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COHOMOLOGY FOR THE LIE ALGEBRA OF TYPE A_2 OVER A FIELD OF CHARACTERISTIC 2

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ABSTRACT. We calculate the cohomology of the classical Lie algebra of type A_2 over an algebraically closed field k of characteristic $p = 2$ with coefficients in simple modules. The obtained results were used to describe the cohomology of the Lie algebra $\mathfrak{gl}_3(k)$ and the cohomology of the restricted Lie algebra of Cartan type $W_3(\mathbf{1})$ with coefficients in the divided power algebra $O_3(\mathbf{1})$.

Keywords: Lie algebra, simple module, cohomology.

1. INTRODUCTION

1.1. Let \mathfrak{g} be a Lie algebra over a field k of characteristic p and M be a \mathfrak{g} -module.

Definition 1. A \mathfrak{g} -module M is said to be peculiar if $H^*(\mathfrak{g}, M) \neq 0$. We will say that M is an n -peculiar module over \mathfrak{g} if $H^n(\mathfrak{g}, M) \neq 0$.

Let $\mathfrak{g} = \mathfrak{sl}_3(k)$. We decompose the space of cochains $C^*(\mathfrak{g}, M)$ into a direct sum of the subspaces with respect to the maximal torus T of the group $G = SL_3(k)$ [1, 4.3]:

$$C^*(\mathfrak{g}, M) = \bigoplus_{\mu \in X(T)} C_{\mu}^*(\mathfrak{g}, M),$$

where $X(T)$ is the additive character group of the torus T . Then

$$H^n(\mathfrak{g}, M) = \bigoplus_{\mu \in X(T)} H_{\mu}^n(\mathfrak{g}, M).$$

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We identify the space $C^n(\mathfrak{g}, M)$ with the space $\bigwedge^n \mathfrak{g}^* \otimes M$. Denote by $\prod(V)$ the set of weights of the subspace V of the G -module $H^*(\mathfrak{g}, M)$.

Since $\prod(H^n(\mathfrak{g}, M)) \subseteq pX(T) \cap \prod(\bigwedge^n \mathfrak{g}^* \otimes M)$, then we will consider elements of the subspace $\overline{C}^n(\mathfrak{g}, M)$ of the space $C^n(\mathfrak{g}, M)$ with weights from the set

$$pX(T) \cap \prod \left(\bigwedge^n \mathfrak{g}^* \otimes M \right).$$

The corresponding subspaces of cocycles and cohomology are denoted by $\overline{Z}^n(\mathfrak{g}, M)$ and $\overline{H}^n(\mathfrak{g}, M)$ respectively. Note that

$$H^n(\mathfrak{g}, M) = \overline{H}^n(\mathfrak{g}, M).$$

We will use the well-known formula

$$(1) \quad \dim H^n(\mathfrak{g}, M) = \dim \overline{Z}^n(\mathfrak{g}, M) + \dim \overline{Z}^{n-1}(\mathfrak{g}, M) - \dim \overline{C}^{n-1}(\mathfrak{g}, M).$$

Since $Tr(adx) = 0$ for all $x \in \mathfrak{g}$, then, according to the main theorem in [2], we get the following formula for the dimensions of the cohomology:

$$(2) \quad \dim H^n(\mathfrak{g}, M) = \dim H^{\dim \mathfrak{g}-n}(\mathfrak{g}, M^*).$$

The weight subspaces are invariant under the coboundary operator action. Therefore, the formula (1) also holds for weight subspaces:

$$(3) \quad \dim H_\mu^n(\mathfrak{g}, M) = \dim \overline{Z}_\mu^n(\mathfrak{g}, M) + \dim \overline{Z}_\mu^{n-1}(\mathfrak{g}, M) - \dim \overline{C}_\mu^{n-1}(\mathfrak{g}, M).$$

1.2. Let M be a $\mathfrak{sl}_n(k)$ -module. We define a $\mathfrak{gl}_n(k)$ -module structure on M . Since $\mathfrak{gl}_n(k) \cong \mathfrak{sl}_n(k) \oplus I$, where I is the subspace generated by the identity $n \times n$ matrix, then a $\mathfrak{gl}_n(k)$ -module structure can be determined by rule

$$(4) \quad (x, a)m = xm + \mu(a)m, \quad (x, a) \in \mathfrak{gl}_n(k), \quad x \in \mathfrak{sl}_n(k), \quad a \in I,$$

where μ is a linear form on I . We denote the obtained $\mathfrak{gl}_m(k)$ -module also via M .

1.3. Let k be an algebraically closed field of characteristic $p > 0$. We will consider the Lie algebra $\mathfrak{gl}_n(k)$ as a subalgebra of the Lie algebra $W_n(\mathbf{m})$. Let $\varepsilon_i = (\delta_{i1}, \dots, \delta_{in})$, where $\delta_{i,j}$ is Kronecker symbol, and m_1, \dots, m_n are positive integers. The divided power algebra $O_n(\mathbf{m})$ with height $\mathbf{m} = \sum_{i=1}^n m_i \varepsilon_i$ is defined as follows:

$$O_n(\mathbf{m}) = \langle x^{(\alpha)} := \prod_{i=1}^n x^{(\alpha_i)} : \alpha \in \Gamma_n(\mathbf{m}), x^{(\alpha)} x^{(\beta)} = \prod_{i=1}^n C_{\alpha_i}^{\alpha_i + \beta_i} x^{(\alpha + \beta)} \rangle_k,$$

$$\Gamma_n(\mathbf{m}) = \left\{ \sum_{i=1}^n \alpha_i \varepsilon_i : \alpha_i \in \mathbb{Z}_0, 0 \leq \alpha_i < p^{m_i}, i = 1, \dots, n \right\},$$

where \mathbb{Z}_0 is the set of integers. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \Gamma_n(\mathbf{m})$, let $|\alpha| = \sum_{i=1}^n \alpha_i$. Let us define the derivations $\partial_i, i = 1, \dots, n$, of the divided power algebra $O_n(\mathbf{m})$ by the formula

