tau + \\
&C\int_0^t \left(|e'(\tau)|^2 + \|\nabla u^n\|_{2,\Omega}^{2(p-1)} + \|u^n\|_{2,\Omega}^{2(\sigma-1)}\right) d\tau \leq \\
&\frac{1}{2}\int_0^t \|u_t^n(\tau)\|_{2,\Omega}^2\, ds + C\left(\int_0^t |e'(\tau)|^2 d\tau + TK_0^{p-1} + TK_0^{\sigma-1}\right) \leq \\
&\frac{1}{2}\int_0^t \|u_t^n(\tau)\|_{2,\Omega}^2\, d\tau + C_2
\end{aligned} \tag{4.26}$$

where

$$C_2 = C\cdot\left(TK_0^{\frac{p-1}{2}} + TK_0^{\frac{\sigma-1}{2}} + \int_0^t |e'(\tau)|^2 ds\right) < \infty$$

and

$$C = \frac{1}{2l_0^2}\sup_t \|g\|_{2,\Omega}^2 \cdot \max\left\{1, C^2(p,\Omega)\|\nabla\omega\|_{p,\Omega}^2, C^2(\sigma,\Omega)|\gamma|^2\|\omega\|_{\sigma,\Omega}^2\right\}.$$

Plugging (4.26) into (4.22) we have

$$\begin{aligned}
&\int_0^t \left(\|u_t^n(\tau)\|_{2,\Omega}^2 + \|\nabla u_t^n(\tau)\|_{2,\Omega}^2\right) d\tau + \frac{1}{p}\|\nabla u^n\|_{L^p(\Omega)}^p + \frac{\gamma}{\sigma}\|u^n\|_{L^\sigma(\Omega)}^\sigma \leq \\
&\quad C(\sigma,\Omega)(1 + \frac{|\gamma|}{\sigma})K_0^{\frac{\sigma}{2}} + \frac{1}{p}\|\nabla u_0\|_{L^p(\Omega)}^p + \frac{|\gamma|}{\sigma}\|u_0\|_{L^\sigma(\Omega)}^\sigma + C_2,
\end{aligned} \tag{4.27}$$

It follows that after maximizing by $t \in [0,T]$ the second estimate

$$\sup_{t\in[0,T]} \Big(\|\nabla u^n\|^p_{L^p(\Omega)} + \|u^n\|^\sigma_{L^\sigma(\Omega)} \Big) + \|u^n_t\|^2_{L^2(Q_T)} + \|\nabla u^n_t\|^2_{L^2(Q_T)} \le K_1 < \infty. \tag{4.28}$$

$\square$

Let now be $1 < p < \infty$ and $\sigma \le 2^*$. In this case we obtain the local a priori estimates.

Lemma 4.2. Assume that

$$u_0(x) \in W_0^{1,2}(\Omega)$$

and the conditions (3.2)-(3.4), (4.9)-(4.10) and (4.1) are fulfiled. Assume that also

$$\sigma \le 2^*,\ 2^* = \frac{2d}{d-2} \text{ if } d > 2; \quad 2^* \in (1,\infty) \text{ if } d = 2 \tag{4.29}$$

and

$$2 < \sigma \quad \text{or} \quad 2 < p. \tag{4.30}$$

Then there exist a constant K_2 and a time $T_1 \in (0,T]$ such that the following a priori estimate is valid for all $t \in (0,T_1)$

$$\|u^n\|^2_{L^\infty\left(0,T_1;W_0^{1,2}(\Omega)\right)} + \|\nabla u^n\|^p_{L^p(Q_{T_1})} + \|u^n\|^\sigma_{L^\sigma(Q_{T_1})} \le K_2 < \infty. \tag{4.31}$$

where T_1 is defined at (4.37) below.

Proof. Using (4.29), estimate the terms on the right hand side of (4.14) except I_{11} :

$$\begin{aligned} &2\gamma \|u^n\|^\sigma_{\sigma,\Omega} \le C_1'(\Omega,\gamma,\sigma,d)\Big(\|\nabla u^n\|^2_{2,\Omega} \Big)^{\frac{\sigma}{2}} \le \\ &C_1'(\Omega,\gamma,\sigma,d)\Big(1 + \|u^n\|^2_{2,\Omega} + \|\nabla u^n\|^2_{2,\Omega} \Big)^{\frac{\sigma}{2}}, \end{aligned} \tag{4.32}$$

$$\begin{aligned} &|I_{12}| \le \frac{1}{2}\|\nabla u^n\|^p_{p,\Omega} + C(p)\left(\frac{1}{l_0}\|g(t)\|_{2,\Omega}\|\nabla\omega\|_{p,\Omega}\|u^n\|_{2,\Omega} \right)^p \le \\ &\frac{1}{2}\|\nabla u^n\|^p_{p,\Omega} + C_1''\big(1 + \|u^n\|^2_{2,\Omega} + \|\nabla u^n\|^2_{2,\Omega}\big)^{\frac{p}{2}} \end{aligned} \tag{4.33}$$

where $C_1'' = C(p)\left(\frac{1}{l_0} \sup_{t\in[0,T]} \|g(t)\|_{2,\Omega}\|\nabla\omega\|_{p,\Omega} \right)^p$.

Using Sobolev and Poincare inequalities, we get also

$$\begin{aligned} &|I_{13}| \le C(\sigma,d,\Omega)|\gamma|\frac{1}{l_0}\|g(t)\|_{2,\Omega}\|\omega\|_{\sigma,\Omega}\|\nabla u^n\|^\sigma_{2,\Omega} \le \\ &C_1'''\big(1 + \|u^n\|^2_{2,\Omega} + \|\nabla u^n\|^2_{2,\Omega}\big)^{\frac{\sigma}{2}} \end{aligned} \tag{4.34}$$

where $C_1''' = C(\sigma, d, \Omega)|\gamma| \frac{1}{l_0} \sup_{t\in[0,T]} \|g(t)\|_{2,\Omega} \|\omega\|_{\sigma,\Omega}$.

