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DESCRIPTION OF MODAL LOGICS WHICH ENJOY CO-COVER PROPERTY

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The abstract. Here we use admissible rules to determine whenever modal logic satisfies weak co-cover property. We prove that logic λ over $S4$ satisfies such property iff the given set of rules are admissible in λ.

Keywords: modal logic, inference rule, Kripke frame and model, admissible rule.

1. Introduction

For modern application of logic in Computer Science and Artificial Intelligence it is often required that the capable language of representing the knowledge about dynamic systems. Distinct non-standard logics (e.g. modal and temporal, logic for multi-agent reasoning) efficiently serve these applications. Firstly they describe a statements by formulas which are peculiar to studied a models in general, and do not take to consideration a variable conditions and a changing assumptions. These conditions and assumptions can be modeled by distinct variations of the notion of logical consequence. The problem of such adequate modeling is one of extremely important problems originated from mathematical logic and mathematics in general. Most important component of our approach consists of the fact that we study logical consequence in terms of inference rules, clauses, but not only the formulas or statements. The formalism concerning description of properties by formulas is well-developed, widely spread and well represented in a scientific literature. It is a representing basis of human reasoning. But formulas describe only a stable, static events; the statement only fix the fact, and isn't able to handle a changing conditions.

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Therefore a studying of (structural) inference rules (or sequents), expressions which have a premise - a given collection of assumptions - and a conclusion, brings us more flexibility and more expressive power to model human reasoning and computing. Premises gives us a current informations collected as assumptions and conclusions represent a knowledge, i.e. facts which we can obtain from our assumptions. The intelligence of reasoning (as a part of AI) again requires the understanding of what are the consistent consequences of observable facts. Within Computer Science these aspects are involved in the analysis of correct instructions for computations, verification of programs and many other areas. Such rules allow us to model a standard situation in a study of logical consequence: given certain assumptions, what does follow from them?

Evidently the notion of inference rule generalizes the notion of formulas, and any formula can be viewed as a structural inference rule without premise and assumption. But admissible inference rules are actually stronger then the clauses, since Harrop (1960, [2]) we know that even the intuitionistic logic H is not structurally complete: it has the admissible inference rules which cannot be represented by formulas, which are not inferred. The same is valid for a broad range of basic modal logics since examples of G. Mints [3] and J. Port [4].

Notion of a admissible inference rule goes back to Lorenzen ([5], 1955). For arbitrary logic admissible rules are exactly those, under which the logic is closed. Clearly any derivable rule is admissible, but, in general, not vice versa. Also, directly from the definition, we see that the set of all rules admissible in a logic λ is the *greatest class* of inference rules by which we can extend an axiomatic system of the logic λ preserving the theorems of λ. Derivable rules may replace some fragments of the fixed length in derivations, thereby linearly shortening them. Admissible rules, which are not derivable, basically may reduce a derivations even more drastically.

The history of studying of an admissible rules could be dated since H. Fridman's question ([6], 1975) about an existence of an algorithm which could distinguish rules admissible in intuitionistic logic H. In the middle of 70-th G. Mints [3] found the strong sufficient conditions for derivability in H admissible rules in special form. The Fridman's question about the existence of an algorithm recognizing admissibility of inference rule was answered affirmatively by V. Rybakov (1984, cf. [7]) for the intuitionistic logic H and a broad class of modal logics (e.g. $K4$, $S4$, GL, cf. [1]).

In 2000-2010 a few results on describing of explicit bases for admissible inference rules for nonstandard logics (S4, K4, H etc.) appeared (see for example [9]). The key condition of these results was weak co-cover property. Possible, weak co-cover property, disjunction property and FMP of logic allow us to describe such basis. Beside the improvement of deductive power in logic, an admissible rule are able to describe some semantic property of given logic. One of the first attempt was the description of intuitionistic logic H by the set of admissible inference rules (cf. R. Iemhoff [8]). That's why this property is interest of article. Here we use admissible rules to determine whenever given modal logic satisfy weak co-cover property. We prove that FMP logic λ over $S4$ satisfy such property iff the given set of rules are admissible in λ.