$$\partial_i x^{(\alpha)} = x^{(\alpha - \varepsilon_i)}.$$

Then the linear space

$$W = W_n(\mathbf{m}) = \left\{ \sum_{j=1}^n a_j \partial_j : a_j \in O_n(\mathbf{m}), i = 1, \dots, n \right\}$$

with multiplication

$$\left[\sum_{j=1}^n a_j \partial_j, \sum_{j=1}^n b_j \partial_j \right] = \sum_{i=1}^n \sum_{j=1}^n (a_j \partial_j(b_i) - b_j \partial_j(a_i)) \partial_i$$

is a Lie algebra. This Lie algebra is called a general Lie algebra of Cartan type or a generalized Jacobson – Witt algebra. It is a Lie algebra of special derivations of $O_n(\mathbf{m})$, hence we can introduce in a natural way the module structure on $O_n(\mathbf{m})$, over $W_n(\mathbf{m})$ by the formula

$$\left(\sum_{j=1}^n a_j \partial_j \right) x^{(\alpha)} = \sum_{j=1}^n a_j x^{(\alpha - \varepsilon_j)}, \quad \sum_{j=1}^n a_j \partial_j \in W_n(\mathbf{m}), \quad x^{(\alpha)} \in O_n(\mathbf{m}).$$

Assuming $W_i = \langle x^{(\alpha)} \partial_j : |\alpha| = i + 1, j = 1, \dots, n \rangle_k$, we obtain the natural grading $W = \bigoplus_{i \geq -1} W_i$ with depth 1. The Lie algebra W is simple, except for the case when $n = 1$ and $p = 2$. The linear map $x^{(\varepsilon_j)} \partial_i \mapsto E_{ij}$ defines an isomorphism between the Lie subalgebra W_0 and the general linear Lie algebra $\mathfrak{gl}_n(k)$, where $E_{i,j}$ is a $n \times n$ matrix with elements $e_{l,q} = \delta_{il} \delta_{jq}$.

If $\mathbf{m} = (1, \dots, 1)$, then $W_n(\mathbf{m})$ is a restricted Lie algebra. It is called the Jacobson–Witt algebra and denoted by $W_n(\mathbf{1})$.

1.4. Main results. Now let \mathfrak{g} be a classical Lie algebra of type A_2 over algebraically closed field k of characteristic $p > 0$ and M is a simple \mathfrak{g} -module. Let $L(r, s)$ denote a simple \mathfrak{g} -module with the highest weight $r\omega_1 + s\omega_2$, where ω_1, ω_2 are fundamental weights. In this paper, we study the cohomology of simple modules for the Lie algebra of type A_2 and for the general linear Lie algebra $\mathfrak{gl}_3(k)$ over an algebraically closed field k of characteristic $p = 2$. We also consider the application of these cohomology to the calculation of the cohomology Jacobson – Witt algebra $W_3(\mathbf{1})$ with coefficients in the divided power algebra $O_3(\mathbf{1})$. To calculate cohomology of $W_3(\mathbf{1})$ we use the general formula obtained in [3, Theorem 0.2].

It is known that the composition of an irreducible representation of the group $SL_3(k)$ on the vector space L with a d -th power of the Frobenius map defines a new representation on which the Lie algebra \mathfrak{g} acts trivially. We denote the obtained module by $L^{(d)}$. To each weight μ of the space L there corresponds the weight $p^d \mu$ of the space $L^{(d)}$. The composition factor of $H^n(\mathfrak{g}, M)$ as a $SL_3(k)$ -module is either a simple twisted module $L^{(d)}$ for some d , or a one-dimensional trivial module k . Let $[H^n(\mathfrak{g}, M) : L^{(d)}]$ denote the composition factor multiplicity of a simple $SL_3(k)$ -module $L^{(d)}$ in the cohomology group $H^n(\mathfrak{g}, M)$. We will also use the following short notation: $H^n(\mathfrak{g}, k) := H^n(\mathfrak{g})$, $\bigoplus_{i=1}^m V := mV$, where V is a decomposable $SL_3(k)$ -module.

Let us formulate the main results:

Theorem 1. *Let \mathfrak{g} be the classical Lie algebra of type A_2 over an algebraically closed field k of characteristic $p = 2$ and M be a simple \mathfrak{g} -module. Then the following isomorphisms of $SL_3(k)$ -modules hold:*

- (a) $H^0(\mathfrak{g}) \cong H^8(\mathfrak{g}) \cong k$, $H^3(\mathfrak{g}) \cong H^5(\mathfrak{g}) \cong L(1, 1)^{(1)} \oplus k$, $H^4(\mathfrak{g}) \cong 2L(1, 1)^{(1)}$;
- (b) $H^1(\mathfrak{g}, L(1, 0)) \cong H^7(\mathfrak{g}, L(1, 0)) \cong L(0, 1)^{(1)}$, $H^2(\mathfrak{g}, L(1, 0)) \cong H^6(\mathfrak{g}, L(1, 0)) \cong 2L(0, 1)^{(1)} \oplus L(1, 0)^{(2)}$, $H^3(\mathfrak{g}, L(1, 0)) \cong H^5(\mathfrak{g}, L(1, 0)) \cong L(0, 1)^{(1)} \oplus L(1, 0)^{(2)}$;
- (c) $H^1(\mathfrak{g}, L(0, 1)) \cong H^7(\mathfrak{g}, L(0, 1)) \cong L(1, 0)^{(1)}$, $H^2(\mathfrak{g}, L(0, 1)) \cong H^6(\mathfrak{g}, L(0, 1)) \cong 2L(1, 0)^{(1)} \oplus L(0, 1)^{(2)}$, $H^3(\mathfrak{g}, L(0, 1)) \cong H^5(\mathfrak{g}, L(0, 1)) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(2)}$.

Otherwise, $H^n(\mathfrak{g}, M) = 0$.