Plugging the inequalities (4.32),(4.17), (4.33) (4.34) into (4.14), and integrating by $\tau \in (0,t)$ we have

$$Y(t) + \int_0^t \left(\|\nabla u^n\|^p_{L^p(\Omega)} + \gamma \|u^n\|^\sigma_{L^\sigma(\Omega)} \right) d\tau \le C_1^0 \int_0^t Y^{\mu_1}(\tau) d\tau + C_1^1 \tag{4.35}$$

where $\mu_1 := \frac{1}{2}\max\{2,\sigma,p\}$ ($\mu_1 > 1$ since (4.30)) and

$$Y(t) := 1 + \|u^n\|^2_{2,\Omega} + \|\nabla u^n\|^2_{2,\Omega},$$

$$C_1^0(t) = \sup_{t\in(0,T)} \left(C_1' + C_1'' + C_1''' + 1\right),$$

$$C_1^1 = \frac{1}{l_0^2} \sup_{t\in(0,T)} \|g(t)\|^2_{2,\Omega} \|e'(t)\|^2_{L^2(0,T)} + \left(1 + \|u_0\|^2_{2,\Omega} + \|\nabla u_0\|^2_{2,\Omega}\right).$$

Omitting the integral terms on the left hand side of (4.35) and applying the Lemma 2.1, we obtain

$$Y(t) \le C_1^1 \left[1 - (\mu_1 - 1) C_1^{1\,\mu_1 - 1} C_1^0 t\right]^{-\frac{1}{\mu_1 - 1}} \tag{4.36}$$

for

$$0 \le t < T_1 := \frac{1}{(\mu_1 - 1) C_1^{1\,\mu_1 - 1} C_1^0}, \tag{4.37}$$

Integrate (4.35) by τ from 0 to t and maximize by t. After using (4.36) we have for all $t \in [0, T_1]$

$$\sup_t \left(\|u^n\|^2_{2,\Omega} + \|\nabla u^n\|^2_{2,\Omega} \right) + \|\nabla u^n\|^p_{L^p(Q_t)} + \gamma \|u^n\|^\sigma_{L^\sigma(Q_t)} \le K_2 < \infty. \tag{4.38}$$

□

Now we get second energy estimates for Galerkin's approximations in the case (4.29).

Lemma 4.3. Assume that all conditions of Lemma 4.2 are hold and

$$u_0(x) \in W_0^{1,2}(\Omega) \cap W^{1,p}(\Omega) \cap L^\sigma(\Omega).$$

Then there is a constant K_3 and a time $T_2 \in (0, T_1]$ such that the following a priori estimate is valid for all $t \in (0, T_2)$

$$\sup_{t\in[0,T_2]} \left(\|\nabla u^n\|^p_{L^p(\Omega)} + \|u^n\|^\sigma_{L^\sigma(\Omega)} \right) + \|u^n_t\|^2_{L^2\left(0,T_2;W_0^{1,2}(\Omega)\right)} \le K_3 < \infty. \tag{4.39}$$

Proof. Analogical way as (4.26), using Holder inequality together with (4.10), (4.38), and

$$\|u^n\|_{L^\sigma(\Omega)} \le C(\sigma,\Omega)\,\|\nabla u^n\|_{L^2(\Omega)} \le C(\sigma,\Omega)K_1 := C_2' < \infty,\ \ t\in[0,T_1] \tag{4.40}$$

we obtain

$$\begin{aligned}
|I_2| &\le \frac{1}{2}\int_0^t \|u_t^n\|_{2,\Omega}^2\,d\tau + \frac{1}{2l_0^2}\cdot\int_0^t \|g(\tau)\|_{2,\Omega}^2\,\|\nabla\omega\|_{p,\Omega}^2\,\|\nabla u^n\|_{p,\Omega}^{2(p-1)}\,d\tau + \\
&\frac{1}{2l_0^2}\int_0^t \|g(\tau)\|_{2,\Omega}^2\left(|e^{'}(\tau)|^2 + C(\Omega)|\gamma|^2\,\|\omega\|_{\sigma,\Omega}^2\cdot\|\nabla u^n\|_{2,\Omega}^{2(\sigma-1)}\right)d\tau \le \\
&\frac{1}{2}\int_0^t \|u_t^n\|_{2,\Omega}^2\,d\tau + C_2^0\int_0^t\left(\frac{1}{p}\|\nabla u^n\|_{p,\Omega}^p\right)^{\mu_2}d\tau + C_2''
\end{aligned} \tag{4.41}$$

where $\mu_2 := \frac{2(p-1)}{p}$, and $C_2^0 = \dfrac{p^{\mu_2}}{2l_0^2}\sup\limits_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2\,\|\nabla\omega\|_{p,\Omega}^2$,

$$C_2'' = \frac{1}{2l_0^2}\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2\int_0^T\left(|e^{'}(\tau)|^2 d\tau + C(\Omega)|\gamma|^2\,\|\omega\|_{\sigma,\Omega}^2\cdot K_2^{\sigma-1}T\right)$$

Plugging (4.41) and (4.40) into (4.22), we get the following ODI

$$\begin{aligned}
&\int_0^t \left(\,\|u_t^n\|_{2,\Omega}^2 + \|\nabla u_t^n\|_{2,\Omega}^2\,\right)d\tau + \frac{1}{p}\|\nabla u^n\|_{L^p(\Omega)}^p \le \\
&C_2^0\int_0^t\left(\frac{1}{p}\|\nabla u^n\|_{L^p(\Omega)}^p\right)^{\mu_2}d\tau + C_2^1
\end{aligned} \tag{4.42}$$

where $C_2^1 = \frac{\gamma}{\sigma}{C_2'}^{\sigma} + \frac{1}{p}\|\nabla u_0\|_{L^p(\Omega)}^p + \frac{|\gamma|}{\sigma}\|u_0\|_{L^\sigma(\Omega)}^\sigma + C_2''$.

Omitting the first two terms on the left hand side of (4.41) and applying the Lemma 2.1 for $\frac{1}{p}\|\nabla u^n\|_{L^p(\Omega)}^p$, we obtain

$$\frac{1}{p}\|\nabla u^n\|_{L^p(\Omega)}^p \le C_2^1\left(1-(\mu_2-1){C_2^1}^{\mu_2-1}C_2^0 t\right)^{-\frac{1}{\mu_2-1}} \tag{4.43}$$

for

$$0\le t < T_2 := \frac{1}{(\mu_2-1){C_2^1}^{\mu_2-1}C_2^0}. \tag{4.44}$$

Substituting (4.43) and (4.40) into (4.42) and taking supremum by $t\in[0,T_2]$, we get the estimate (4.39).

