ON THE UNIQUENESS OF ℐ-LIMITS OF SEQUENCES

A. BLALI, A. EL AMRANI, R.A. HASSANI, A. RAZOUKI

ABSTRACT. We define the $\mathcal{I}$-sequential topology on a topological space where $\mathcal{I}$ denotes an ideal of the set of positive integers. We also study the relationship between $\mathcal{I}$-separatedness and uniqueness of $\mathcal{I}$-limits of sequences. Furthermore, we give a characterization of uniqueness of $\mathcal{I}$-limits of sequences by $\mathcal{I}$-closedness of sequentially $\mathcal{I}$-compact subset.

Keywords: $\mathcal{I}$-convergence, $\mathcal{I}$-sequential topology, $\mathcal{I}$-separated, sequentially $\mathcal{I}$-compact, $\mathcal{I}$-bounded, sequentially $\mathcal{I}$-continuity.

1. INTRODUCTION

The concept of statistical convergence was introduced by Fast [17] and Steinhaus [44], and it was developed in [17, 39, 40, 9, 10, 22, 12, 7, 24]. In 2000, P. Kostyrko and T. Šalát [30], gave a generalization of statistical convergence by using the notion of ideals of subsets of positive integers, which were named as $\mathcal{I}$-convergence, and studied some of its fundamental properties. Several results and properties [41, 43, 29, 18, 34, 14, 45, 13, 33, 15, 26, 37, 25, 38, 23] connected with $\mathcal{I}$-convergence were given in different spaces, like real numbers space, matrix space, n-normed space, metric space, locally convex space, Fréchet-Uryshon space, probabilistic normed space and sequence space. Recently from 2012 until 2019 the authors of [31, 13, 36, 2, 46], extended the idea of $\mathcal{I}$-convergence of sequences to any topological space and derived several basic properties of these concepts in topological space. It is well known that any continuous mapping is sequentially continuous, i.e., sequential continuity is a weaker condition than ordinary continuity. But, contrariwise, in Theorem 8 a sequentially $\mathcal{I}$-continuous mapping can be $\mathcal{I}$-continuous.

The following property is called $\mathcal{I}$-Fréchet-Urysohn; Let $A \subset X$ and $x \in X$, $x \in \overline{A}^{\mathcal{I}}$ if and only if $(x_n)_n$ $\mathcal{I}$-converges to x in X for some sequence $(x_n)_n$ in A, where $\overline{A}^{\mathcal{I}}$ is the $\mathcal{I}$-closure of A. A space X is called an $\mathcal{I}$-Fréchet-Urysohn space if it verifies the above property. Any topological space X has the property of $\mathcal{I}$-Fréchet-Urysohn, by construction of its $\mathcal{I}$-sequential topology.

We recall that a space X is a sequential space if each sequentially open subset of X is open. X is called $\mathcal{I}$-sequential space if each $\mathcal{I}$-open subset of X is open. In the case where $\mathcal{I}$ is an admissible ideal the following holds [37]:

$$\text{Fréchet-Urysohn space} \implies \mathcal{I}\text{-Fréchet-Urysohn space}$$
$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$
$$\text{Sequential space} \implies \mathcal{I}\text{-sequential space}$$

It is easy to verify that X is separated (Hausdorff) if and only if X has unique limits for a first-countable space. In general, a separated topological space X has unique sequential limits, but the converse is not true. In [19], an example of a non first-countable Fréchet space having unique limits of sequences but is not separated is presented. It is well known that every compact subset of a separated space is closed. In [28] A. J. Insel showed that a first-countable space X is separated if and only if every compact subset of X is closed.

In the first section, we define the $\mathcal{I}$-sequential topology and compare it with the sequential topology. In the second section, we give some properties and examples of sequentially $\mathcal{I}$-compact subsets. In the last section, we study the relationship between $\mathcal{I}$-separatedness and uniqueness of $\mathcal{I}$-limits of sequences, and also we prove that a $\mathcal{I}$-topological space X has unique $\mathcal{I}$-limits of sequences if and only if every sequentially $\mathcal{I}$-compact subset of X is $\mathcal{I}$-closed.

2. $\mathcal{I}$-SEQUENTIAL TOPOLOGY

Recall the notion of statistical convergence in a topological space X (See [4, 5, 20]). For each subset A of $\mathbb{N}$ the asymptotic density of A, denoted $\delta(A)$, is given by

$$\delta(A) = \lim_{n \to \infty} \frac{1}{n} |\{k \in A : k \leqslant n\}|,$$

if it exists, is in $[0, 1]$. We recall also that $\delta(\mathbb{N} \setminus A) = 1 - \delta(A)$ for each $A \subset \mathbb{N}$.

A sequence $(x_n)_n$ in X converges statistically to a point $x \in X$ if for each neighborhood U of x in X, $\delta(\{n \in \mathbb{N} : x_n \notin U\}) = 0$ [16] i.e. $\delta(\{n \in \mathbb{N} : x_n \in U\}) = 1$ which is denoted by $s - \lim_{n \to \infty} x_n = x$ or $x_n \xrightarrow{s} x$.

For a sequence $(x_n)_n$ and a subset U in X, let $A_U(x_n) = \{n \in \mathbb{N} : x_n \notin U\}$, which is denoted by A_U if non confusion arises. It is easy to see that a sequence $(x_n)_n$ in a topological space X converges statistically to a point $x \in X$ if and only if for any neighborhood U of x in X, we have $\delta(A_U) = 0$ [46].

Let $\mathcal{P}(\mathbb{N})$ be the family of all subsets of $\mathbb{N}$. An ideal $\mathcal{I}$ of $\mathbb{N}$ is a family of subsets of $\mathbb{N}$ which is hereditary (For all $(A, B) \in \mathcal{I}^2$, $B \subseteq A \Rightarrow B \in \mathcal{I}$) and additive (For all $(A, B) \in \mathcal{I}^2$, $A \cup B \in \mathcal{I}$) [30]. An ideal $\mathcal{I}$ is said to be non-trivial, if $\mathcal{I} \neq \emptyset$ and $\mathbb{N} \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is called admissible if $\mathcal{I}$ contains every finite subset of $\mathbb{N}$ ($\{\{n\} : n \in \mathbb{N}\} \subseteq \mathcal{I}$). Clearly, every non-trivial ideal $\mathcal{I}$ defines a dual filter

$F_{\mathcal{I}} = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \in \mathcal{I}\}$ on $\mathbb{N}$ [46].

Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is an admissible ideal of $\mathbb{N}$. Let $\mathcal{I}_\delta$ be the family of subsets $A \subseteq \mathbb{N}$ with $\delta(A) = 0$ i.e. $\mathcal{I}_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$. Then $\mathcal{I}_\delta$ is an admissible ideal, and its dual filter is $F_{\mathcal{I}_\delta} = \{A \subseteq \mathbb{N} : \delta(A) = 1\}$ [41].

Throughout this paper, we take $\mathcal{I}$ an admissible non-trivial ideal of $\mathbb{N}$.

Definition 1 (Definition 2.1, [46]). *A sequence $(x_n)_n$ in a topological space X is said to be $\mathcal{I}$-convergent to a point $x \in X$, provided for any neighborhood U of x we have $A_U \in \mathcal{I}$, which is denoted by $\mathcal{I}\text{-}\lim\limits_{n\to\infty} x_n = x$ or $x_n \xrightarrow{\mathcal{I}} x$, and the point x is called the $\mathcal{I}$-limit of the sequence $(x_n)_n$.*

Lemma 1 (Lemma 2.5, [46]). *Let $\mathcal{I} \subseteq \mathcal{J}$ be two ideals of $\mathbb{N}$. If $(x_n)_n$ is a sequence in a topological space X such that $x_n \xrightarrow{\mathcal{I}} x$, then $x_n \xrightarrow{\mathcal{J}} x$.*

Proposition 1. *Let $(x_n)_n$ be a sequence in a topological space X. If $(x_n)_n$ is convergent to x in X, then it is also $\mathcal{I}$-convergent to x.*

Proof. Since $\mathcal{I}$ is admissible, then $\mathcal{I}_f \subset \mathcal{I}$ and by Lemma 1 the result is true. $\square$

The converse of the previous proposition is false as shown by the following example.