Theorem 2. Let $\mathfrak{g} = \mathfrak{gl}_3(k)$ be the general linear Lie algebra over an algebraically closed field k of characteristic $p = 2$ and M be a simple \mathfrak{g} -module defined by the formula (4). Then the following isomorphisms of $SL_3(k)$ -modules hold:

$$(a) H^0(\mathfrak{g}) \cong H^1(\mathfrak{g}) \cong H^8(\mathfrak{g}) \cong H^9(\mathfrak{g}) \cong k, H^3(\mathfrak{g}) \cong H^6(\mathfrak{g}) \cong L(1, 1)^{(1)} \oplus k, \\ H^4(\mathfrak{g}) \cong H^5(\mathfrak{g}) \cong 3L(1, 1)^{(1)} \oplus k;$$

$$(b) H^1(\mathfrak{g}, L(1, 0)) \cong H^8(\mathfrak{g}, L(1, 0)) \cong L(0, 1)^{(1)}, H^2(\mathfrak{g}, L(1, 0)) \cong H^7(\mathfrak{g}, L(1, 0)) \cong \\ 3L(0, 1)^{(1)} \oplus L(1, 0)^{(2)}, H^3(\mathfrak{g}, L(1, 0)) \cong H^6(\mathfrak{g}, L(1, 0)) \cong 3L(0, 1)^{(1)} \oplus 2L(1, 0)^{(2)}, \\ H^4(\mathfrak{g}, L(1, 0)) \cong H^5(\mathfrak{g}, L(1, 0)) \cong L(0, 1)^{(1)} \oplus L(1, 0)^{(2)};$$

$$(c) H^1(\mathfrak{g}, L(0, 1)) \cong H^8(\mathfrak{g}, L(0, 1)) \cong L(1, 0)^{(1)}, H^2(\mathfrak{g}, L(0, 1)) \cong H^7(\mathfrak{g}, L(0, 1)) \cong \\ 3L(1, 0)^{(1)} \oplus L(0, 1)^{(2)}, H^3(\mathfrak{g}, L(0, 1)) \cong H^6(\mathfrak{g}, L(0, 1)) \cong 3L(1, 0)^{(1)} \oplus 2L(0, 1)^{(2)}, \\ H^4(\mathfrak{g}, L(0, 1)) \cong H^5(\mathfrak{g}, L(0, 1)) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(2)}.$$

Otherwise, $H^n(\mathfrak{g}, M) = 0$.

Proposition 1. Let $\mathfrak{g} = W_3(\mathbf{1})$ be the Jacobson–Witt algebra over an algebraically closed field k of characteristic $p = 2$ and $M = O_3(\mathbf{1})$ is a \mathfrak{g} -module. Then the following isomorphisms of $SL_3(k)$ -modules hold:

- (a) $H^0(\mathfrak{g}, M) \cong H^{12}(\mathfrak{g}, M) \cong k$;
- (b) $H^1(\mathfrak{g}, M) \cong H^{11}(\mathfrak{g}, M) \cong 4k$;
- (c) $H^2(\mathfrak{g}, M) \cong H^{10}(\mathfrak{g}, M) \cong 6k$;
- (d) $H^3(\mathfrak{g}, M) \cong H^9(\mathfrak{g}, M) \cong L(1, 1)^{(1)} \oplus 5k$;
- (e) $H^4(\mathfrak{g}, M) \cong H^8(\mathfrak{g}, M) \cong 6L(1, 1)^{(1)} \oplus 5k$;
- (f) $H^5(\mathfrak{g}, M) \cong H^7(\mathfrak{g}, M) \cong 15L(1, 1)^{(1)} \oplus 7k$;
- (g) $H^6(\mathfrak{g}, M) \cong 20L(1, 1)^{(1)} \oplus 8k$.

Otherwise, $H^n(\mathfrak{g}, M) = 0$.

The cohomology of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_3(k)$ with coefficients in simple modules are known in the following cases:

- 1) $p > 0$, $H^1(\mathfrak{g}, M)$ for all simple \mathfrak{g} -modules M [4];
- 2) $p > 3$, $H^2(\mathfrak{g}, M)$ for all simple \mathfrak{g} -modules M [5];
- 3) $p = 2$, $H^1(\mathfrak{g}, \mathfrak{g}) = 0$ [6];
- 4) $p = 2$, $H^2(\mathfrak{g}, \mathfrak{g}) = 0$ [7].

2. COHOMOLOGY OF THE LIE ALGEBRA OF TYPE A_2

This section is devoted to the proof of Theorem 1.

Proof. As basis vectors for the Lie algebra \mathfrak{g} , we choose the special derivations of the divided power algebra $O_3(\mathbf{1})$: $h_1 = x_1\partial_1 - x_2\partial_2$, $h_2 = x_2\partial_2 - x_3\partial_3$, $e_1 = x_1\partial_2$, $e_2 = x_2\partial_3$, $e_3 = x_1\partial_3$, $f_1 = x_2\partial_1$, $f_2 = x_3\partial_2$, $f_3 = x_3\partial_1$. Then there are following natural descriptions for nontrivial restricted simple modules of the Lie algebra \mathfrak{g} :

$$L(1, 0) = \langle x_1, x_2, x_3 \rangle_k, \\ L(0, 1) = \langle \partial_1, \partial_2, \partial_3 \rangle_k, \\ L(1, 1) \cong \mathfrak{g} = \langle x_1\partial_1 - x_2\partial_2, x_2\partial_2 - x_3\partial_3, x_1\partial_2, x_2\partial_3, x_1\partial_3, x_2\partial_1, x_3\partial_2, x_3\partial_1 \rangle_k.$$

It is known that the peculiar modules of the Lie algebra \mathfrak{g} are restricted [8]. According to Lemma 3.1 in [5], only the following simple restricted modules are peculiar: $L(0, 0) \cong k$, $L(1, 0)$, $L(0, 1)$. Consider each of these modules separately.

Let $M = L(0, 0) \cong k$.