□

4.3. Passage to the limit as $n \to \infty$

By means of reflexivity and up to some subsequences, the estimates (4.12) and (4.31) imply that

$$u^n \rightharpoonup u \quad \text{weakly-}* \text{ in } L^\infty(0,T;W_0^{1,2} \cap L^\sigma(\Omega)), \quad \text{as } n \to \infty, \tag{4.45}$$
$$u^n \rightharpoonup u \quad \text{weakly in} \quad L^2(Q_T) \cap L^\sigma(Q_T), \quad \text{as } n \to \infty, \tag{4.46}$$
$$\nabla u^n \rightharpoonup \nabla u \quad \text{weakly-}* \text{ in } L^\infty(0,T;L^2 \cap L^p(\Omega)), \text{ as } n \to \infty, \tag{4.47}$$
$$\nabla u^n \rightharpoonup \nabla u \quad \text{weakly in} \quad L^2(Q_T) \cap L^p(Q_T), \text{ as } n \to \infty, \tag{4.48}$$
$$u_t^n \rightharpoonup u_t \quad \text{weakly in } L^2(0,T;W_0^{1,2}(\Omega)), \quad \text{as } n \to \infty. \tag{4.49}$$

On the other hand, (4.31) and (4.13) imply the existence of functions S and R such that, respectively,

$$|\nabla u^n|^{p-2}\nabla u^n \rightharpoonup S \quad \text{weakly in } L^{p'}(Q_T), \quad \text{as } n \to \infty, \tag{4.50}$$
$$|u^n|^{\sigma-2}u^n \rightharpoonup R \quad \text{weakly in } L^{\sigma'}(Q_T), \quad \text{as } n \to \infty. \tag{4.51}$$

From (4.47) and (4.49), due to the compact and continuous imbedding,

$$W_0^{1,s}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega) \hookrightarrow L^2(\Omega), \qquad \forall\, r : 2 \le r < s^*, \quad s = \max\{2,p\}$$

and by Aubin-Lions compactness lemma, we obtain

$$u^n \longrightarrow u \quad \text{strongly in } L^s(0,T;L^r(\Omega)), \quad 2 \le r < s^*, \quad \text{as } n \to \infty, \tag{4.52}$$

and in particular,

$$u^n \longrightarrow u \quad \text{strongly in } L^2(0,T;L^2(\Omega)), \quad \text{as } n \to \infty. \tag{4.53}$$

As a consequence of (4.53) and Riesz-Fischer's theorem, we have up to some subsequence,

$$u^n \longrightarrow u \quad \text{a.e. in } Q_T, \quad \text{as } n \to \infty, \tag{4.54}$$

which together with (4.51) yields (see Lemma 1.3 in [20, p. 12])

$$|u^n|^{\sigma-2}u^n \rightharpoonup |u|^{\sigma-2}u \quad \text{weakly in } L^{\sigma'}(Q_T), \quad \text{as } n \to \infty. \tag{4.55}$$

Under the assumption (4.5) and (4.49), (4.52), we have also that

$$u^n \longrightarrow u \quad \text{strongly in} \quad L^\sigma(Q_T), \quad \text{as} \quad n \to \infty, \quad \text{for} \quad \sigma < 2^*.$$

and consequently

$$\|u^n\|_{\sigma,Q_t} \longrightarrow \|u\|_{\sigma,Q_t} \quad \text{as} \quad n \to \infty. \tag{4.56}$$

Let now $\eta(t)$ be a continuously differentiable function on $[0,T]$ such that $\eta(T) = 0$, where T is the maximal time such that above first and second estimates are hold. Multiplying (4.8) by η and integrating by $t \in [0,T]$, we obtain

$$\int_{Q_T} (u_t^n \cdot z_k + \nabla u_t^n \cdot \nabla z_k) dx dt + \int_{Q_T} |\nabla u^n|^{p-2} \nabla u^n \cdot \nabla z_k \, dx dt =$$
$$\gamma \int_{Q_T} |u^n|^{\sigma-2} u^n \cdot z_k \, dx dt + \int_0^T \left[\frac{1}{g_0(t)} (e'(t) + \right. \tag{4.57}$$
$$\left. \int_\Omega |\nabla u^n|^{p-2} \nabla u^n \cdot \nabla \omega \; dx - \gamma \int_\Omega |u^n|^{\sigma-2} u^n \cdot \omega dx \right) \int_\Omega g \cdot z_k \, dx \Bigg] \, dt,$$

and using above convergence results (4.49), (4.50), (4.55), and (4.46), we obtain

$$\int_{Q_T} (u_t \cdot z_k + \nabla u_t \cdot \nabla z_k) dx dt + \int_{Q_T} S \cdot \nabla z_k \, dx dt =$$
$$\gamma \int_{Q_T} |u|^{\sigma-2} u \cdot z_k \, dx dt + \int_0^T \left[\frac{1}{g_0(t)} (e'(t) + \right. \tag{4.58}$$
$$\left. + \int_\Omega S \cdot \nabla \omega \; dx - \gamma \int_\Omega |u|^{\sigma-2} u \cdot \omega dx \right) \int_\Omega g \cdot z_k \, dx \Bigg] \, dt.$$

for all $z_k = \psi_k(x)\eta(t), \quad k \in \{1, \dots, n\}$.