Example 1. *Let X be a Hausdorff topological space containing at least two distinct points x and y. Let $(x_n)_n$ be a sequence in X defined by*

$$x_n = \begin{cases} x, & \text{if } n \text{ is a square} \\ y, & \text{if not} \end{cases}$$

By Definition 1 $(x_n)_n$ is $\mathcal{I}_\delta$-convegent to y; indeed, for all neighborhoods U of y, we have $A_U \subset A$, where $A = \{n \in \mathbb{N} : n \text{ is a square}\}$, and $A \in \mathcal{I}_\delta$ (because $\delta(A) = 0$). But $(x_n)_n$ is not convergent to y.

Definition 2. *Let $\mathcal{I}$ be an ideal of $\mathbb{N}$ and X a topological space.*

  (i) For each subset A of X, we define $\overline{A}^{\mathcal{I}}$, the $\mathcal{I}$-closure of A by:

$$\overline{A}^{\mathcal{I}} = \{x \in X / \exists (x_n)_n \subset A : x_n \xrightarrow{\mathcal{I}} x\},$$

  it is clear that $\overline{\emptyset}^{\mathcal{I}} = \emptyset$ and $A \subset \overline{A}^{\mathcal{I}}$ for all $A \subset X$.

  (ii) A subset $F \subseteq X$ is said to be $\mathcal{I}$-closed if $\overline{F}^{\mathcal{I}} = F$.

  (iii) A subset $U \subseteq X$ is said to be $\mathcal{I}$-open if $X \setminus U$ is $\mathcal{I}$-closed.

Proposition 2. *Let U be a subset of a topological space X. If U is an open subset then it is also an $\mathcal{I}$-open.*

Proof. Let U be an open subset of X. Suppose, to derive a contradiction, that U is not $\mathcal{I}$-open i.e. U^c is not $\mathcal{I}$-closed. Let $(x_n)_n$ be a sequence in U^c such that $x_n \xrightarrow{\mathcal{I}} x$ and $x \in U$. Then for any neighborhood V of X, $\{n \in \mathbb{N} : x_n \notin V\} \in \mathcal{I}$ (Definition 1). For $V = U$, $\{n \in \mathbb{N} : x_n \notin U\} = \mathbb{N} \in \mathcal{I}$. which is a contradiction, because $\mathcal{I}$ is not trivial. Thus U is $\mathcal{I}$-open. $\square$

Definition 3. *Let A be a subset of X. The $\mathcal{I}$-interior of A is the set $\overset{\circ\mathcal{I}}{A} = A \setminus \overline{(X \setminus A)}^{\mathcal{I}}$.*

Proposition 3. *Let A be a subset of X. A is $\mathcal{I}$-open if and only if $\overset{\circ\mathcal{I}}{A} = A$.*

Proof. If A is $\mathcal{I}$-open, then $X \setminus A$ is $\mathcal{I}$-closed. Therefore $\overset{\circ\mathcal{I}}{A} = A \setminus \overline{(X \setminus A)}^{\mathcal{I}}$. Hence $\overset{\circ\mathcal{I}}{A} = A \setminus (X \setminus A)$. Thus $\overset{\circ\mathcal{I}}{A} = A$. Conversely, if $A = \overset{\circ\mathcal{I}}{A}$, then $A = A \setminus \overline{(X \setminus A)}^{\mathcal{I}}$. Therefore $A \cap \overline{(X \setminus A)}^{\mathcal{I}} = \emptyset$. Hence $\overline{X \setminus A}^{\mathcal{I}} \subset X \setminus A$. Thus $\overline{X \setminus A}^{\mathcal{I}} = X \setminus A$ i.e. $X \setminus A$ is $\mathcal{I}$-closed. Thus A is $\mathcal{I}$-open. $\qquad\square$

Theorem 1. *Let A be a subset of a topological space X. The following assertions are equivalent:*

(i) A is $\mathcal{I}$-closed.
(ii) $A = \cap\{F/F$ is $\mathcal{I}$-closed and $A \subset F\}$.

Proof. $(i) \Rightarrow (ii)$ It is obvious.
$(ii) \Rightarrow (i)$ Assume that $A = \cap\{F/F$ is $\mathcal{I}$-closed and $A \subset F\}$. Let show that $\overline{A}^{\mathcal{I}} \subset A$. Let $x_0 \notin A$. Then there exists a $\mathcal{I}$-closed subset F containing A such that $x_0 \notin F$. Suppose for the sake of contradiction that $x_0 \in \overline{A}^{\mathcal{I}}$. Then there exists $(x_n)_n$ in A such that $x_n \overset{\mathcal{I}}{\to} x_0$. Since $A \subset F$, then $(x_n)_n$ is in F. Thus $x_0 \in \overline{F}^{\mathcal{I}} = F$, which is a contradiction. So $x_0 \notin \overline{A}^{\mathcal{I}}$. $\qquad\square$

Theorem 2. *Let A be a subset of a topological space X. The following assertions are equivalent:*

(i) A is $\mathcal{I}$-open.
(ii) $A = \cup\{U/U$ is $\mathcal{I}$-open and $U \subset A\}$.

Proof. $(i) \Rightarrow (ii)$ It is obvious.
$(ii) \Rightarrow (i)$ Assume that $A = \cup\{U/U$ is $\mathcal{I}$-open and $U \subset A\}$ and show that A is $\mathcal{I}$-open. It suffices to prove that $A \subset \overset{\circ\mathcal{I}}{A}$. Let $x_0 \notin \overset{\circ\mathcal{I}}{A}$. Suppose for the sake of contradiction that $x_0 \in A$. Then there exists an $\mathcal{I}$-open subset U of A such that $x_0 \in U$. Since $U \subset A$, then $x_0 \in \overset{\circ\mathcal{I}}{A}$; which is a contradiction. Thus, $A \subset \overset{\circ\mathcal{I}}{A}$ i.e. A is $\mathcal{I}$-open. $\qquad\square$

Using Zorn's lemma, one can show that in the family of all admissible ideals $\mathcal{I}$ of $\mathbb{N}$, there exists a maximal ideal (with respect to inclusion). If $\mathcal{I}$ is a maximal ideal of $\mathbb{N}$, then for each $A \subset \mathbb{N}$ we have either $A \in \mathcal{I}$, or $A \in \mathbb{N} \setminus \mathcal{I}$ [30].

Definition 4 (Definition 3.15, [46])**.** *Let $\mathcal{I}$ be an ideal of $\mathbb{N}$, and U a subset of X. A sequence $(x_n)_n$ in X is $\mathcal{I}$-eventually in U, if there is $E \in \mathcal{I}$ such that for all $n \in \mathbb{N} \setminus E$, $x_n \in U$.*

Proposition 4 (Proposition 3.16, [46])**.** *Let $\mathcal{I}$ be a maximal ideal of $\mathbb{N}$ and X a topological space. Then, $U \subseteq X$ is $\mathcal{I}$-open if and only if each $\mathcal{I}$-convergent sequence in X, to a point of U, is $\mathcal{I}$-eventually in U.*

Theorem 3. [46, Theorem 3.17] *Let $\mathcal{I}$ be a maximal ideal of $\mathbb{N}$ and X be a topological space. If U, V are two $\mathcal{I}$-open subsets of X, then $U \cap V$ is $\mathcal{I}$-open.*

We have the following characterization of $\mathcal{I}$-convergence of a sequence.

Theorem 4. *Let $\mathcal{I}$ be a maximal ideal of $\mathbb{N}$ and X a topological space. A sequence $(x_n)_n$ in X is $\mathcal{I}-$convergent to an element $x \in X$ if and only if for any $\mathcal{I}$-open subset U of X with $x \in U$, there is $E \in \mathcal{I}$ such that for all $n \in \mathbb{N} \setminus E$, $x_n \in U$.*

Proof. The result is necessary by Proposition 4.