Lemma 1. *The following isomorphisms of $SL_3(k)$ -modules hold:*

- (a) $H^0(\mathfrak{g}) \cong k$;
- (b) $H^3(\mathfrak{g}) \cong L(1, 1)^{(1)} \oplus k$;
- (c) $H^4(\mathfrak{g}) = 2L(1, 1)^{(1)}$;
- (d) $H^5(\mathfrak{g}) \cong L(1, 1)^{(1)} \oplus k$;
- (e) $H^8(\mathfrak{g}, k) \cong k$.

Otherwise, $H^n(\mathfrak{g}) = 0$.

Proof. The statements (a) and (e) are obvious. The triviality of the cohomology of $H^1(\mathfrak{g})$ in characteristic $p = 2$ was proved in [4]. Let us prove the triviality of the cohomology $H^2(\mathfrak{g})$. Obviously, the set of all weights of the subspace $\overline{C}^2(\mathfrak{g})$ consists only one zero weight. Then, if the space $H^2(\mathfrak{g})$ is nontrivial, then as a $SL_3(k)$ -module it can contain only composition factors isomorphic to a one-dimensional trivial module, and is generated by the classes of cocycles with weight 0. The subspace $\overline{C}_0^2(\mathfrak{g})$ is four-dimensional and generated by the cochains $\psi_1^2 = h_1^* \wedge h_2^*$, $\psi_2^2 = e_1^* \wedge f_1^*$, $\psi_3^2 = e_2^* \wedge f_2^*$, $\psi_4^2 = e_3^* \wedge f_3^*$. Let $\sum_{i=1}^4 b_i \psi_i^2 \in Z^2(\mathfrak{g})$, where $b_1, b_2, b_3, b_4 \in k$. Using the cocycle condition, we obtain $b_1 = 0, b_2 + b_3 + b_4 = 0$. Whence it follows that $\dim Z_0^1(\mathfrak{g}) = 2$, and then, by the formula (3), $\dim H_0^2(\mathfrak{g}, M) = 2 + 0 - 2 = 0$. Therefore, $H^2(\mathfrak{g}) = H_0^2(\mathfrak{g}) = 0$. Moreover, according to the formula (2), the triviality of $H^2(\mathfrak{g})$ implies the triviality of $H^6(\mathfrak{g})$.

Thus, it remains to prove the statements (b) – (d).

(b) Since $\prod(\overline{C}^3(\mathfrak{g}) = \{0, \pm 2(\omega_1 + \omega_2), \pm 2(2\omega_1 - \omega_2), \pm 2(-\omega_1 + 2\omega_2)\})$, then composition factors of $H^3(\mathfrak{g})$ can only be a twisted simple module $L(1, 1)^{(1)}$ and the trivial one-dimensional module k . They are generated by the classes of cocycles having weights $2\omega_1 + \omega_2$ and 0, respectively. Let's consider each of these cases separately.

The subspace $\overline{C}_0^3(\mathfrak{g})$ is eight-dimensional and generated by vectors

$$\begin{aligned} & h_1^* \wedge e_1^* \wedge f_1^*, h_2^* \wedge e_1^* \wedge f_1^*, h_1^* \wedge e_2^* \wedge f_2^*, h_2^* \wedge e_2^* \wedge f_2^*, \\ & h_1^* \wedge e_3^* \wedge f_3^*, h_2^* \wedge e_3^* \wedge f_3^*, e_3^* \wedge f_1^* \wedge f_2^*, e_1^* \wedge e_2^* \wedge f_3^*. \end{aligned}$$

Suppose that a linear combination of these vectors with the coefficients $b_i, i = 1, \dots, 8$, respectively, is a 3-cocycle. Then the cocycle condition implies that

$$\begin{cases} b_1 + b_2 + b_5 + b_7 - b_8 = 0, \\ b_2 + b_3 - b_7 + b_8 = 0, \\ b_3 + b_4 + b_6 + b_7 - b_8 = 0, \\ 2b_4 + 2b_7 - 2b_8 = 0, \\ 2b_5 + 2b_6 + 2b_7 - 2b_8 = 0. \end{cases}$$

The solution space of this linear system with respect to $b_i, i = 1, \dots, 8$, is five-dimensional. Hence, $\dim Z_0^3(\mathfrak{g}) = 5$.

The subspace $\overline{C}_0^2(\mathfrak{g})$ is 4-dimensional and generated by vectors

$$h_1^* \wedge h_2^*, e_1^* \wedge f_1^*, e_2^* \wedge f_2^*, e_3^* \wedge f_3^*.$$

If $a_1 h_1^* \wedge h_2^* + a_2 e_1^* \wedge f_1^* + a_3 e_2^* \wedge f_2^* + a_4 e_3^* \wedge f_3^* \in \overline{Z}^2(\mathfrak{g})$, then

$$\begin{cases} a_1 = 0, \\ a_4 = a_2 + a_3. \end{cases}$$

Hence, $\dim \overline{Z}_0^2(\mathfrak{g}) = 2$.

According to the formula (3),

$$\dim H_0^3(\mathfrak{g}) = \dim \overline{Z}_0^3(\mathfrak{g}) + \dim \overline{Z}_0^2(\mathfrak{g}, V) - \dim \overline{C}_0^2(\mathfrak{g}, V) = 5 + 2 - 4 = 3.$$

All other weight subspaces $\overline{C}_\mu^3(\mathfrak{g})$, $\mu \in \prod(\overline{C}^3(\mathfrak{g})) \setminus \{0\}$, are one-dimensional, and the corresponding weight subspaces $\overline{C}_\mu^2(\mathfrak{g})$ are zero subspaces in $C^2(\mathfrak{g})$. In addition, according to the cocycle condition, any nonzero element of each of these weight subspaces $\overline{C}_\mu^3(\mathfrak{g})$ is a 3-cocycle. Therefore, $\dim \overline{H}_\mu^3(\mathfrak{g}) = 1$, if $\mu \in \prod(\overline{C}^3(\mathfrak{g})) \setminus \{0\}$.

Thus, the composition factors of $H^3(\mathfrak{g})$ as $SL_3(k)$ -modules are $L(1, 1)^{(1)}$ and k , and the multiplicities of both modules are equal to 1. Therefore, $H^3(\mathfrak{g}) \cong L(1, 1)^{(1)} \oplus k$. The proof of the statement (b) is completed.