By linearity and by a continuity argument, the equation (4.58) is still true for any

$$z \in Z := \{ z = \psi\zeta : \psi \in \mathcal{V}, \; \zeta \in C_0^\infty(0,T) \}.$$

Let us now prove that

$$S = |\nabla u|^{p-2} \nabla u \tag{4.59}$$

by using monotonicity. Firstly, we observe that since the set Z is dense in $L^\infty(0,T; W_2^1) \cap L^p(0,T; V_p) \cap L^\sigma(Q_T)$, we can take $z = u_n$ and $z = u$ as test functions in (4.57) and (4.58), respectively

$$\int_{Q_T} (u_t^n \cdot u^n + \nabla u_t^n \cdot \nabla u^n) dx dt + \int_{Q_T} |\nabla u^n|^p \, dx dt =$$
$$\gamma \int_{Q_T} |u^n|^\sigma \, dx dt + \int_0^T \left[\frac{1}{g_0(t)} (e'(t) + \right. \tag{4.60}$$
$$\left. \int_\Omega |\nabla u^n|^{p-2} \nabla u^n \nabla \omega \; dx - \gamma \int_\Omega |u^n|^{\sigma-2} u^n \omega dx \right) \int_\Omega g u^n dx \Bigg] \, dt.$$

and

$$
\begin{aligned}
&\int\limits_{Q_T} (u_t \cdot u + \nabla u_t \cdot \nabla u) dxdt + \int\limits_{Q_T} S \cdot \nabla u \, dxdt = \\
&\gamma \int\limits_{Q_T} |u|^\sigma \, dxdt + \int\limits_0^T \left[\frac{1}{g_0(t)} \left(e'(t) + \right.\right. \\
&\left.\left. \int\limits_\Omega S \cdot \nabla \omega \; dx - \gamma \int\limits_\Omega |u|^{\sigma-2} u \cdot \omega dx \right) \int\limits_\Omega g \cdot u \, dx \right] dt.
\end{aligned}
\tag{4.61}
$$

Next, using assumption (4.1) together with the monotonicity property of the operator $F(\xi) = |\xi|^{r-2}\xi$ for $r = p$ (see e.g. [11]), we have

$$
X_n := \int\limits_{Q_T} \left(|\nabla u^n|^{p-2} \nabla u^n - |\nabla z|^{p-2} \nabla z \right) \cdot (\nabla u^n - \nabla z) dxdt \geq 0 \tag{4.62}
$$

for all $z \in Z$. Expanding (4.62) and using the identity (4.60),

$$
\begin{aligned}
&0 \leq X_n = \gamma \int\limits_{Q_T} |u_n|^\sigma \, dxdt + \int\limits_0^T \int\limits_\Omega g u^n \, dx \left[\frac{1}{g_0(t)} \left(e'(t) + \right.\right. \\
&\left.\left. \int\limits_\Omega |\nabla u^n|^{p-2} \, \nabla u^n \cdot \nabla \omega \; dx - \gamma \int\limits_\Omega |u^n|^{\sigma-2} u^n \cdot \omega dx \right) \right] dt - \\
&\int\limits_{Q_T} \left(u_t^n u^n + \nabla u_t^n \nabla u^n \right) \, dxdt - \int\limits_{Q_T} |\nabla u^n|^{p-2} \nabla u^n \cdot \nabla z \, dxdt - \\
&\int\limits_{Q_T} |\nabla z|^{p-2} \nabla z \nabla u^n \, dxdt + \int\limits_{Q_T} |\nabla z|^p \, dxdt := \int\limits_{Q_T} |\nabla z|^p \, dxdt \\
&+ X_n^1 + X_n^2 + X_n^3 - X_n^4 - X_n^5 - X_n^6 - X_n^7 - X_n^8.
\end{aligned}
\tag{4.63}
$$

Regarding the convergence of the terms X_n^5 and X_n^6, we first note that (see, [8]),

$$
u \in C_w([0,T]; W_0^{1,2}(\Omega)), \tag{4.64}
$$

where $C_w([0,T]; W_0^{1,2}(\Omega))$ denotes the subspace of $L^\infty([0,T]; W_0^{1,2}(\Omega))$ formed by weakly continuous functions from $[0,T]$ onto $W_0^{1,2}(\Omega)$. Hence, (4.64) implies the quantities $u(0)$, $u(T)$, $\nabla(u(0))$ and $\nabla(u(T)))$ are meaningful.

Taking the $\limsup$ of (4.63) and using the properties of $\limsup$ and $\liminf$, and the convergence results (4.56) for X_n^1, (4.46) for X_n^2 and X_n^8, (4.50) for X_n^3 and X_n^7, (4.55) for X_n^4, and

(4.49) for X_n^5 and X_n^6, we obtain

$$0 \leq \gamma \int\limits_{Q_T} |u|^\sigma \, dxdt + \int\limits_0^T \left[\frac{1}{g_0(t)} \left(e'(t) + \int\limits_\Omega S \cdot \nabla \omega \, dx - \gamma \int\limits_\Omega |u|^{\sigma-2} u \cdot \omega dx \right) \int\limits_\Omega g \cdot u \, dx \right] dt - \int\limits_{Q_T} (u_t \cdot u + \nabla u_t \cdot \nabla u) \, dxdt - \int\limits_{Q_T} S \cdot \nabla z \, dxdt - \int\limits_{Q_T} |\nabla z|^{p-2} \nabla z \cdot \nabla u \, dxdt + \int\limits_{Q_T} |\nabla z|^p \, dxdt. \tag{4.65}$$

Combining (4.61) with (4.65), we achieve to

$$\int\limits_{Q_T} \left(S - |\nabla z|^{p-2} \nabla z \right) \cdot (\nabla u - \nabla z) \, dxdt \geq 0 \qquad \forall \, z \in Z. \tag{4.66}$$

By density, (4.66) still holds for all $z \in L^p(0,T;V_p) \cap L^\sigma(Q_T)$. Thus, taking $z = u \pm \delta \xi$ for an arbitrary $\xi \in L^p(0,T;V_p) \cap L^\sigma(Q_T)$ and $\delta > 0$, it follows from (4.66)

$$\pm \int\limits_{Q_T} \left(S - |\nabla u \mp \delta \nabla \xi|^{p-2} \left(\nabla u \mp \delta \nabla \xi \right) \right) \cdot \nabla \xi \, dxdt \geq 0. \tag{4.67}$$

Letting $\delta \to 0$ in (4.67), we obtain

$$\pm \int\limits_{Q_T} \left(S - |\nabla u|^{p-2} \nabla u \right) \cdot \nabla \xi \, dxdt \geq 0.$$

Due to the arbitrariness of ξ, we see that it must be $S = |\nabla u|^{p-2} \nabla u$ which proves (4.59). □

5. The case $\gamma \leq 0$: global and local existence

In this section, we consider the problem (3.6)-(3.9) in the case $1 < \sigma = const < \infty$ and

$$\gamma \leq 0. \tag{5.1}$$

The following theorems are hold.