Conversely, let U be an open subset of X with $x \in U$. By Proposition 2, U is $\mathcal{I}$-open. Therefore, by hypothesis, there is $E \in \mathcal{I}$ with for all $n \in \mathbb{N} \setminus E$, $x_n \in U$. Then $\mathbb{N} \setminus E \subset \{n \in \mathbb{N} : x_n \in U\}$. Therefore $\{n \in \mathbb{N} : x_n \notin U\} \subset E$ and $E \in \mathcal{I}$. Thus $x_n \xrightarrow{\mathcal{I}} x$. $\qquad\square$

Theorem 5. *Let $\mathcal{I}$ be a maximal ideal of $\mathbb{N}$ and X be a topological space. The family $\tau_{\mathcal{I}} = \{U \in \mathcal{P}(X) / U \text{ is } \mathcal{I}\text{-open}\}$ is a topology on X. We call it the $\mathcal{I}$-sequential topology.*

Proof. It is obvious that $\emptyset \in \tau_{\mathcal{I}}$. $X \in \tau_{\mathcal{I}}$ because $\emptyset$ is an $\mathcal{I}$-closed subset of X so the complement of $\emptyset$ is $\mathcal{I}$-open that is X. Let $U, V \in \tau_{\mathcal{I}}$, by Theorem 3, $U \cap V \in \tau_{\mathcal{I}}$. Let $(U_\alpha)_{\alpha \in \Delta}$ be a family of elements of $\tau_{\mathcal{I}}$. We must show that $\bigcup\limits_{\alpha \in \Delta} U_\alpha \in \tau_{\mathcal{I}}$. Since $(\bigcup\limits_{\alpha \in \Delta} U_\alpha)^c = \bigcap\limits_{\alpha \in \Delta} U_\alpha^c$, it suffices to prove that $\bigcap\limits_{\alpha \in \Delta} U_\alpha^c$ is $\mathcal{I}$-closed i.e. $\overline{\bigcap\limits_{\alpha \in \Delta} U_\alpha^c}^{\mathcal{I}} = \bigcap\limits_{\alpha \in \Delta} U_\alpha^c$. It is obvious that $\overline{\bigcap\limits_{\alpha \in \Delta} U_\alpha^c}^{\mathcal{I}} \supset \bigcap\limits_{\alpha \in \Delta} U_\alpha^c$. Conversely let $x \in \overline{\bigcap\limits_{\alpha \in \Delta} U_\alpha^c}^{\mathcal{I}}$. Then there is a sequence $(x_n)_n$ in $\bigcap\limits_{\alpha \in \Delta} U_\alpha^c$ with $x_n \xrightarrow{\mathcal{I}} x$. Therefore, for all $\alpha \in \Delta$, $(x_n)_n$ is in U_α^c with $x_n \xrightarrow{\mathcal{I}} x$. Since U_α^c is $\mathcal{I}$-closed, then for all $\alpha \in \Delta$, $x \in U_\alpha^c$. Therefore $x \in \bigcap\limits_{\alpha \in \Delta} U_\alpha^c$. Thus $\bigcap\limits_{\alpha \in \Delta} U_\alpha^c$ is $\mathcal{I}$-closed. $\qquad\square$

In the following, we distinguish between sequential topology and $\mathcal{I}$-sequential topology on a topological space X.

Remark 1.   *(i) We define the sequential topology as the set of sequentially open subset of X. We denote it by*

$$\tau^s = \{U \in \mathcal{P}(X) / (\forall (x_n)_n \subset U^c) \ x_n \to x \Rightarrow x \in U^c\}.$$

In the same way we can define the toplology $\tau_{\mathcal{I}}$ by

$$\tau_{\mathcal{I}} = \{U \in \mathcal{P}(X) / (\forall (x_n)_n \subset U^c) \ x_n \xrightarrow{\mathcal{I}} x \Rightarrow x \in U^c\},$$

which is equivalent to the notation in Theorem 5.

(ii) A sequence $(x_n)_n$ in X is convergent to an element x if and only if it is τ^s-convergent to x (See [1]). However, this property is not true for the topology $\tau_{\mathcal{I}}$ by Example 1.

The following examples show that an $\mathcal{I}$-open subset can't be open.

Example 2. *Let $X \neq \emptyset$ be a non countable set, τ the co-countable topology on X and $\mathcal{I}$ an ideal of $\mathbb{N}$. All $\mathcal{I}$-convergent sequences in X are constant. Suppose that, on the contrary, there is a nonconstant sequence $(x_n)_n$ in X which is $\mathcal{I}$-convergent to an element $x \in X$. We put $N = \{x_n, n \in \mathbb{N}\} \setminus \{x\}$ which is countable. Then $X \setminus N$ is an open subset of X which contains x. By Definition 1 of $\mathcal{I}$-convergence,*

$\{n \in \mathbb{N} : x_n \notin X \setminus N\} = \mathbb{N}$, but $\mathbb{N} \notin \mathcal{I}$ (not trivial), which is not the case. Therefore $(x_n)_n$ is constant. All subsets of X are $\mathcal{I}$-open; Let A be a subset of X and $(x_n)_n$ be a sequence in A which is $\mathcal{I}$-convergent to an element $x \in X$. Then $(x_n)_n$ is constant, so x is in A. Thus, by Definition 2 A is $\mathcal{I}$-closed. Therefore A is also $\mathcal{I}$-open. We deduce that all subsets of X are both $\mathcal{I}$-closed and $\mathcal{I}$-open. But the singleton $\{x\}$ is not open.

Example 3. *We recall that Ω denotes the first uncountable ordinal with the usual ordering. Let $[0, \Omega]$ be the topological space such as its open subsets are $[0, \lambda[=\{\alpha/\alpha < \lambda\}$ where $\lambda \leq \Omega$ (the order topology). All sequences of countable ordinals have a countable limit ω (first infinite ordinal). But, there isn't a sequence of countable ordinals that converges to Ω. Let $\mathcal{I}$ be an ideal of $\mathbb{N}$. Let $(x_n)_n$ be a sequence in $[0, \Omega[$ which is $\mathcal{I}$-convergent to x. Then $x \in [0, \Omega[$ because Ω is the first uncountable ordinal. Thus $[0, \Omega[$ is $\mathcal{I}$-closed, but $[0, \Omega[$ is not a closed subset in $[0, \Omega]$, .i.e, $\{\Omega\}$ is an $\mathcal{I}$-open subset which is not open.*

Theorem 6. *Let $\mathcal{I}$ be a maximal ideal of $\mathbb{N}$ and X a topological space. Then $\tau \preccurlyeq \tau_{\mathcal{I}} \preccurlyeq \tau^s$.*

Proof. By Proposition 2 an open subset of X is $\mathcal{I}$-open.

Now let U be an $\mathcal{I}$-open subset of X. We show that U is an τ^s-open. Suppose that, on the contrary, U is not an τ^s-open. Then $X \setminus U$ is not τ^s-closed. Therefore there is a sequence $(x_n)_n$ in $X \setminus U$ with $x_n \xrightarrow{\tau^s} x \in U$. By previous remark we have $x_n \xrightarrow{\tau} x$. Since $\mathcal{I}$ is admissible, then $x_n \xrightarrow{\mathcal{I}} x$ and $x \in U$ (Proposition 1). Thus $X \setminus U$ is not $\mathcal{I}$-closed i.e. U is not $\mathcal{I}$-open, which is a contradiction. So U is an τ^s-open. $\qquad \square$

The refinement $\tau \preccurlyeq \tau_{\mathcal{I}}$ is strict by Example 2 and Example 3. The refinement $\tau_{\mathcal{I}} \preccurlyeq \tau^s$ is also strict by the following example.