(c) Note that the sets of weights of the subspaces $\overline{C}^4(\mathfrak{g})$ and $\overline{C}^3(\mathfrak{g})$ coincide, that is, $\prod(\overline{C}^4(\mathfrak{g})) = \prod(\overline{C}^3(\mathfrak{g}))$. Therefore, the composition factors of $H^3(\mathfrak{g})$ can also be only a twisted simple module $L(1, 1)^{(1)}$ and a one-dimensional trivial module k . Their highest weights are $2(\omega_1 + \omega_2)$ and 0 respectively. Let us consider each of these cases separately.

The subspace $\overline{C}_0^4(\mathfrak{g})$ is 10-dimensional and generated by vectors

$$\begin{aligned} &h_1^* \wedge h_2^* \wedge e_1^* \wedge f_1^*, h_1^* \wedge h_2^* \wedge e_2^* \wedge f_2^*, h_1^* \wedge h_2^* \wedge e_3^* \wedge f_3^*, h_1^* \wedge e_1^* \wedge e_2^* \wedge f_3^*, \\ &h_2^* \wedge e_1^* \wedge e_2^* \wedge f_3^*, h_1^* \wedge e_3^* \wedge f_1^* \wedge f_2^*, h_2^* \wedge e_3^* \wedge f_1^* \wedge f_2^*, \\ &e_1^* \wedge e_2^* \wedge f_1^* \wedge f_2^*, e_1^* \wedge e_3^* \wedge f_1^* \wedge f_3^*, e_2^* \wedge e_3^* \wedge f_2^* \wedge f_3^*. \end{aligned}$$

Suppose that a linear combination of these vectors with coefficients b_i , $i = 1, \dots, 10$, respectively, is a 4-cocycle. Then the cocycle condition implies that $b_1 = b_4 + b_6$, $b_2 = b_5 + b_7$, $b_3 = b_4 + b_5 + b_6 + b_7$. The solution space of this linear system with respect to b_i , $i = 1, \dots, 10$, is seven-dimensional. Hence, $\dim H_0^4(\mathfrak{g}) = 7 + 5 - 8 = 4$.

The subspace $\overline{C}_{2(\omega_1 + \omega_2)}^4(\mathfrak{g}, M)$ is two-dimensional and generated by the cochains

$$\psi_1^4 = h_1^* \wedge f_1^* \wedge f_2^* \wedge f_3^*, \psi_2^4 = h_2^* \wedge f_1^* \wedge f_2^* \wedge f_3^*.$$

Then $\sum_{i=1}^2 a_i \psi_i^4 \in Z^4(\mathfrak{g})$ for all $a_1, a_2 \in k$. Therefore, $\dim Z_{2(\omega_1 + \omega_2)}^4(\mathfrak{g}) = 2$, and, by the formula (3), $\dim H_{2(\omega_1 + \omega_2)}^4(\mathfrak{g}) = 2 + 1 - 1 = 2$. Then

$$[H^4(\mathfrak{g}, M) : L(\omega_1 + \omega_2)^{(1)}] = 2.$$

Thus, $H^4(\mathfrak{g}) \cong 2L(1, 1)^{(1)}$. So, the statement (c) holds.

Using the formula (2) and statement (b), we obtain statement (d). The proof of Lemma 1 is complete. \square

Let $M = L(1, 0) = \langle x_1, x_2, x_3 \rangle_k$.

Lemma 2. *The following isomorphisms of $SL_3(k)$ -modules hold:*

- (a) $H^0(\mathfrak{g}, L(1, 0)) = H^8(\mathfrak{g}, L(1, 0)) = 0$;
- (b) $H^1(\mathfrak{g}, L(1, 0)) \cong L(0, 1)^{(1)}$;
- (c) $H^2(\mathfrak{g}, L(1, 0)) \cong 2L(0, 1)^{(1)} \oplus L(1, 0)^{(2)}$;
- (d) $H^3(\mathfrak{g}, L(1, 0)) \cong L(0, 1)^{(1)} \oplus L(1, 0)^{(2)}$;
- (e) $H^4(\mathfrak{g}, L(1, 0)) = 0$;
- (f) $H^5(\mathfrak{g}, L(1, 0)) \cong L(0, 1)^{(1)} \oplus L(1, 0)^{(2)}$;
- (g) $H^6(\mathfrak{g}, L(1, 0)) \cong 2L(0, 1)^{(1)} \oplus L(1, 0)^{(2)}$;

(h) $H^7(\mathfrak{g}, L(1, 0)) \cong L(0, 1)^{(1)}$.

Otherwise, $H^n(\mathfrak{g}, L(1, 0)) = 0$.

Proof. (a) It's obvious that $\overline{C}^0(\mathfrak{g}, M) = \overline{C}^8(\mathfrak{g}, M) = 0$. Therefore, $H^0(\mathfrak{g}, L(1, 0)) = H^8(\mathfrak{g}, L(1, 0)) = 0$.

(b) The set of all weights of the subspace $\overline{C}^1(\mathfrak{g}, M)$ consists the following three weights: $2\omega_2$, $2(\omega_1 - \omega_2)$ and $-2\omega_1$. Among them, only $2\omega_2$ is dominant. Then, if the space $H^1(\mathfrak{g}, M)$ is nontrivial, then as a $SL_3(k)$ -module it contains only composition factors isomorphic to the twisted simple module $L(0, 1)^{(1)}$, and is generated by the classes of cocycles of the weight $2\omega_2$. The subspace $\overline{C}_{2\omega_1}^1(\mathfrak{g}, M)$ is two-dimensional and generated by the cochains $f_2^* \otimes x_1, f_3^* \otimes x_2$. Suppose that $a_1 f_2^* \otimes x_1 + a_2 f_3^* \otimes x_2 \in Z^1(\mathfrak{g}, M)$, where $a_1, a_2 \in k$. Using the cocycle condition, we obtain $a_1 = a_2$. Therefore, $\dim Z_{2\omega_2}^1(\mathfrak{g}, M) = 1$, and, by the formula (3), $\dim H_{2\omega_2}^1(\mathfrak{g}, M) = 1$. This means that the multiplicity of $L(0, 1)^{(1)}$ in $H^1(\mathfrak{g}, M)$ is equal to 1. So, $H^1(\mathfrak{g}, M) \cong L(0, 1)^{(1)}$.