Theorem 5.1. [Global existence] Let the conditions (3.2)-(3.4) and (5.1) be fulfilled and

$$1 < \sigma, \; p \leq 2. \tag{5.2}$$

Then the problem (3.6)-(3.9) has at least one weak solution $u(x,t)$ for all $t \in [0,T]$. Moreover, the weak solutions to the problem (3.6)-(3.9) satisfy the following estimates

$$\sup_{t\in[0,T]} \left(\|u\|_{2,\Omega}^2 + \|\nabla u\|_{2,\Omega}^2\right) + \|\nabla u\|_{p,Q_T}^p + \|u\|_{\sigma,Q_T}^\sigma \le C_1, \tag{5.3}$$

$$\sup_{t\in[0,T]} \left(\|\nabla u\|_{p,\Omega}^p + \|u(t)\|_{\sigma,\Omega}^\sigma\right) + \|u_t\|_{2,Q_T}^2 + \|\nabla u_t\|_{2,Q_T}^2 \le C_2. \tag{5.4}$$

Theorem 5.2. [Local existence] Let the conditions (1.5), (3.2)-(3.4) and (5.1) be fulfilled and one of the assumption is hold

$$2 < \sigma \quad \text{or} \quad 2 < p. \tag{5.5}$$

Then there exists a time $T_* \in (0,T]$ such that the problem (3.6)-(3.9) has at least one weak solution $u(x,t)$ for all $t \in [0,T_*]$. Moreover, for all $t \in [0,T_*]$ the estimates (5.3) and (5.4) are hold. Here $T_* := T_4$ and T_4 is defined at (5.25) below.

Remark 5.1. In the case $\gamma \le 0$ we do not need the condition $\sigma \le 2^*$, which have used in Theorem 4.2. Because, we use the monotonicity for

$$\begin{aligned} X_n := \int\limits_{Q_T} \left(|\nabla u^n|^{p-2}\nabla u^n - |\nabla z|^{p-2}\nabla z\right)\cdot(\nabla u^n - \nabla z)dxdt + \\ |\gamma|\left(|u^n|^{\sigma-2}u^n - |z|^{\sigma-2}z\right)\cdot(u^n - z)dxdt \ge 0 \end{aligned} \tag{5.6}$$

in order to pass the limit, since γ is a positive in (5.6). Therefore the expression (4.63) has the following form

$$\begin{aligned} &0 \le X_n = \int\limits_0^T \left[\frac{1}{g_0(t)}\left(e'(t) + \int\limits_\Omega |\nabla u^n|^{p-2}\nabla u^n\cdot\nabla\omega\, dx + \right.\right. \\ &\left.\left.|\gamma|\int\limits_\Omega |u^n|^{\sigma-2}u^n\cdot\omega dx\right)\right]\int\limits_\Omega g u^n\, dxdt - \int\limits_{Q_T}\left(u_t^n u^n + \nabla u_t^n \nabla u^n\right) dxdt - \\ &\int\limits_{Q_T} |\nabla u^n|^{p-2}\nabla u^n\cdot\nabla z\, dxdt - \int\limits_{Q_T}|\nabla z|^{p-2}\nabla z\nabla u^n\, dxdt + \int\limits_{Q_T}|\nabla z|^p\, dxdt - \\ &-|\gamma|\int\limits_{Q_T}|u^n|^{\sigma-2}u^n\cdot z\, dxdt - |\gamma|\int\limits_{Q_T}|z|^{\sigma-2}z\cdot u^n\, dxdt + \int\limits_{Q_T}|z|^\sigma\, dxdt := \\ &X_n^1 + X_n^2 + X_n^3 - X_n^4 - X_n^5 - X_n^6 - X_n^7 + \int\limits_{Q_T}|\nabla z|^p\, dxdt - X_n^8 - X_n^9 + \int\limits_{Q_T}|z|^\sigma\, dxdt. \end{aligned} \tag{5.7}$$

Proof. For to prove these theorems, it is enough to establish analogical as above the first and second estimates. Then we repeat the arguments we have used to pass to limit in Section 4.3. □

5.1. Global and local a priori estimates

Lemma 5.1. Assume that

$$u_0 \in W_0^{1,2}(\Omega) \cap W^{1,p}(\Omega) \cap L^\sigma(\Omega) \tag{5.8}$$

and the conditions (3.2) -(3.4), and (5.1) are hold. If the condition (5.2) is fulfilled, then the following estimates (global in time) are valid for all $t \in (0, T]$

$$\|u^n\|^2_{L^\infty\left(0,T;W_0^{1,2}(\Omega)\right)} + \|\nabla u^n\|^p_{L^p(Q_T)} + \|u^n\|^\sigma_{L^\sigma(Q_T)} \le K_4 < \infty \tag{5.9}$$

$$\sup_{t\in[0,T]} \left(\|\nabla u^n\|^p_{L^p(\Omega)} + \|u^n\|^\sigma_{L^\sigma(\Omega)}\right) + \|u^n_t\|^2_{L^2\left(0,T;W_0^{1,2}(\Omega)\right)} \le K_5 < \infty. \tag{5.10}$$

If (5.5) holds instead of (5.2), then there exist a time $T_3 \in (0, T]$ and $T_4 \in (0, T_3]$ such that the estimate (5.9) is hold are for all $t \in (0, T_3]$ and (5.10) for all $t \in (0, T_4]$. Here T_3 and T_4 are defined at (5.20) and (5.26) respectively.