Example 4. *Let $\mathcal{I}$ be a maximal ideal of $\mathbb{N}$ and $[0, 1]$ the unit interval of $\mathbb{R}$ with the topology*

$$\tau = \{U \subset [0, 1] : 0 \notin U \text{ or } card([0, 1] \setminus U) \text{ is countable}\}.$$

$]0, 1]$ *is τ^s-closed because there is no sequence in $]0, 1]$ convergent to 0. Assume for the sake of contradiction that there is a sequence $(x_n)_n$ in $]0, 1]$ convergent to 0. Then, for any neighborhood U of 0, there is some $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in U$. Consider $U = [0, 1] \setminus \{x_n : n \in \mathbb{N}\}$ which is below to τ because $card\big([0, 1] \setminus ([0, 1] \setminus \{x_n : n \in \mathbb{N}\})\big)$ is countable. But, by construction of U, for all $n \in \mathbb{N}$, $x_n \notin U$, which is not the case. Thus $]0, 1]$ is τ^s-closed, i.e., $\overline{]0, 1]}^s =]0, 1]$.*

Suppose, to derive a contradiction, that $\tau_{\mathcal{I}} = \tau^s$, i.e., for any subset A of $[0, 1]$, $\overline{A}^{\mathcal{I}} = \overline{A}^s$. We choose $A =]0, 1]$ which is τ^s-closed by the above. Since $\tau_{\mathcal{I}} \preccurlyeq \tau^s$, then $]0, 1]$ is also $\mathcal{I}$-closed. Consider the sequence

$$x_n = \begin{cases} \frac{1}{2}, & \text{if } n \text{ is a square} \\ 0, & \text{if not} \end{cases}$$

with $x_n \xrightarrow{\mathcal{I}_\delta} 0$ (see Example 1). Then $\overline{]0, 1]}^{\mathcal{I}} = [0, 1]$. Therfore $\overline{]0, 1]}^{\mathcal{I}} \neq \overline{]0, 1]}^s$. So $\tau^s \succ \tau_{\mathcal{I}}$.

Theorem 7. *Let $\mathcal{I}$ be a maximal ideal of $\mathbb{N}$ and X a first countable topological space. Then $\tau_{\mathcal{I}} = \tau$.*

Proof. An open subset is always $\mathcal{I}$-open by Proposition 2.

Let U be an $\mathcal{I}$-open subset of X, i.e., U^c is an $\mathcal{I}$-closed subset of X. We now show that U^c is closed. Suppose that, on the contrary, U^c is not closed, i.e., there are some $x \in \overline{U^c} \setminus U^c$. Let (V_n) be a decreasing sequence of basis of neighborhoods of x. For all $n \in \mathbb{N}$, $U^c \cap (V_n \setminus \{x\}) \neq \emptyset$ (because $x \notin U^c$). Let $(x_n)_n$ be a sequence in $U^c \cap V_n$ with $x_n \neq x$. Let W be a neighborhood of x. Then there is $n_0 \in \mathbb{N}$ with $V_{n_0} \subset W$. Thus, for $n \geq n_0$, $V_n \subset W$. Therefore, for all $n \geq n_0$, $x_n \in W$. So, $x_n \to x$ in X. By Proposition 1, $x_n \xrightarrow{\mathcal{I}} x$. Therefore $x \in U^c$ because U^c is $\mathcal{I}$-closed, which is not the case. Thus U^c is closed. $\qquad\square$

Definition 5. *Let $\mathcal{I}$ be an ideal of $\mathbb{N}$. Let (X, τ) and (Y, ρ) be topological spaces and $f : X \to Y$ a mapping:*

(i) *f is called $\mathcal{I}$-continuous provided for if U is $\mathcal{I}$-open in Y, then $f^{-1}(U)$ is $\mathcal{I}$-open in X.*

(ii) *f is called sequentially $\mathcal{I}$-continuous provided for each sequence $(x_n)_n$ in X with $x_n \xrightarrow{\mathcal{I}} x$, we have $f(x_n) \xrightarrow{\mathcal{I}} f(x)$.*

Theorem 8. *Let $\mathcal{I}$ be an ideal of $\mathbb{N}$. Let (X, τ), (X', ρ) be two topological spaces and $f : X \to X'$ a function. If f is sequentially $\mathcal{I}$-continuous, then f is $\mathcal{I}$-continuous.*

Proof. Let $V \in \rho_{\mathcal{I}}$. Suppose that $f^{-1}(V) \notin \tau_{\mathcal{I}}$. Then, there exists a sequence $(x_n)_n$ in $[f^{-1}(V)]^c$ such that $x_n \xrightarrow{\mathcal{I}} x$ in X and $x \in f^{-1}(V)$. Since f is sequentially $\mathcal{I}$-continuous, $f(x_n) \xrightarrow{\mathcal{I}} f(x)$ in X'. Therefore $(f(x_n))_n$ is in V^c. Hence $f(x) \in V^c$, which is not the case. Thus, $f^{-1}(V) \in \tau_{\mathcal{I}}$. $\qquad\square$

3. Sequentially $\mathcal{I}$-compact subsets

Definition 6. *Let $\mathcal{I}$ be an ideal of $\mathbb{N}$ and (X, τ) be a topological space:*

(i) *X is called $\mathcal{I}$-separated if for any two distinct points $x, y \in X$ there exist two $\mathcal{I}$-open subsets U and V containing respectively x and y such that $U \cap V = \emptyset$.*

(ii) *A subset F in X is called $\mathcal{I}$-compact if for every $\mathcal{I}$-open cover of F there exists a finite subcover of F.*

(iii) *A subset $F \subset X$ is said to be sequentially compact (respectively, sequentially $\mathcal{I}$-compact) if any sequence in F has a subsequence converging (respectively, $\mathcal{I}$-converging) to a point in F.*

(iv) *X is said to be countably $\mathcal{I}$-compact if for every countable $\mathcal{I}$-open cover of X has a finite subcover.*

Theorem 9. *Any sequentially compact subset of a topological space is sequentially $\mathcal{I}$-compact.*

Proof. It is obvious by Proposition 1. $\qquad\square$

In the following, a subset of a topological space can be neither sequentially compact nor sequentially $\mathcal{I}$-compact.

Example 5. $\mathbb{R}$ *is not a sequentially $\mathcal{I}$-compact set, since the sequence $(n)_{n \in \mathbb{N}}$ has no $\mathcal{I}$-convergent subsequence. Also, $]0, 1[$ is not a sequentially $\mathcal{I}$-compact set, since the sequence $(1/n)_{n \in \mathbb{N}^*}$ has no subsequence which is $\mathcal{I}$-convergent in $]0, 1[$.*

A sequentially $\mathcal{I}$-compact topological space can be not $\mathcal{I}$-compact by the following example.

Example 6. *We take the same hypothesis as in Example 3. Let $\mathcal{E} = \{[0, \alpha[/\alpha < \Omega\}$ be the open covering of Ω. Any finite union $\mathcal{F}$ of subset of $\mathcal{E}$ is a countable ordinal which has a successor in $[0, \alpha[$. Therefore $\mathcal{E}$ is not covered by $\mathcal{F}$, i.e., Ω is not $\mathcal{I}$-compact. On the other side, all sequences of countable ordinals have a countable limit ω (first infinite ordinal). Thus, Ω is sequentially $\mathcal{I}$-compact but not $\mathcal{I}$-compact.*

Theorem 10. *Let Y be a subspace of X. If Y is $\mathcal{I}$-compact (resp. countably $\mathcal{I}$-compact), then Y is compact (resp. countably compact).*

Proof. Any cover (resp. countable cover) of Y with open sets is also a cover (resp. countable cover) of Y with $\mathcal{I}$-open sets by Proposition 2. The finite subcover given by $\mathcal{I}$-compactness is then a finite subcover of open sets. $\square$