(c) Since $\prod(\overline{C}^2(\mathfrak{g}, M)) = \{2\omega_2, 2(\omega_1 - \omega_2), -2\omega_1, 4\omega_1, 4(-\omega_1 + \omega_2), -4\omega_2\}$, then composition factors of $H^2(\mathfrak{g}, M)$ can only be twisted simple modules $L(1, 0)^{(1)}$ and $L(4\omega_1) = L(2^2\omega_1) = L(1, 0)^{(2)}$. They are generated by the classes of cocycles with weights $2\omega_2$ and $2^2\omega_1$, respectively. Let us consider each of these cases separately.

The subspace $\overline{C}_{2\omega_2}^2(\mathfrak{g}, M)$ is seven-dimensional and generated by the cochains $\phi_1^2 = h_1^* \wedge f_2^* \otimes x_1, \phi_2^2 = h_2^* \wedge f_2^* \otimes x_1, \phi_3^2 = h_1^* \wedge f_3^* \otimes x_2, \phi_4^2 = h_2^* \wedge f_3^* \otimes x_2, \phi_5^2 = e_1^* \wedge f_3^* \otimes x_1, \phi_6^2 = f_1^* \wedge f_2^* \otimes x_2, \phi_7^2 = f_2^* \wedge f_3^* \otimes x_3$. Suppose that $\sum_{i=1}^7 a_i \phi_i^2 \in Z^2(\mathfrak{g}, M)$, where $a_i \in k$ for all $i = 1, 2, \dots, 7$. Using the cocycle condition, we get $a_1 = a_3, a_2 = a_6 + a_7, a_4 = a_6 + a_7, a_5 = a_3 + a_6$. Therefore $\dim Z_{2\omega_2}^2(\mathfrak{g}, M) = 3$, and, by the formula (3), $\dim H_{2\omega_2}^2(\mathfrak{g}, M) = 3 + 1 - 2 = 2$. So, $[H^2(\mathfrak{g}, M) : L(0, 1)^{(1)}] = 2$.

It's obvious that $\overline{C}_{4\omega_1}^1(\mathfrak{g}, M) = 0$. The subspace $\overline{C}_{4\omega_1}^2(\mathfrak{g}, M)$ is one-dimensional and generated by the cochain $f_1^* \wedge f_3^* \otimes x_1$. It is easy to see that $a f_1^* \wedge f_3^* \otimes x_1 \in Z^2(\mathfrak{g}, M)$ for all $a \in k$. Therefore $\dim Z_{4\omega_1}^2(\mathfrak{g}, M) = 1$, and, by (3), $\dim H_{4\omega_1}^2(\mathfrak{g}, M) = 1$. So, $[H^2(\mathfrak{g}, M) : L(1, 0)^{(2)}] = 1$.

Thus, $H^2(\mathfrak{g}, M) \cong 2L(0, 1)^{(1)} \oplus L(1, 0)^{(2)}$.

(d) Note that $\prod(\overline{C}^3(\mathfrak{g}, M)) = \prod(\overline{C}^2(\mathfrak{g}, M))$. Therefore, composition factors of $H^3(\mathfrak{g}, M)$ can also be only twisted simple modules $L(0, 1)^{(1)}$ and $L(1, 0)^{(2)}$. Their highest weights are $2\omega_2$ and $2^2\omega_1$, respectively. Consider each of these cases separately.

The subspace $\overline{C}_{2\omega_2}^3(\mathfrak{g}, M)$ generated by the cochains

$$\begin{aligned} \phi_1^3 &= h_1^* \wedge h_2^* \wedge f_2^* \otimes x_1, \phi_2^3 = h_1^* \wedge h_2^* \wedge f_3^* \otimes x_2, \phi_3^3 = h_1^* \wedge e_1^* \wedge f_3^* \otimes x_1, \\ \phi_4^3 &= h_2^* \wedge e_1^* \wedge f_3^* \otimes x_1, \phi_5^3 = h_1^* \wedge f_1^* \wedge f_2^* \otimes x_2, \phi_6^3 = h_2^* \wedge f_1^* \wedge f_2^* \otimes x_2, \\ \phi_7^3 &= h_1^* \wedge f_2^* \wedge f_3^* \otimes x_3, \phi_8^3 = h_2^* \wedge f_2^* \wedge f_3^* \otimes x_3, \phi_9^3 = e_1^* \wedge f_1^* \wedge f_2^* \otimes x_1, \\ \phi_{10}^3 &= e_3^* \wedge f_2^* \wedge f_3^* \otimes x_1, \phi_{11}^3 = e_2^* \wedge f_2^* \wedge f_3^* \otimes x_2, \phi_{12}^3 = e_1^* \wedge f_1^* \wedge f_3^* \otimes x_2. \end{aligned}$$

Let $\sum_{i=1}^{12} a_i \phi_i^3 \in Z^3(\mathfrak{g}, M)$, where $a_i \in k$ for all $i = 1, 2, \dots, 12$. Using the cocycle condition, we obtain $a_1 = a_5 + a_7, a_2 = a_8, a_3 = a_5, a_4 = a_5 + a_7 + a_8, a_6 = a_5 + a_7, a_9 = a_7 + a_{12}, a_{10} = a_7 + a_{11} + a_{12}$. From this we get $\dim Z_{2\omega_2}^3(\mathfrak{g}, M) = 5$, and, by (3), $\dim H_{2\omega_2}^3(\mathfrak{g}, M) = 5 + 3 - 7 = 1$. Therefore, $[H^3(\mathfrak{g}, M) : L(0, 1)^{(1)}] = 1$.