Proof. In the case (5.1), the first and second energy equalities (4.14) and (4.22) have the following form, respectively

$$\frac{1}{2}\frac{d}{dt}\left(\|u^n\|^2_{2,\Omega} + \|\nabla u^n\|^2_{2,\Omega}\right) + \|\nabla u^n\|^p_{L^p(\Omega)} + |\gamma|\,\|u^n\|^\sigma_{L^\sigma(\Omega)} = I_1', \tag{5.11}$$

$$\begin{aligned}&\int_0^t \left(\|u^n_t(\tau)\|^2_{2,\Omega} + \|\nabla u^n_t(\tau)\|^2_{2,\Omega}\right) d\tau + \frac{1}{p}\|\nabla u^n\|^p_{L^p(\Omega)} + \frac{|\gamma|}{\sigma}\|u^n\|^\sigma_{L^\sigma(\Omega)} = \\ &\frac{1}{p}\|\nabla u_0\|^p_{L^p(\Omega)} + \frac{|\gamma|}{\sigma}\|u_0\|^\sigma_{L^\sigma(\Omega)} + I_2',\end{aligned} \tag{5.12}$$

where

$$\begin{aligned}I_1' =& \frac{1}{g_0(t)}\left(e'(t) + \int_\Omega |\nabla u^n|^{p-2}\,\nabla u^n \cdot \nabla\omega\; dx\right.\\ &\left. + |\gamma| \int_\Omega |u^n|^{\sigma-2} u^n \cdot \omega dx\right) \int_\Omega g(x,t) u^n(x) dx = \sum_{i=1}^3 I_{1i}',\end{aligned} \tag{5.13}$$

$$\begin{aligned}I_2' =& \frac{1}{g_0(t)}\left(e'(t) + \int_\Omega |\nabla u^n|^{p-2}\,\nabla u^n \cdot \nabla\omega\; dx\right.\\ &\left. + |\gamma| \int_\Omega |u^n|^{\sigma-2} u^n \cdot \omega dx\right) \int_\Omega g(x,t) u^n_t(x) dx.\end{aligned} \tag{5.14}$$

Let us estimate I_{13}', using Hölder and Young's inequalities. For the terms I_{11}' and I_{12}' we use the inequalities (4.17) and (4.18) which valid for any $\sigma, p > 1$, respectively.

$$|I'_{13}| \leq \frac{1}{l_0}|\gamma| \left\|g\right\|_{2,\Omega} \left\|u^n\right\|_{2,\Omega} \left\|\omega\right\|_{\sigma,\Omega} \left\|u^n\right\|^{\sigma-1}_{\sigma,\Omega} \leq \frac{|\gamma|}{2} \left\|u^n\right\|^{\sigma}_{\sigma,\Omega} + C_3 \left(\left\|u^n\right\|^2_{2,\Omega}\right)^{\frac{\sigma}{2}}. \tag{5.15}$$

where $C_3 = \frac{|\gamma|}{2l_0^{\sigma}} \sup\limits_{t\in[0,T]} \left\|g(t)\right\|^{\sigma}_{2,\Omega} \left\|\omega\right\|^{\sigma}_{\sigma,\Omega}$.

Plugging inequalities (4.17), (4.33), (5.15) into (5.11), and adding 1 into derivative, we have

$$\begin{aligned} &\frac{d}{dt}\left(1 + \left\|u^n\right\|^2_{2,\Omega} + \left\|\nabla u^n\right\|^2_{2,\Omega}\right) + \left\|\nabla u^n\right\|^p_{L^p(\Omega)} + |\gamma| \left\|u^n\right\|^{\sigma}_{\sigma,\Omega} \leq \\ &\frac{1}{4}\left\|u^n\right\|^2_{2,\Omega} + \frac{1}{l_0^2}\left\|g(t)\right\|^2_{2,\Omega} |e'(t)|^2 + 2C_{13}\left(\left\|u^n\right\|^2_{2,\Omega}\right)^{\frac{\sigma}{2}} + \\ &2C''_1(t)\big(1 + \left\|u^n\right\|^2_{2,\Omega} + \left\|\nabla u^n\right\|^2_{2,\Omega}\big)^{\frac{p}{2}} \leq C_3^0\big(1 + \left\|u^n\right\|^2_{2,\Omega} + \left\|\nabla u^n\right\|^2_{2,\Omega}\big)^{\mu_3} + C'_3, \end{aligned} \tag{5.16}$$

where $\mu_3 = \max\{1, \frac{p}{2}, \frac{\sigma}{2}\}$, $C_3^0 = \frac{1}{4} + 2C_3 + 2C''_1$, and $C'_3 = \frac{1}{l_0^2}\left\|g(t)\right\|^2_{2,\Omega} |e'(t)|^2$. Integrating (5.16) by $\tau \in (0, t)$, we get

$$Y(t) + \left\|\nabla u^n\right\|^p_{L^p(Q_t)} + |\gamma| \left\|u^n\right\|^{\sigma}_{L^{\sigma}(Q_t)} \leq C_3^0 \int\limits_0^t Y^{\mu_3}(\tau)d\tau + C_3^1, \tag{5.17}$$

where $Y(t) = 1 + \left\|u^n\right\|^2_{2,\Omega} + \left\|\nabla u^n\right\|^2_{2,\Omega}$ and

$$C_3^1 := 1 + \left\|u_0\right\|^2_{2,\Omega} + \left\|\nabla u_0\right\|^2_{2,\Omega} + \frac{1}{l_0^2} \sup_{t\in[0,T]} \left\|g(t)\right\|^2_{2,\Omega} \int\limits_0^T |e'(t)|^2 dt.$$

Apply the classical Gronwall's lemma to (5.17) in the case $\mu_3 = 1$, i.e. if (5.2) holds, and the Lemma 2.1 in the case $\mu_3 > 1$, i.e. if (5.5) holds. As result, we obtain the following estimates, respectively

$$Y(t) \leq C_3^1 \cdot e^{C_3^0 T} < \infty, \quad \forall t \in [0, T] \text{ in the case } \mu_3 = 1 \tag{5.18}$$

and

$$Y(t) \leq C_3^1 \left[1 - (\mu_3 - 1){C_3^1}^{\mu_3-1} C_3^0 t\right]^{-\frac{1}{\mu_3-1}} < \infty, \tag{5.19}$$

for

$$0 \leq t < T_3 := \frac{1}{(\mu_3 - 1){C_3^1}^{\mu_3-1} C_3^0} \text{ in the case } \mu_3 > 1. \tag{5.20}$$

Substituting (5.18) and (5.19) into (5.17), and taking supremum by t, we obtain the following first energy estimate

$$\sup_{t\in(0,T'_*]} \left(\left\|u^n\right\|^2_{2,\Omega} + \left\|\nabla u^n\right\|^2_{2,\Omega}\right) + \left\|\nabla u^n\right\|^p_{L^p(Q_{T'_*})} + \left\|u^n\right\|^{\sigma}_{L^{\sigma}(Q_{T'_*})} \leq K_4, \tag{5.21}$$

where $T'_* = T$ if (5.2) holds and $T'_* = T_3$ if (5.5) holds.