Definition 7. [42, Definition 4.1]. *Let X be a normed space. A sequence $(x_n)_n$ in X is $\mathcal{I}$-bounded if there is a real number $M > 0$ such that $\{n \in \mathbb{N} : \|x_n\| > M\} \in \mathcal{I}$.*

Theorem 11. *Every bounded sequence in a normed space is $\mathcal{I}$-bounded.*

Proof. Let $(x_n)_n$ be a bounded sequence in a normed space X. Then, there is a real number $M > 0$, such that for all $n \in \mathbb{N}$, $\|x_n\| < M$. Therefore $\{n \in \mathbb{N} : \|x_n\| > M\} = \emptyset$, then $\{n \in \mathbb{N} : \|x_n\| > M\} \in \mathcal{I}$. Thus, $(x_n)_n$ is $\mathcal{I}$-bounded. $\square$

Theorem 12. *Let (X, τ) be a topological sapce, $\mathbb{R}$ the real line under its usual topology and $f : X \to \mathbb{R}$ a sequentially $\mathcal{I}$-continuous mapping. If A is sequentially $\mathcal{I}$-compact subset of X, then $f(A)$ is $\mathcal{I}$-bounded.*

Proof. Suppose, to derive a contradiction, that $f(A)$ is not $\mathcal{I}$-bounded. Then there is a sequence $(y_n)_n$ in $f(A)$ such that for all real number $M > 0$, $\{n \in \mathbb{N} : |y_n| > M\} \notin \mathcal{I}$. Therefore there is a sequence $(x_n)_n$ in A such that for all $n \in \mathbb{N}$, $f(x_n) = y_n$. Since A is sequentially $\mathcal{I}$-compact, the sequence $(x_n)_n$ has a $\mathcal{I}$-convergent subsequence $(x_{n_k})_k$ to an element x_0 of A. So, $f(x_{n_k}) \xrightarrow{\mathcal{I}} f(x_0)$. Then, for any neighborhood U of $f(x_0)$, $\{n \in \mathbb{N} : f(x_n) \notin U\} \in \mathcal{I}$, i.e. there is a real number $M > 0$ such that $\{n \in \mathbb{N} : |f(x_n)| > M\} \in \mathcal{I}$, which is not the case. Thus $f(A)$ is $\mathcal{I}$-bounded. $\square$

Theorem 13. *Let X be a topological space. If X is sequentially $\mathcal{I}$-compact and Y is an $\mathcal{I}$-closed subset of X, then Y is sequentially $\mathcal{I}$-compact.*

Proof. Let $(y_n)_n$ be a sequence in Y. By sequential $\mathcal{I}$-compactness of X, $(y_n)_n$ has a subsequence which is $\mathcal{I}$-convergent to some y in X. Since Y is $\mathcal{I}$-closed, then $y \in Y$.

$\square$

Theorem 14. *Let X and Y be topological spaces. If X is sequentially $\mathcal{I}$-compact and $f : X \to Y$ is a sequentially $\mathcal{I}$-continuous mapping, then $f(X)$ is sequentially $\mathcal{I}$-compact.*

Proof. Let $(y_n)_n$ be any sequence in $f(X)$. Then there is a sequence $(x_n)_n$ in X such that $y_n = f(x_n)$. By sequential $\mathcal{I}$-compactness of X, there is an $\mathcal{I}$-converging subsequence $(x_{n_k})_k$ to x. Then $(f(x_{n_k}))_k$ is a subsequence of $(y_n)_n$ which is $\mathcal{I}$-convergent to $f(x)$. $\square$

4. $\mathcal{I}$-SEPARATED SPACES AND THE UNIQUENESS OF $\mathcal{I}$-LIMITS OF SEQUENCES

Proposition 5. *Let (X,τ) be a separated topological space. Then X is $\mathcal{I}$-separated.*

Proof. It is obvious by Proposition 2. $\qquad\square$

Theorem 15. *Let (X,τ) be a sequential space. If $\mathcal{I}$ is maximal, then X is $\mathcal{I}$-separated if and only if X is separated.*

Proof. Since (X,τ) is sequential, then $\tau = \tau^s$. Therefore, by Theorem 7, $\tau_\mathcal{I} = \tau$. Thus, X is $\mathcal{I}$-separated if and only if X is separated. $\qquad\square$

Through the following example, there is a topological space X which is $\mathcal{I}$-separated but not separated.

Example 7. *We take the same hypothesis as in Example 2. X is $\tau_\mathcal{I}$-separated but not separated. Let $x,y \in X$, with $x \neq y$. Let U and V be two open neighborhoods of x and y respectively. Then $U = X \setminus U'$ and $V = X \setminus V'$ with U' and V' are countable subsets of X. Therefore, $U \cap V = (X \setminus U') \cap (X \setminus V') = X \setminus (U' \cup V')$. Since X is uncountable and $U' \cup V'$ is countable, then $X \setminus (U' \cup V') \neq \emptyset$. Thus, X is not separated. On the other hand, $G = \{x\}$ and G^c are $\mathcal{I}$-open subsets of X such that $x \in G$ and $y \in G^c$. Since $G \cap G^c = \emptyset$, then X is $\mathcal{I}$-separated.*

Theorem 16. *Let (X,τ) be an $\mathcal{I}$-separated topological space. If $\mathcal{I}$ is maximal, then every $\mathcal{I}$-convergent sequence in X has one $\mathcal{I}$-limit.*

Proof. Suppose, towards a contradiction, that $(x_n)_n$ is a sequence in X, $\mathcal{I}$-convergent to x and y such that $x \neq y$. By Theorem 4, for any $\mathcal{I}$-open subset U of X with $x \in U$, there is $E_1 \in \mathcal{I}$ such that for all $n \in \mathbb{N} \setminus E_1$, $x_n \in U$. Likewise, for any $\mathcal{I}$-open subset V of X such that $y \in V$ there is $E_2 \in \mathcal{I}$ such that for all $n \in \mathbb{N} \setminus E_2$, $x_n \in V$. Then, $\{n \in \mathbb{N} : x_n \notin U\} \subset E_1$ and $\{n \in \mathbb{N} : x_n \notin V\} \subset E_2$. Since $\{n \in \mathbb{N} : x_n \notin U\} = \{n \in \mathbb{N} : x_n \in U^c\}$ and $\{n \in \mathbb{N} : x_n \notin V\} = \{n \in \mathbb{N} : x_n \in V^c\}$, then $\{n \in \mathbb{N} : x_n \in U^c\} \cup \{n \in \mathbb{N} : x_n \in V^c\} \subset E_1 \cup E_2$. We can choose U and V such that $U \cap V = \emptyset$ (X is $\mathcal{I}$-separated). Therefore $U^c \cup V^c = X$, then $\{n \in \mathbb{N} : x_n \in U^c\} \cup \{n \in \mathbb{N} : x_n \in V^c\} = \mathbb{N}$. Hence $\mathbb{N} \subset E_1 \cup E_2$. Since $E_1 \in \mathcal{I}$ and $E_2 \in \mathcal{I}$ then $E_1 \cup E_2 \in \mathcal{I}$. Therefore $\mathbb{N} \in \mathcal{I}$, which is a contradiction because $\mathcal{I}$ is not trivial. Thus, $x = y$. $\qquad\square$

Theorem 17. *Let $\mathcal{I}$ be a maximal ideal and (X,τ) be an $\mathcal{I}$-separated topological space. If A is a sequentially $\mathcal{I}$-compact subset of X, then A is $\mathcal{I}$-closed.*

Proof. Let $(x_n)_n$ be a sequence in A such that $x_n \xrightarrow{\mathcal{I}} x$. Since A is sequentially $\mathcal{I}$-compact, there exists $(x_{n_k})_k$ a subsequence of $(x_n)_n$ such that $x_{n_k} \xrightarrow{\mathcal{I}} y$ with $y \in A$. By previous theorem we have $y = x$, so $x \in A$. Thus A is $\mathcal{I}$-closed. $\qquad\square$