The subspace $\overline{C}_{4\omega_1}^3(\mathfrak{g}, M)$ is two-dimensional and generated by the cochains. Then from the cocycle condition $\sum_{i=1}^2 a_i \psi_i^3 \in Z^3(\mathfrak{g}, M)$ it follows that $a_2 = 0$. Therefore, $\dim Z_{4\omega_1}^3(\mathfrak{g}, M) = 1$, and, by the formula (3), $\dim H_{4\omega_1}^3(\mathfrak{g}, M) = 1 + 1 - 1 = 1$. Then $[H^3(\mathfrak{g}, M) : L(1, 0)^{(2)}] = 1$.

Thus, $H^2(\mathfrak{g}, M) \cong L(0, 1)^{(1)} \oplus L(1, 0)^{(2)}$.

(e) The set of weights of the subspace $\overline{C}^4(\mathfrak{g}, M)$ coincides with the set of weights of the subspace $\overline{C}^3(\mathfrak{g}, M)$. Therefore, the composition factors $H^3(\mathfrak{g}, M)$ can only be twisted simple modules $L(0, 1)^{(1)}$ and $L(1, 0)^{(2)}$. Let us consider each of these cases separately.

The subspace $\overline{C}_{2\omega_2}^4(\mathfrak{g}, M)$ generated by the cochains

$$\begin{aligned} \phi_1^4 &= h_1^* \wedge h_2^* \wedge e_1^* \wedge f_3^* \otimes x_1, \phi_2^4 = h_1^* \wedge h_2^* \wedge f_1^* \wedge f_2^* \otimes x_2, \phi_3^4 = h_1^* \wedge h_2^* \wedge f_2^* \wedge f_3^* \otimes x_3, \\ \phi_4^4 &= h_1^* \wedge e_1^* \wedge f_1^* \wedge f_2^* \otimes x_1, \phi_5^4 = h_2^* \wedge e_1^* \wedge f_1^* \wedge f_3^* \otimes x_1, \phi_6^4 = h_1^* \wedge e_3^* \wedge f_2^* \wedge f_3^* \otimes x_1, \\ \phi_7^4 &= h_2^* \wedge e_3^* \wedge f_2^* \wedge f_3^* \otimes x_1, \phi_8^4 = h_1^* \wedge e_1^* \wedge f_1^* \wedge f_3^* \otimes x_2, \phi_9^4 = h_2^* \wedge e_1^* \wedge f_1^* \wedge f_3^* \otimes x_2, \\ \phi_{10}^4 &= h_1^* \wedge e_2^* \wedge f_2^* \wedge f_3^* \otimes x_2, \phi_{11}^4 = h_2^* \wedge e_2^* \wedge f_2^* \wedge f_3^* \otimes x_2, \phi_{12}^4 = e_1^* \wedge e_2^* \wedge f_2^* \wedge f_3^* \otimes x_1, \\ \phi_{13}^4 &= e_3^* \wedge f_1^* \wedge f_2^* \wedge f_3^* \otimes x_2, \phi_{14}^4 = e_1^* \wedge f_1^* \wedge f_2^* \wedge f_3^* \otimes x_3. \end{aligned}$$

Suppose that $\sum_{i=1}^{14} a_i \phi_i^4 \in Z^4(\mathfrak{g}, M)$, where $a_i \in k$ for all $i = 1, 2, \dots, 14$. Using the cocycle condition, we get $a_1 = a_3, a_2 = a_3, a_4 = a_8, a_6 = a_3 + a_8 + a_{10}, a_7 = a_5 + a_{11}, a_9 = a_3 + a_5, a_{13} = a_3 + a_5 + a_{10} + a_{12} + a_{14}$. From this we get $\dim Z_{2\omega_2}^4(\mathfrak{g}, M) = 7$, and, by formula (3), $\dim H_{2\omega_2}^4(\mathfrak{g}, M) = 7 + 5 - 12 = 0$. Therefore, $[H^3(\mathfrak{g}, M) : L(0, 1)^{(1)}] = 0$.

The subspace $\overline{C}_{4\omega_1}^4(\mathfrak{g}, M)$ is two-dimensional and generated by the cochains $\psi_1^4 = h_1^* \wedge h_2^* \wedge f_1^* \wedge f_3^* \otimes x_1, \psi_2^4 = e_2^* \wedge f_1^* \wedge f_2^* \wedge f_3^* \otimes x_1$. Then from the cocycle condition $\sum_{i=1}^2 a_i \psi_i^4 \in Z^4(\mathfrak{g}, M)$ it follows that $a_1 = 0$. Therefore, $\dim Z_{4\omega_1}^4(\mathfrak{g}, M) = 1$, and, by (3), $\dim H_{4\omega_1}^4(\mathfrak{g}, M) = 1 + 1 - 2 = 0$. Then $[H^3(\mathfrak{g}, M) : L(1, 0)^{(2)}] = 0$. Thus, $H^4(\mathfrak{g}, M) = 0$.

Other statements are proved similarly to the previous ones. The results of calculation are shown in Tables 1 and 2.

Table 1. Dimension table for $M = L(1, 0)$ and $\mu = 2^2\omega_2$

| n | $\dim H_\mu^n(\mathfrak{g}, M)$ | $\dim \overline{Z}_\mu^n(\mathfrak{g}, M)$ | $\dim \overline{Z}_\mu^{n-1}(\mathfrak{g}, M)$ | $\dim \overline{C}_\mu^{n-1}(\mathfrak{g}, M)$ |
|-----|---------------------------------|--|--|--|
| 1 | 1 | 1 | 0 | 0 |
| 2 | 2 | 3 | 1 | 2 |
| 3 | 1 | 5 | 3 | 7 |
| 4 | 0 | 7 | 5 | 12 |
| 5 | 1 | 8 | 7 | 14 |
| 6 | 2 | 6 | 8 | 12 |
| 7 | 1 | 2 | 6 | 7 |