2. Denotation, Preliminary Facts

We assume the reader to be aware of the algebraic and Kripke semantics for modal logics and to have a certain initial knowledge concerning the basic facts on inference rules and their admissibility (though we briefly recall all necessary facts below). As a good entry point to the subject we would recommend among modern literature Rybakov [1] for a general technique and for advanced technique concerning modal logics and inference rules. According to modern trends by a *logic* we understand the set of all theorems provable in a given axiomatic system, or the set of valid formulas for a certain class of Kripke frames. In particular, a normal modal logic λ is a set of modal formulas which is closed under substitution, modus ponens and necessitation rule $A \;/\; \Box A$, and including all theorems of the minimal propositional modal logic K. In the following definitions and results we mean an algebraic propositional logic extending logic $S4$ with finite model property by a modal logic (cf. [1]).

A frame $\mathcal{F} := \langle F, R\rangle$ is a pair, where F is a nonempty set and R is a binary relation on F. The basis set of a frame and a frame itself are often denoted by the same letter for simplicity. Further we consider only frames where R is transitive and reflexive relation.

A model is a triple $\mathcal{M} = \langle W,\ R,\ V\rangle$, where $\mathcal{F} := \langle F, R\rangle$ is a frame and V is a valuation of a set of propositional letters P in the frame $\mathcal{F}$ that is $V : P \to 2^W$. $Dom(V) = P$ is called the domain of V.

A frame $\mathcal{F} = \langle F,\ R\rangle$ is called open subframe of frame $\mathcal{G} = \langle G,\ R\rangle$ (denote $\mathcal{F} \sqsubseteq \mathcal{G}$) if $F \subseteq G$ and $\forall a \in F\, \forall b \in G\, (aRb \Longrightarrow b \in F)$ holds. If $\mathcal{M}_1 = \langle W_1,\ R_1,\ V_1\rangle$, $\mathcal{M}_2 = \langle W_2,\ R_2,\ V_2\rangle$ are models then we call $\mathcal{M}_1$ an open submodel of $\mathcal{M}_2$ (denote $\mathcal{M}_1 \sqsubseteq \mathcal{M}_2$) if : 1) $\langle W_1,\ R_1\rangle$ is open subframe of $\langle W_2,\ R_2\rangle$; 2) $Dom(V_1) = Dom(V_2)$ and $\forall p \in Dom(V_1)\ V_1(p) = V_2(p) \cap W_1$.

A mapping $f : \langle F,\ R\rangle \to \langle G,\ S\rangle$ is called p-morphism if (1) $aRb \Longrightarrow f(a)Sf(b)$; (2) $f(x)Sz \Longrightarrow \exists y \in F : f(y) = z \,\&\, xRy$.

We say a mapping $f : \mathcal{M}_1 = \langle W_1,\ R_1,\ V_1\rangle \to \mathcal{M}_2 = \langle W_2,\ R_2,\ V_2\rangle$ is a p-morfism of the model $\mathcal{M}_1$ into the $\mathcal{M}_2$ if 1) f is a p-morfism of the frame $\mathcal{F}_1 = \langle W_1,\ R_1\rangle$ into the frame $\mathcal{F}_2 = \langle W_2,\ R_2\rangle$; 2) the valuations $V_1,\ V_2$ are defined on the same set of propositional letters; 3) $\forall p \in Dom(V_1), \forall a \in W_1 (a \models_{V_1} p \iff f(a) \models_{V_2} p)$.

The primary property of open submodels and p-morfisms consist of the fact that they preserve the truth of formulas:

Proposition 1. *(cf.* [1]*) 1) If $\mathcal{M}_1$ is an open submodel of a model $\mathcal{M}_2$ then for every formula α, $\mathcal{M}_2 \models \alpha$ implies $\mathcal{M}_1 \models \alpha$;*

2) if f is a p-morfism of model $\mathcal{M}_1 = \langle W_1,\ R_1,\ V_1\rangle$ onto model $\mathcal{M}_2 = \langle W_2,\ R_2,\ V_2\rangle$ then for any formula α which is built out of letters from the domain $Dom(V_1)$, then $\forall a \in W_1 (a \models_{V_1} \alpha \iff f(a) \models_{V_2} \alpha)$.