Theorem 18. *Let (X,τ) be a first countable topological space and $\mathcal{I}$ a maximal ideal. If each $\mathcal{I}$-convergent sequence in X has unique $\mathcal{I}$-limit, then X is separated.*

Proof. Let x and y be a distinct points of a first countable topological space (X,τ). Let $(U_i)_{i\in\mathbb{N}}$ and $(V_j)_{j\in\mathbb{N}}$ be a local basis of neighborhoods of x and y respectively ($\tau_\mathcal{I} = \tau$, since X is first countable, Theorem 7). Striving for a contradiction, suppose that X is not separated. Then every neighborhood of x intersects every

neighborhood of y on a non-empty subset. Define $W_1 = U_1$, $W_2 = U_1 \cap V_1$, $W_3 = U_1 \cap V_1 \cap U_2, \ldots, W_{2n} = U_1 \cap V_1 \cap \ldots U_n \cap V_n$, $W_{2n+1} = U_1 \cap V_1 \cap \ldots U_n \cap V_n \cap U_{n+1}$. Therefore, $W_1 \supset W_2 \supset \ldots W_n \supset \ldots$. Consider the sequence $(x_n)_{n \geqslant 1}$ such that x_n is in W_n, for all $n \in \mathbb{N}^\star$. Let G_1 be an open set containing x, and G_2 be an open set containing y. There exists a natural number N, such that for all $n > N$, $W_n \subset G_1$ and $W_n \subset G_2$. Then both $\{n \in \mathbb{N} : x_n \notin G_1\}$ and $\{n \in \mathbb{N} : x_n \notin G_2\}$ are finite. Since $\mathcal{I}$ is admissible and by Proposition 2, $x_n \xrightarrow{\mathcal{I}} x$ and $x_n \xrightarrow{\mathcal{I}} y$, which is not the case. Thus, X is separated. $\qquad \square$

Recall that if $\varphi : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, then $(x_{\varphi(n)})_n$ is a subsequence of the sequence $(x_n)_n$.

Lemma 2. *Every subsequence of an $\mathcal{I}$-convergent sequence in a topological space is also $\mathcal{I}$-convergent.*

Proof. Let $(x_n)_n$ be an $\mathcal{I}$-convergent sequence to x, in a topological space X. Then, for all neighborhoods U of x, $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$. Let $(x_{\varphi(n)})_n$ be a subsequence of $(x_n)_n$. Since $\{\varphi(n) \in \mathbb{N} : x_{\varphi(n)} \notin U\} \subset \{n \in \mathbb{N} : x_n \notin U\}$, then $\{\varphi(n) \in \mathbb{N} : x_{\varphi(n)} \notin U\} \in \mathcal{I}$. Thus, $x_{\varphi(n)} \xrightarrow{\mathcal{I}} x$. $\qquad \square$

Theorem 19. *Let X a topological space and $(x_n)_n$ a sequence in X with $x_n \xrightarrow{\mathcal{I}} x$ in X. Then $A = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ is a sequentially $\mathcal{I}$-compact subset of X.*

Proof. Let $(y_n)_n$ be a sequence in A. If $\{y_n : n \in \mathbb{N}\}$ is finite, then there is an element y_{n_0} in $\{y_n : n \in \mathbb{N}\}$ such that $\{n \in \mathbb{N} : y_n = y_{n_0}\}$ is infinite. Then $(y_n)_n$ has a convergent subsequence, which is $\mathcal{I}$-convergent by Proposition 1.

If $\{y_n : n \in \mathbb{N}\}$ is infinite, we can construct a subsequence $(y_{\beta_n})_n$ of $(y_n)_n$ by the following: For all $p \in \mathbb{N}^\star$

$\alpha_1 = \min\{n \in \mathbb{N} : x_n \in \{y_n : n \in \mathbb{N}\}\}$ and $\beta_1 = \max\{n \in \mathbb{N} : x_{\alpha_1} = y_n\}$

$\alpha_2 = \min\{n \in \mathbb{N} : x_n \in \{y_n : n > \beta_1\}\}$ and $\beta_2 = \max\{n \in \mathbb{N} : x_{\alpha_2} = y_n, n > \beta_1\}$

$\vdots$

$\alpha_p = \min\{n \in \mathbb{N} : x_n \in \{y_n : n > \beta_{p-1}\}\}$ and $\beta_p = \max\{n \in \mathbb{N} : x_{\alpha_p} = y_n, n > \beta_{p-1}\}$

All subsets $\{n \in \mathbb{N} : x_n \in \{y_n : n > \beta_{p-1}\}\}$ are not empty, because $\{y_n : n \in \mathbb{N}\}$ is not finite. Then, clearly, this sequence $(y_{\beta_n})_n$ is a subsequence of $(y_n)_n$ and also a subsequence of $(x_n)_n$. By Lemma 2 , $y_{\beta_n} \xrightarrow{\mathcal{I}} x$ because $x_n \xrightarrow{\mathcal{I}} x$. Thus, A is a sequentially $\mathcal{I}$-compact subset of X. $\qquad \square$

In [28], A. J. Insel showed that a first-countable space X is separated if and only if every compact subset of X is closed. This equivalence is not valid for uncountable spaces, which is proved by the following example.

Example 8. *We take the same hypothesis as in Example 7. We have seen that X is not separated. We next observe that the only compact subsets of X are the finite subsets. For that let K be any infinite subset of X. Then, there exists a sequence $(x_n)_n$ of distinct elements in K. For each positive integer n define $U_n = X \setminus \{x_j : j > n\}$. Then the collection $\{U_n : n \in \mathbb{N}\}$ is an open cover of K which has no finite subcover. Since every finite subset of X is countable and hence closed. Thus, every compact subset of X is closed.*

Theorem 20. *Let (X,τ) be a topological space. Then, X has unique $\mathcal{I}$-limits of sequences if and only if every sequentially $\mathcal{I}$-compact subset of X is $\mathcal{I}$-closed.*

Proof. Assume that every sequentially $\mathcal{I}$-compact subset of X is $\mathcal{I}$-closed. To obtain a contradiction, suppose that there is a sequence $(x_n)_n$ in X such that $x_n \xrightarrow{\mathcal{I}} x$ and $x_n \xrightarrow{\mathcal{I}} y$ with $x \neq y$. Let $\{n \in \mathbb{N} : x_n = x\}$ or $\{n \in \mathbb{N} : x_n = y\}$ is infinite. Then, the constant subsequence (x) is a subsequence of $(x_n)_n$ which is convergent to x and y. By Proposition 1, (x) is $\mathcal{I}$-convergent to x and y. Since $\{x\}$ is sequentially $\mathcal{I}$-compact, then $\{x\}$ is $\mathcal{I}$-closed. Therefore $\{x\} = \overline{\{x\}}^{\mathcal{I}}$, which contradicts $x \neq y$.

If $\{n \in \mathbb{N} : x_n = x\}$ and $\{n \in \mathbb{N} : x_n = y\}$ are finite. Then, there exists a subsequence $(x_{\varphi(n)})_n$ of $(x_n)_n$ such that $x_{\varphi(n)} \neq x$ and $x_{\varphi(n)} \neq y$ for all $n \in \mathbb{N}$. Since $(x_n)_n$ converges to x and y, then $(x_{\varphi(n)})_n$ is convergent to x and y which is $\mathcal{I}$-convergent to x and y by Proposition 1. Let $A = \{x_{\varphi(n)} : n \in \mathbb{N}\}$. By previous lemma, $A \cup \{x\}$ is sequentially $\mathcal{I}$-compact. By hypothesis, $A \cup \{x\}$ is $\mathcal{I}$-closed, i.e., $A \cup \{x\} = \overline{A \cup \{x\}}^{\mathcal{I}}$. It follows that $y \notin \overline{A \cup \{x\}}^{\mathcal{I}}$ and hence there does not exist a sequence $(y_n)_n$ in $A \cup \{x\}$ such that $y_n \xrightarrow{\mathcal{I}} y$, which contradicts the fact that the sequence $(x_{\varphi(n)})_n$ in A is $\mathcal{I}$-convergent to x and y with $x \neq y$. Thus, X has unique $\mathcal{I}$-limits of sequences.