Table 2. Dimension table for $M = L(1, 0)$ and $\mu = 2^2\omega_1$

| n | $\dim H_\mu^n(\mathfrak{g}, M)$ | $\dim \overline{Z}_\mu^n(\mathfrak{g}, M)$ | $\dim \overline{Z}_\mu^{n-1}(\mathfrak{g}, M)$ | $\dim \overline{C}_\mu^{n-1}(\mathfrak{g}, M)$ |
|-----|---------------------------------|--|--|--|
| 1 | 0 | 0 | 0 | 0 |
| 2 | 1 | 1 | 0 | 0 |
| 3 | 1 | 1 | 1 | 1 |
| 4 | 0 | 1 | 1 | 2 |
| 5 | 1 | 2 | 1 | 2 |
| 6 | 1 | 1 | 2 | 2 |
| 7 | 0 | 0 | 1 | 1 |

The proof of Lemma 2 is complete. \square

Using the duality of the modules $L(1, 0)$, $L(0, 1)$, the formula (2), and Lemma 2, we obtain the following result for the simple module $M = L(0, 1)$:

Lemma 3. *The following isomorphisms of $SL_3(k)$ -modules hold:*

- (a) $H^1(\mathfrak{g}, L(0, 1)) \cong L(1, 0)^{(1)}$;
- (b) $H^2(\mathfrak{g}, L(0, 1)) \cong 2L(1, 0)^{(1)} \oplus L(0, 1)^{(2)}$;
- (c) $H^3(\mathfrak{g}, L(0, 1)) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(2)}$;
- (d) $H^5(\mathfrak{g}, L(0, 1)) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(2)}$;
- (e) $H^6(\mathfrak{g}, L(0, 1)) \cong 2L(1, 0)^{(1)} \oplus L(0, 1)^{(2)}$;
- (f) $H^7(\mathfrak{g}, L(0, 1)) \cong L(1, 0)^{(1)}$.

Otherwise, $H^n(\mathfrak{g}, L(0, 1)) = 0$.

Combining the results of Lemmas 1 – 3, we obtain all the statements of Theorem 1. \square

3. COHOMOLOGY FOR $\mathfrak{gl}_3(k)$ AND $W_3(\mathbf{1})$

In this section, we consider some applications of Theorem 1. We calculate the cohomology of simple modules for $\mathfrak{gl}_3(k)$ and the cohomology of $W_3(\mathbf{1})$ with coefficients in $O_3(\mathbf{1})$ over an algebraically closed field of characteristic $p = 2$.

3.1. Cohomology for $\mathfrak{gl}_3(k)$. This subsection is devoted to the proof of Theorem 2. In the next proposition we give the connections between the cohomology of simple modules of Lie algebras $\mathfrak{sl}_3(k)$ and $\mathfrak{gl}_3(k)$.

Proposition 2. *Let $\mathfrak{g} = \mathfrak{gl}_3(k)$ be the general linear Lie algebra over an algebraically closed field k of characteristics $p = 2$ and M is a $\mathfrak{gl}_3(k)$ -module defined by the formula (4). Then, if $\mu = 0$, then*

$$(5) \quad H^n(\mathfrak{gl}_3(k), M) \cong H^n(\mathfrak{sl}_3(k), M) \oplus H^{n-1}(\mathfrak{sl}_3(k), M),$$

otherwise $H^n(\mathfrak{gl}_3(k), M) = 0$.

Proof. Since I is an ideal of the Lie algebra $\mathfrak{gl}_3(k)$, then we can use the Serre-Hochschild spectral sequence $\{E_r^{l,q}\}$. In particular, for the cohomology $H^m(\mathfrak{gl}_n(k), M)$ we get

$$E_2^{l,q} = H^l(\mathfrak{gl}_n(k)/I, H^q(I, M)) \cong H^l(\mathfrak{sl}_n(k), H^q(I, M)).$$

Suppose that $\mu = 0$, then $H^0(I, M) \cong H^1(I, M) \cong M$, and $H^q(I, M) = 0$, if $q \geq 2$. Therefore $E_2^{lq} = 0$, if $q \geq 2$. Let $l + q = n$, then

$$E_2^{n,0} \cong H^n(\mathfrak{sl}_3(k), M), E_2^{n-1,1} \cong H^{n-1}(\mathfrak{sl}_3(k), M)$$

and $E_2^{lq} = 0$ in other integer points of the first quadrant. Therefore,

$$H^n(\mathfrak{gl}_3(k), M) \cong E_2^{n,0} \oplus E_2^{n-1,1} \cong H^n(\mathfrak{sl}_3(k), M) \oplus H^{n-1}(\mathfrak{sl}_3(k), V).$$

If $\mu \neq 0$, then $H^q(I, M) = 0$ for all $q \geq 0$. Then for all integer points of the first quadrant $E_2^{lq} = 0$. So, in this case,

$$H^n(\mathfrak{gl}_3(k), M) = 0.$$

□

Proof of Theorem 2. Follows from Proposition 2 and Theorem 1.

3.2. Cohomology for $W_3(\mathbf{1})$. In this subsection we prove Proposition 1.

Proof. According to Theorem 0.2 in [3], for any $n \in \mathbb{Z}_0$

$$H^n(W_3(\mathbf{1}), O_3(\mathbf{1})) \cong \bigoplus_{i=0}^n (\bigwedge^i(3k) \otimes_k H^{n-i}(\mathfrak{gl}_3(k))).$$

Since $\bigwedge^0(3k) = \bigwedge^3(3k) \cong k$, $\bigwedge^1(3k) = \bigwedge^2(3k) \cong 3k$, and $\bigwedge^i(3k) = 0$ in all other cases, then

$$H^n(W_3(\mathbf{1}), O_3(\mathbf{1})) \cong H^n(\mathfrak{gl}_3(k)) \oplus 3H^{n-1}(\mathfrak{gl}_3(k)) \oplus 3H^{n-2}(\mathfrak{gl}_3(k)) \oplus H^{n-3}(\mathfrak{gl}_3(k), k).$$

Then Proposition 1 follows from the last formula and Theorem 2. □

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