Let $\mathcal{F}_i = \langle W_i,\ R_i\rangle,\ i \in I$ be a family of pairwise disjoint frames, i.e. $W_i \cup W_j = \emptyset$ for $i \neq j \in I$. The disjoin union of this family is the frame $\sqcup_{i\in I}\mathcal{F}_i = \langle W,\ R\rangle$, where $W = \cup_{i\in I} W_i,\ R = \cup_{i\in I} R_i$. Disjoint union of models is defined analogously.

By Lemma 2.5.26 [1] disjoint union of frames (or models) preserves the truth of formulas: $\sqcup_{i\in I}\mathcal{F}_i \models \alpha \iff \forall i(\mathcal{F}_i \models \alpha)$. A disjoint union of λ-frames is λ-frame.

Any subset C of a frame F, which either a single irreflexive point or a set C satisfying the following properties: (1) $\forall x, y \in C(xRy \,\&\, yRx)$; (2) $\forall x \in C \forall y \in W(xRy \,\&\, yRx \Longrightarrow y \in C)$, is called a *cluster C of the frame F*. A cluster is *proper* if $|C| > 1$, otherwise that cluster is called degenerated, singleton. For any element $a \in F$ a cluster generated by a is denoted by $C(a)$. Any set of clusters of F which are non-compared by R is called an antichain. The antichain $\mathcal{A}$ is nontrivial if it consists of two clusters at least; otherwise $\mathcal{A}$ is trivial.

We call the maximal number of clusters in chains of clusters, generated-up by an element (or a cluster) the *depth of an element (or a cluster)*. For any transitive frame F (or a transitive Kripke model M) *n-slice* $S_n(F)$ ($S_n(M)$) is the set of all elements of depth n from F (M respectively). And $S_{\leq n}(F)$ is the set of all elements from F with a depth not more than n.

We say a frame F is an λ–frame for a logic λ if all theorems of λ are valid at F, and $\lambda(F)$ – the set of all formulas valid in F – is the logic generated by F. A frame F is rooted if $\exists\, a \in F$ such that $\forall\, b \in F\ aRb$. Then we say $C(a)$ is the root of F.

We put $b^R := \{x | \exists y \in C(b) : yRx\} \cup C(b)$; $b^{<R} := \{x | \exists y \in C(b) : yRx \wedge \neg xRy\}$, where $C(b)$ is a cluster of a frame F containing the element b. For any subset $X \subseteq F$ X^R is $\cup\{x^R | x \in X\}$. That is subframe b^R of F is upwards cone generated by b, and X^R is subframe of F generated by X respectively. An element $b \in F$ or cluster $C(b)$ is *a co-cover* for a set $X \subseteq F$, if $b^R \backslash C(b) = X^R$. We understand a cluster C as λ-co-cover for X if it generate λ-frame $C^R := X^R \cup \{C\}$ as the root. We'll identify one-element cluster $C(a)$ and reflexive element a (generating this cluster) as co-cover since the set of R-accessible elements is the same for both.

A subset $\mathcal{X}$ of the given model $\mathcal{M}$ is definable if there is a formula α such $\forall\, x \in \mathcal{M}\, [\, x \in \mathcal{X} \Longleftrightarrow x \models_V \alpha]$. And a valuation V is definable in a model $\mathcal{M}$ if for any letter p, the set $V(p)$ is definable.

Let $\alpha_1, ..., \alpha_n, \beta$ to be some formulas. We understand the figure r, where

$$r := \frac{\alpha_1(x_1, \ldots, x_n), ..., \alpha_n(x_1, \ldots, x_n)}{\beta(x_1, \ldots, x_n)},$$

is the (structural) inference rule, which derives $s(\beta)$ from $s(\alpha_1), ..., s(\alpha_n)$ for every substitution s. We say r is *derivable* in a logic λ if there is a derivation β in λ from the set of assumptions $\{\alpha_1, ..., \alpha_n\}$.

An *inference rule* $r = \{\alpha_1(x_1, \ldots, x_n), \ldots, \alpha_k(x_1, \ldots, x_n)/\beta(x_1, \ldots, x_n)\}$ is called *admissible in a logic* λ, if for any formulas $\delta_1, \ldots, \delta_n$ the assertion

$$((\alpha_1(\delta_1, \ldots, \delta_n) \in \lambda) \cdots (\alpha_k(\delta_1, \ldots, \delta_n) \in \lambda) \implies \beta(\delta_1, \ldots, \delta_n) \in \lambda)$$

holds.