Conversely, assume that X has unique $\mathcal{I}$-limits of sequences and let A be a sequentially $\mathcal{I}$-compact subset of X. It is enough to show that $\overline{A}^{\mathcal{I}} \subset A$. Let $x \in \overline{A}^{\mathcal{I}}$. By $\mathcal{I}$-Fréchet-Urysohn property, there exists a sequence $(x_n)_n$ in A such that $x_n \xrightarrow{\mathcal{I}} x$. Since A is sequentially $\mathcal{I}$-compact, then every sequence in A has a $\mathcal{I}$-convergent subsequence which is $\mathcal{I}$-convergent to a point in A. By uniquness of the $\mathcal{I}$-limit x we have $x \in A$. Thus A is $\mathcal{I}$-closed. $\qquad\square$

Corollary 1. *Let $\mathcal{I}$ be a maximal ideal and X a first-countable space. X is $\mathcal{I}$-separated, if and only if every sequentially $\mathcal{I}$-compact subset of X is $\mathcal{I}$-closed.*

Proof. Let X be a first-countable space. Since $\mathcal{I}$ is an admissible maximal ideal, then by Theorem 16, Theorem 18 and Proposition 5, X is $\mathcal{I}$-separated if and only if X has only one $\mathcal{I}$-limit point. Therefore by previous theorem, X has only one limit point in X if and only if every sequentially $\mathcal{I}$-compact subset of X is $\mathcal{I}$-closed. $\quad\square$

Lemma 3. *Let X and Y be topological spaces. X and Y are sequentially $\mathcal{I}$-compact if and only if $X \times Y$ is.*

Proof. Assume that X and Y are sequentially $\mathcal{I}$-compact. If $(x_n, y_n)_n$ is a sequence in $X \times Y$, then there is a subsequence $(x_{n_k})_k$ of $(x_n)_n$ which is $\mathcal{I}$-convergent to x in X. In the sequence $(y_{n_k})_k$, there is also a subsequence $(y_{n_j})_j$ which $\mathcal{I}$-converges to y in Y. Since $(x_{n_j})_j$ is a subsequence of $(x_{n_k})_k$, it is also $\mathcal{I}$-convergent to x. Thus, $(x_{n_j}, y_{n_j}) \xrightarrow{\mathcal{I}} (x,y)$.

Conversely, let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the two projections on $X \times Y$. By Theorem 14 , $p_X(X \times Y)$ and $p_Y(X \times Y)$ are sequentially $\mathcal{I}$-compact, i.e., X and Y are sequentially $\mathcal{I}$-compact. $\qquad\square$

Theorem 21. *Let (X,τ) be a topological space and $\mathcal{I}$ a maximal ideal. If X is $\mathcal{I}$-separated, then the diagonal set of $X \times X$ is $\mathcal{I}$-closed.*

Proof. Let (X, τ) be a $\mathcal{I}$-separated topological space, $\mathcal{I}$ a maximal ideal and Δ the diagonal set of $X \times X$. By Theorem 16, X has unique $\mathcal{I}$-limits of sequences. To obtain a contradiction, we suppose that Δ is not $\mathcal{I}$-closed. Then there is a sequence $(x_n, x_n)_n$ in Δ which is $\mathcal{I}$-convergent to (x, y) with $(x, y) \notin \Delta$, i.e. $x_n \xrightarrow{\mathcal{I}} x$ and $x_n \xrightarrow{\mathcal{I}} y$ such that $x \neq y$, which contradicts the fact that X has unique $\mathcal{I}$-limits of sequences. Thus, Δ is $\mathcal{I}$-closed. $\qquad\square$

Theorem 22. *Let (X, τ) be a topological space. If the diagonal set of $X \times X$ is $\mathcal{I}$-closed, then X has unique $\mathcal{I}$-limits of sequences.*

Proof. Let Δ be the diagonal set of $X \times X$. Striving for a contradiction, suppose that there exists a sequence $(x_n)_n$ in X such that $x_n \xrightarrow{\mathcal{I}} x$ and $x_n \xrightarrow{\mathcal{I}} y$ with $x \neq y$. Then $(x_n, x_n)_n$ $\mathcal{I}$-converges to (x, y) with $(x, y) \notin \Delta$ which contradicts the fact that Δ is $\mathcal{I}$-closed. Thus, X has unique $\mathcal{I}$-limits of sequences. $\qquad\square$

Theorem 23. *Let (X, τ), (Y, τ) be topological spaces, $f : X \to Y$ and $g : X \to Y$ be sequentially $\mathcal{I}$-continuous mappings. If Y is $\mathcal{I}$-separated, then $A = \{x \in X : f(x) = g(x)\}$ is $\mathcal{I}$-closed.*

Proof. Assume for contradiction that A is not $\mathcal{I}$-closed. Then by the $\mathcal{I}$-Fréchet-Urysohn proprety there is a sequence $(x_n)_n$ in A which is $\mathcal{I}$-convergent to x with $x \notin A$. Since $f : X \to Y$ and $g : X \to Y$ are sequentially $\mathcal{I}$-continuous, then $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$. By Theorem 16, X has unique $\mathcal{I}$-limits of sequences, therefore $f(x) = g(x)$ and $x \in A$, which is not the case. Thus, A is $\mathcal{I}$-closed. $\qquad\square$

Lemma 4 (Theorem 4.2, [46]). *Let X, Y be topological spaces and $f : X \to Y$ be a mapping. If f is continuous, then f is sequentially $\mathcal{I}$-continuous.*

Theorem 24. *Let (X, τ), (Y, τ) be topological spaces, $f : X \to Y$ and $g : X \to Y$ be sequentially $\mathcal{I}$-continuous mappings. If $A = \{x \in X : f(x) = g(x)\}$ is $\mathcal{I}$-closed, then Y has unique $\mathcal{I}$-limits of sequences.*

Proof. We replace X by $Y \times Y$, we also replace f and g by the two continuous projections on $Y \times Y$, that they are also sequentially $\mathcal{I}$-continuous by previous lemma. Then, $\{(x, y) \in Y \times Y : f(x) = g(x)\}$ is the diagonal subset of $Y \times Y$, which is $\mathcal{I}$-closed by hypothesis. Therefore, Y has unique $\mathcal{I}$-limits of sequences, by Theorem 22. $\qquad\square$

REFERENCES

[1] A.El. Amrani, *Sequentially spaces and the finest locally $\mathbb{K}$-convex of topologies having the same onvergent sequences*, Proyecciones (Antofagasta, On line), **37**:1 (2018), 153–169.

[2] A.K. Banerjee, A. Banerjee, *$\mathcal{I}$-convergence classes of sequences and nets in topological spaces*, arXiv:1608.03381, 2016.

[3] J.R. Boone, F. Siwiec, *Sequentially quotient mappings*, Czech. Math. J., **26** (1976), 174–182. Zbl 0334.54003

[4] R.C. Buck, *The measure theoretic approach to density*, Am. J. Math., **68** (1946), 560–580. Zbl 0061.07503

[5] R.C. Buck, *Generalized asymptotic density*, Am. J. Math., **75** (1953), 335–346. Zbl 0050.05901

[6] H. Çakallı, *On G-continuity*, Comput. Math. Appl., **61**:2 (2011), 313–318. Zbl 1211.40002

[7] J. Červeňanský, *Statistical convergence and statistical continuity*, Zborník vedeckých prác MtF STU, **6** (1998), 207–212.