Next, by using Holder's and Cauchy's inequalities estimate I'_2:

$$|I'_2| \leq \frac{1}{2}\int_0^t \|u_t^n\|_{2,\Omega}^2 \, d\tau + \frac{1}{2l_0^2} \cdot \int_0^t \|g(\tau)\|_{2,\Omega}^2 \|\nabla\omega\|_{p,\Omega}^2 \|\nabla u^n\|_{p,\Omega}^{2(p-1)} \, d\tau +$$

$$\frac{1}{2l_0^2}\int_0^t \|g(\tau)\|_{2,\Omega}^2 \left(|e^{'}(\tau)|^2 + C(\Omega)|\gamma|^2 \|\omega\|_{\sigma,\Omega}^2 \cdot \|\nabla u^n\|_{\sigma,\Omega}^{2(\sigma-1)}\right) d\tau \leq \tag{5.22}$$

$$\frac{1}{2}\int_0^t \|u_t^n\|_{2,\Omega}^2 \, d\tau + C_4^0 \int_0^t \left(1 + \frac{1}{p}\|\nabla u^n\|_{p,\Omega}^p + \frac{\gamma}{\sigma}\|u^n\|_{\sigma,\Omega}^\sigma\right)^{\mu_4} d\tau$$

where

$$\mu_4 := \max\{\frac{2(p-1)}{p}, \frac{2(\sigma-1)}{\sigma}\}$$

$$C_4^0 := \frac{1}{2l_0^2}\sup_{t\in[0,T]} \|g(t)\|_{2,\Omega}^2 \cdot \max\{\|\nabla\omega\|_{p,\Omega}^2, |e'(t)|^2 + C(\Omega)|\gamma|^2 \|\omega\|_{\sigma,\Omega}^2\}$$

Plugging inequality (5.22) into (5.12) and adding 1 to both sides, we have for $Z(t) := 1 + \frac{1}{p}\|\nabla u^n\|_{p,\Omega}^p + \frac{|\gamma|}{\sigma}\|u^n\|_{\sigma,\Omega}^\sigma > 1$

$$Z(t) + \int_0^t \left(\|u_t^n(\tau)\|_{2,\Omega}^2 + \|\nabla u_t^n(\tau)\|_{2,\Omega}^2\right) d\tau \leq C_4^1 + C_4^0 \int_0^t Z^{\mu_4}(\tau) d\tau, \tag{5.23}$$

where $C_4^1 := 1 + \frac{1}{p}\|\nabla u_0\|_{L^p(\Omega)}^p + \frac{|\gamma|}{\sigma}\|u_0\|_{L^\sigma(\Omega)}^\sigma$. Omitting the integrals on left hand side of (5.23) and applying the classical Granwall's lemma to the obtained inequality in the case $\mu_4 \leq 1$, i.e. if (5.2) holds, and the Lemma 2.1, in the case $\mu_4 > 1$, i.e. if (5.5) holds, we get the following estimates, respectively

$$Z(t) \leq C_4^1 \cdot e^{C_4^0 T} < \infty, \quad \forall t \in [0, T] \tag{5.24}$$

and

$$Z(t) \leq C_4^1 \left(1 - (\mu_4 - 1)C_4^0 (C_4^1)^{\mu_4 - 1} T\right)^{-\frac{1}{\mu_4 - 1}} \tag{5.25}$$

$$\text{for } \ 0 < t < T_4 := \frac{1}{(\mu_4 - 1)C_4^0 (C_4^1)^{\mu_4 - 1}}. \tag{5.26}$$

Substituting (5.24) and (5.25) into (5.23) separately, and taking esssupremum by t to the result, we obtain the following second estimate

$$\sup_{t\in[0,T_{max}]} \left(\|\nabla u^n\|_{L^p(\Omega)}^p + \|u^n\|_{L^\sigma(\Omega)}^\sigma\right) + \|u_t^n\|_{L^2\left(0,T_{max};W_0^{1,2}(\Omega)\right)}^2 \leq K_5 < \infty. \tag{5.27}$$

where $T_{max} = T$, in the case (5.2) and $T_{max} = T_4 \leq T_3$, in the case (5.5). $\square$

6. Uniqeness

Theorem 6.1. Assume that the following conditions hold

$$1 < p \leq 4 \tag{6.1}$$

$$\nabla \omega \in L^{\frac{2p}{4-p}}(\Omega) \tag{6.2}$$

and

$$\sigma \leq \frac{2d}{d-s}, d > s, \quad \text{where } s \in \{2, p\}. \tag{6.3}$$

If $\gamma \leq 0$, assume addition to (6.1)-(6.3) that all conditions of Lemmas 4.1 and 4.2 are fulfilled. If $\gamma > 0$, assume addition to (6.1)-(6.3) that all conditions of Lemmas 3.2 and 3.3 are fulfilled. Then the weak solution of (3.6)-(3.9) is unique. Here 2^* and p^* are Sobolev conjugates of 2 and p respectively.

Remark 6.1. The condition (6.3) means that take $s = \max\{2, p\}$, if we want to get the range of σ to be large, and take $s = \min\{2, p\}$, if the range of dimension d to be large.