The admissible inference rules have the following algebraic description

Proposition 2. [1] *An inference rule $r = \{\alpha_1, \ldots, \alpha_k/\beta\}$ is admissible in a logic λ iff quasi-identity $r^* = \{\alpha_1 = 1\ \& \ldots \&\ \alpha_k = 1 \Longrightarrow \beta = 1\}$ is true on free algebra of countable rank $\mathcal{F}_w(\lambda)$ from the variety $Var(\lambda)$ generated by λ.*

Any derivable rule is admissible, but not obligatory conversely. Also directly from the definition we see that the set of all rules admissible in a logic λ is the *greatest* class of inference rules by which we can extend axiomatic system of the logic λ preserving the theorems of λ. Derivable rules may replace some fragments of the fixed length in derivations, thereby shortening them linearly. Admissible rules, which are not derivable, in principle may reduce derivations even more drastically.

For a given frame $\mathcal{F}$, valuation V and inference rule $r := \alpha_1, ..., \alpha_n/\beta$, we say r *is valid on $\mathcal{F}$ wrt V*, and write $\mathcal{F} \models_V r$, if as soon as $\forall x \in \mathcal{F}$ and $\forall i\ (x \models_V \alpha_i)$ holds, we have $\forall x \in \mathcal{F}(x \models_V \beta)$. A rule r *is valid on a frame* $\mathcal{F}$ if r is valid at $\mathcal{F}$ under any valuation, we write then $\mathcal{F} \models r$.

Given logic λ has got finite model property (FMP for short), if for arbitrary formula α holds: if $\alpha \notin \lambda$ than finite λ-model refuting α exists.

A Kripke Model $\langle F, R, V\rangle$, where $V : P_n \to 2^F$, $P_n = \{p_1, p_2, \dots, p_n\}$, is *$n$-characteristic for a logic λ* iff for any formula α which is built up on $p_1, \dots, p_n$ formula $\alpha \in \lambda$ iff $\langle F, R, V\rangle \models \alpha$.

The admissibility of inference rules in modal logic λ over $S4$ can be described through their validness in certain special n-characteristic Kripke models. The description of these models $Ch_n(\lambda)$ and criteria for recognizing admissibility in λ by means of them are given in [1], for instance. As we will strongly occupy these techniques in the sequel, we briefly recall the construction of $Ch_n(\lambda)$ for FMP logic λ over $S4$ and the semantic criterion for recognizing admissibility.

Given a FMP logic λ over $S4$, a set $P_n := \{p_1, ..., p_n\}$ of propositional letters, we construct the first slice $S_1(Ch_n(\lambda))$ as that follows. It consists of the collection of all clusters with all possible valuations V of letters from P_n which does not have doubling – clusters with the same valuation, and clusters which are isomorphic as Kripke models.

Assuming $S_{\leq m}(Ch_n(\lambda))$ to be constructed, we put in $S_{m+1}(Ch_n(\lambda))$ the clusters as that follows. We take an arbitrary antichain $\mathcal{X}$ of clusters from $S_{\leq m}(Ch_n(\lambda))$ having one cluster of depth m at least and put all clusters C from $S_1(Ch_n(\lambda))$ in $S_{m+1}(Ch_n(\lambda))$, assuming any C to be an immediate predecessor for all elements from $\mathcal{X}$ (co-cover for $\mathcal{X}$), such that:

(i) $C^R = \mathcal{X}^R \cup C$ is a λ–frame and

(ii) if $\mathcal{X} := \{C_1\}$ then C is not a Kripke submodel of C_1.

Iterating this procedure we get the model $Ch_n(\lambda)$ as the result. To recall a model $\mathcal{M}$ is n-characteristic for a logic λ if, for any formula α, which is built up out of letters from P_n, $\alpha \in \lambda$ iff $\mathcal{M} \models \alpha$. We need the following facts:

Theorem 3. *(cf. [1]) For any FMP logic λ over $S4$ the model $Ch_n(\lambda)$ is n-characteristic for λ.*

Theorem 4. *(cf. [1]) For any inference rule r, r is admissible in FMP logic λ over $S4$ iff r is valid in the frame of $Ch_n(\lambda)$ wrt any definable valuation for any given n.*

3. Main results

We say that a logic λ, extending logic $S4$, has *weak co-cover property* (WCP for short) whenever for every finite rooted λ-frame $\mathcal{F}$ and an arbitrary nontrivial antichain $\mathcal{X}$ of clusters from $\mathcal{F}$, the frame $\mathcal{F}_1$ which is result of adjoining a singleton reflexive co-covering as the root to the frame $\bigcup_{c\in\mathcal{X}^R} c^R$ is a λ-frame as well.