[8] K. Chandrasekharan, *Topological preliminaries*, in *A course on topological groups*, Hindustan Book Agency, New Delhi, 1996, 1–18. Zbl 0959.22001

[9] J. Connor, *The statistical and strong p-Cesaro convergence of sequences*, Analysis, **8**:1-2 (1988), 47–63. Zbl 0653.40001

[10] J. Connor, *Two valued measures and summability*, Analysis, **10**:4 (1990), 373–385. Zbl 0726.40009

[11] J. Connor, K.-G. Grosse-Erdmann, *Sequential definitions of continuity for real functions*, Rocky Mt. J. Math., **33**:1 (2003), 93–121. Zbl 1040.26001

[12] J. Connor, J. Kline, *On statistical limit points and the consistency of statistical convergence*, J. Math. Anal. Appl., **197**:2 (1996), 392–399. Zbl 0867.40001

[13] P. Das, *Some further results on ideal convergence in topological spaces*, Topology Appl., **159**:10-11 (2012), 2621–2626. Zbl 1250.40002

[14] P. Das, P. Kostyrko, W. Wilczyński, P. Malik, *$\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences*, Math. Slovaca, **58**:5 (2008), 605–620. Zbl 1199.40026

[15] P. Das, E. Savaş, *On $\mathcal{I}$-convergence of nets in locally solid Riesz spaces*, Filomat, **27**:1 (2013), 89–94. Zbl 1340.40015

[16] G. Di Maio, L.D. Kočinac, *Statistical convergence in topology*, Topology Appl., **156**:1 (2008), 28–45. Zbl 1155.54004

[17] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2**, (1951), 241–244. Zbl 0044.33605

[18] R. Filipów, N. Mrożek, I. Recław, P. Szuca, *Ideal convergence of bounded sequences*, J. Symb. Log., **72**:2 (2007), 501–512. Zbl 1123.40002

[19] S. Franklin, *Spaces in which sequences suffice II*, Fundam. Math., **61** (1967), 51–56. Zbl 0168.43502

[20] A. Freedman, J. Sember, *Densities and summability*, Pac. J. Math., **95** (1981), 293–305. Zbl 0504.40002

[21] W. Freedman, *Convergence preserving mappings on topological groups*, Topology Appl., **154**:6 (2007), 1089–1096. Zbl 1120.22001

[22] J. Fridy, *Statistical limit points*, Proc. Am. Math. Soc., **118**:4 (1993), 1187–1192. Zbl 0776.40001

[23] D.N. Georgiou, A.C. Megaritis, G.A. Prinos, *A study on convergence and ideal convergence classes*, Topology Appl., **241** (2018), 38–49. Zbl 1395.54004

[24] D. Georgiou, A. Megaritis, G. Prinos, F. Sereti, *On statistical convergence of sequences of closed sets in metric spaces*, Math. Slovaca, **71**:2 (2021), 409–422.

[25] D. Georgiou, S. Iliadis, A. Megaritis, G. Prinos, *Ideal-convergence classes*, Topology Appl., **222** (2017), 217–226. Zbl 1373.54012

[26] B. Hazarika, A. Esi, *Some ideal-convergent generalized difference sequences in a locally convex space defined by a Musielak-Orlicz function*, Mat. Stud., **42**:2 (2014), 195–208. Zbl 1334.46005

[27] W.-C. Hong, S.-H. Kweon, *On the uniqueness of sequential limits in sequential convergence spaces*, East Asian Math. J., **10**:2 (1994), 271–277.

[28] A.J. Insel, *A note on the Hausdorff separation property in first countable spaces*, Am. Math. Mon., **72** (1965), 289–290. Zbl 0139.40302

[29] P. Kostyrko, M. Máčaj, T. Šalát, M. Sleziak, *$\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points*, Math. Slovaca, **55**:4 (2005), 443–464. Zbl 1113.40001

[30] P. Kostyrko, T. Šalát, W. Wilczyński, *$\mathcal{I}$-convergence*, Real Anal. Exch., **26**:2 (2000), 669–685. Zbl 1021.40001

[31] B.K. Lahiri, P. Das, *$\mathcal{I}$ and $\mathcal{I}^*$-convergence in topological spaces*, Math. Bohem., **130**:2 (2005), 153–160. Zbl 1111.40001

[32] O. Mucuk, T. Şahan, *On g-sequential Continuity*, arXiv:1201.1795, (2011). (*On G-sequential continuity*, Filomat, **28**:6, (2014), 1181–1189. Zbl 06704838)

[33] M. Mursaleen, S. Mohiuddine, *On ideal convergence in probabilistic normed spaces*, Math. Slovaca, **62**:1 (2012), 49–62. Zbl 1274.40034

[34] A. Nabiev, S. Pehlivan, M. Gürdal, *On $\mathcal{I}$-Cauchy sequences*, Taiwanese J. Math., **11**:2 (2007), 569–576. Zbl 1129.40001

[35] H.-H. Ostmann, *Additive Number Theory: Part One General Investigations*, Springer-Verlag, Berlin etc., 1956. Zbl 0072.03101

[36] S.K. Pal, *$\mathcal{I}$-sequential topological spaces*, Appl. Math. E-Notes, **14** (2014), 236–241. Zbl 1320.54006

[37] V. Renukadevi, B. Prakash, *$\mathcal{I}$-Fréchet-Urysohn spaces*, Math. Morav., **20**:2 (2016), 87–97. Zbl 06750236

[38] V. Renukadevi, P. Vijayashanthi, *On $\mathcal{I}$-Fréchet-Urysohn spaces and sequential $\mathcal{I}$-convergence groups*, Math. Morav., **23**:1 (2019), 119–129. MR3979962

[39] T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca, **30** (1980), 139–150. Zbl 0437.40003

[40] T. Šalát, R. Tijdeman, *On density measures of sets of positive integers*, in *Topics in Classical Number Theory. Colloq. Math. Soc. Janos Bolyai*, **34**, 1984, 1445–1457. Zbl 0548.10037

[41] T. Šalát, B.C. Tripathy, M. Ziman, *On some properties of $\mathcal{I}$-convergence*, Tatra Mt. Math. Publ, **28**:2 (2004), 279–286. Zbl 1110.40002

[42] T. Šalát, B.C. Tripathy, M. Ziman, *On $\mathcal{I}$-convergence field*, Ital. J. Pure Appl. Math, **17** (2005), 45–54. Zbl 1099.40005

[43] T. Šalát, B.C. Tripathy, M. Ziman, *On $\mathcal{I}$-convergence field*, Ital. J. Pure Appl. Math, **17** (2005), 45–54. Zbl 1099.40005

[44] I. Schoenberg, *The integrability of certain functions and related summability methods. I,II*, Am. Math. Mon., **66** (1959), 361–775. Zbl 0089.04002

[45] B.C. Tripathy, M. Sen, S. Nath, *$\mathcal{I}$-convergence in probabilistic n-normed space*, Soft Comput., **16**:6 (2012), 1021–1027. Zbl 1264.40006

[46] X. Zhou, L. Liu, S. Lin, *On topological spaces defined by $\mathcal{I}$-convergence*, Bull. Iran. Math. Soc., **46**:3 (2020), 675–692. Zbl 1454.54005

Aziz Blali
Department of mathematics,
École Normale Supérieure,
Sidi Mohamed Ben Abdellah University,
B.P. 5206 Bensouda-Fès, Morocco
Email address: aziz.blali@usmba.ac.ma

Abdelkhalek El Amrani
Department of mathematics and computer science,
Faculty of Sciences Dhar El mahraz,
Sidi Mohamed Ben Abdellah University,
B.P. 1769-Fès Atlas, Morocco
Email address: abdelkhalek.elamrani@usmba.ac.ma

Rachid Ameziane Hassani
Department of mathematics and computer science,
Faculty of Sciences Dhar El mahraz,
Sidi Mohamed Ben Abdellah University,
B.P. 1769-Fès Atlas, Morocco
Email address: ra.ameziane@yahoo.fr

Abdelhak Razouki
Department of mathematics and computer science,
Faculty of Sciences Dhar El mahraz,
Sidi Mohamed Ben Abdellah University,
B.P. 1769-Fès Atlas, Morocco
Email address: razoukiabdelhak@gmail.com