Proof. Let u_1 and u_2 be two weak solutions to the problem (3.6)-(3.8) in the sense of Definition 3.1. Using $u := u_1 - u_2$ as a test function in (3.15), it follows, by subtracting the equation for u_2 to the equation for u_1, that

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{2,\Omega}^2 + \|\nabla u\|_{2,\Omega}^2\right) + D = G + F, \tag{6.4}$$

where

$$\begin{aligned} D =& \int_\Omega \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right)\cdot \nabla u \; dx; \\ G =& \gamma \int_\Omega \left(|u_1|^{\sigma-2}u_1 - |u_2|^{\sigma-2}u_2\right)\cdot u \; dx; \\ F =& \frac{1}{g_0(t)}\left(\int_\Omega \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right)\cdot \nabla \omega \; dx \right. \\ & \left. - \gamma \int_\Omega \left(|u_1|^{\sigma-2}u_1 - |u_2|^{\sigma-2}u_2\right)\cdot \omega \; dx\right)\int_\Omega g(x,t) u dx. \end{aligned} \tag{6.5}$$

Let be $\gamma \leq 0$. Then by the monotonicity, i.e. the inequality (2.2), we have

$$D \geq 0, \quad G \leq 0. \tag{6.6}$$

Using Holder, Minkovskii inequalities and (2.1) in Lemma 2.2 with $\delta = 0$, we estimate F.

$$
\begin{aligned}
&|F| \le \frac{1}{k_0}\|g\|_{2,\Omega}\|u\|_{2,\Omega}\left(\int_\Omega C\,|\nabla u|\,(|\nabla u_1| + |\nabla u_2|)^{p-2}\,|\nabla\omega|\,dx\right.\\
&\left.+|\gamma|\int_\Omega \left(|u|\,(|u_1| + |u_2|)^{\sigma-2}\,|\omega|\,dx\right)\right) \le \frac{1}{k_0}\|g\|_{2,\Omega}\|u\|_{2,\Omega}\\
&\|\nabla u\|_{2,\Omega}\left[\left(\|\nabla u_1 + \nabla u_2\|_{p,\Omega}\right)^{p-2}\|\nabla\omega\|_{L^{\frac{2p}{4-p}}(\Omega)}\right.\\
&\left.+\,|\gamma|\,\|u\|_{2^*,\Omega}\left(\|u_1 + u_2\|_{\frac{(\sigma-2)d}{2},\Omega}\right)^{\sigma-2}\|\omega\|_{2^*,\Omega}\right] \le \\
&\frac{1}{k_0}\|g\|_{2,\Omega}\|u\|_{2,\Omega}\|\nabla u\|_{2,\Omega}\left[\left(\|\nabla u_1\|_{p,\Omega} + \|\nabla u_2\|_{p,\Omega}\right)^{p-2}\|\nabla\omega\|_{L^{\frac{2p}{4-p}}(\Omega)} + \right.\\
&\left.C(\Omega)\,|\gamma|\left(\|\nabla u_1\|_{s,\Omega} + \|\nabla u_2\|_{s,\Omega}\right)^{\sigma-2}\|\nabla\omega\|_{2,\Omega}\right] \le \\
&\frac{1}{2}b_1(t)\left(\|u\|_{2,\Omega}^2 + \|\nabla u\|_{2,\Omega}^2\right), p \le 4, \quad \sigma \le \frac{2d}{d-s},\\
&\frac{(\sigma-2)d}{2} \le \frac{ds}{d-s} := s^* \Leftrightarrow \sigma \le \frac{2d}{d-s}
\end{aligned}
\tag{6.7}
$$

where where

$$
\begin{aligned}
b_1(t) = \frac{1}{k_0}\|g\|_{2,\Omega}&\left[\left(\|\nabla u_1\|_{p,\Omega} + \|\nabla u_2\|_{p,\Omega}\right)^{p-2}\|\nabla\omega\|_{L^{\frac{2p}{4-p}}(\Omega)}\right.\\
&\left.+\,C(\Omega)|\gamma|\left(\|\nabla u_1\|_{s,\Omega} + \|\nabla u_2\|_{s,\Omega}\right)^{\sigma-2}\|\nabla\omega\|_{2,\Omega}\right].
\end{aligned}
$$

2. Let consider now the case $\gamma > 0$. In this case we get from (6.4)

$$
\frac{d}{dt}y(t) \le |G| + |F|, \tag{6.8}
$$

where $y(t) = \|u\|_{2,\Omega}^2 + \|\nabla u\|_{2,\Omega}^2$. Using the second assertion of Lemma 2.2 with $\delta = 0$ we estimate

$$
\begin{aligned}
&|G| = \left|\gamma\int_\Omega \left(|u_1|^{\sigma-2}u_1 - |u_2|^{\sigma-2}u_2\right)\cdot u\right| dx \le \\
&|\gamma|\int_\Omega |u|^2\,(|u_1| + |u_2|)^{\sigma-2} \le |\gamma|\,\|u\|_{L^{\frac{2d}{d-2}}(\Omega)}^2\,\||u_1| + \|u_2|\|_{L^{\frac{(\sigma-2)d}{2}}(\Omega)}^{\sigma-2} \le \\
&C(\sigma,p,\Omega)|\gamma|\left(\|\nabla u_1\|_{s,\Omega} + \|\nabla u_2\|_{s,\Omega}\right)^{\sigma-2}\|\nabla u\|_{2,\Omega}^2 \le b_2(t)Y(t), \quad \sigma \le \frac{2d}{d-s}
\end{aligned}
\tag{6.9}
$$

where $b_2(t) := 2C(\sigma,p,\Omega)|\gamma| \left(\|\nabla u_1\|_{s,\Omega} + \|\nabla u_2\|_{s,\Omega}\right)^{\sigma-2}$ and $\sigma \leq \frac{2d}{d-s}$. ($m \leq \frac{2d}{d-p}$ if $p > 2$ or $m \leq \frac{2d}{d-2}$ if $p < 2$). For F we use the inequality (6.7).

Plugging (6.6) and (6.7) into (6.4) in the case $\gamma \leq 0$, and (6.7) and (6.9) into (6.8) in the case $\gamma \leq 0$, we arrive to the following Cauchy problem

$$\begin{cases} \dfrac{d}{dt} y(t) \leq a(t)y(t), \\ \\ y(0) = 0, \end{cases} \tag{6.10}$$

where

$$a(t) := b_1(t) \quad in \quad the \quad case \quad \gamma \leq 0;$$
$$a(t) := b_2(t) \quad in \quad the \quad case \quad \gamma > 0.$$

Due to the conditions to the Theorem 6.1. $a(t) \in L^1(0, T_{max})$ in any case $\gamma \leq 0$ and $\gamma > 0$, where T_{max} is a maximal time, such that the weak solution to the problem (3.6)-(3.9) is exist. It follows from (6.10) that $y(t) \equiv 0$ for all $t \in [0, T_{max}]$, and consequently that $u_1 \equiv u_2$. □

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