Given $n \in N$ with $n > 1$, define the formulas:

$$\pi_i := p_i \wedge \bigwedge_{j\neq i} \neg p_j;\ 1 \leq i, j \leq n, \quad A_n := \bigwedge_{1\leq i\leq n} \Diamond \pi_i;$$

$$A_{n,1} := \Box\Big[\bigwedge_{1\leq i\leq n} (p_i \to \neg\Diamond q)\Big];\quad B := q \vee \neg\Diamond q.$$

Also we define the rules:

$$\mathcal{R}_n := \frac{\Box\big(A_{n,1} \wedge \neg(A_n \wedge B)\big)}{\Box\neg A_n};\ n = 2, 3, \dots$$

Note, these rules are special case of those from [9] which gives an explicit basis for admissible rules of logic S4. Next theorem is almost the same as in Lemma 3.1

[9]. The WCP of logic is a key condition in proof of this statement. Let's reproduce common part of that proof.

Theorem 1. *The rules $\mathcal{R}_n,\ n > 1$, are admissible in every FMP logic λ over $S4$ that enjoys the weak co-cover property.*

Proof Assume not. Let for some n the rule

$$\mathcal{R}_n := \frac{\Box\big(A_{n,1} \wedge \neg(A_n \wedge B)\big)}{\Box\neg A_n}$$

is not admissible in λ. Hence there is a definable valuation V of variables from $\mathcal{R}_n$ in a certain constructive k–characteristic model $Ch_\lambda(k)$. Therefore

$$Ch_\lambda(k) \models_V \Box\big(A_{n,1} \wedge \neg(A_n \wedge B)\big) \ \&\ Ch_\lambda(k) \not\models_V \Box\neg A_n. \qquad (\mathbf{1})$$

Consequently there exists element $a \in Ch_\lambda(k)$ such that $a \not\models_V \Box\neg A_n$. Then there are elements $b_1, \ldots, b_n \in Ch_\lambda(k)$ such that aRb_i & $b_i \models_V p_i$. By the weak co-cover property there exists a reflexive element $b \in Ch_\lambda(k)$ which is a co-cover for the set of R-minimal clusters from the set $\{C(b_1), \ldots, C(b_n)\}$, that is:

$$\{b\}^R := \{b\} \cup \bigcup_{1\le i\le n} (b_i)^R.$$

By $(\mathbf{1})$ it follows that $b \models_V A_{n,1}$ and $b \models_V A_n$. Since b is a co-cover for $\{b_1, \ldots, b_n\}$ that's clear that $b \models_V B$. Indeed $b_i \models_V p_i$ and $b \models_V A_{n,1}$ holds, therefore $\forall i \le n\ b_i \models_V \neg\Diamond q$. From this we conclude $b \models_V q$ or $b \models_V \neg q$ and hence $b \models_V \neg\Diamond q$. Therefore we obtain $b \models_V A_n \wedge B$ which contradicts to $b \models_V \Box\neg(A_n \wedge B)$ by the assumption $(\mathbf{1})$. ■

Theorem 2. *If $\forall n$ the rules $\{\mathcal{R}_n, n > 1\}$ are admissible in FMP logic λ, over $S4$ then logic λ enjoys weak co-cover property.*

Proof. Let's suppose all rules $\mathcal{R}_n,\ n > 1, n \in N$, which are admissible in FMP logic λ over $S4$, but λ doesn't enjoy weak co-cover property. By definition there exists finite rooted λ-frame $G = b^R$ and nontrivial antichain of clusters $\mathcal{X} \subset G$ such that frame $\varepsilon^R := \bigcup_{c\in\mathcal{X}^R} c^R \cup \{\varepsilon\}$ which is obtained by adjoining a singleton reflexive co-cover ε as a root to the frame $\bigcup_{c\in\mathcal{X}^R} c^R$, is not λ-frame. We'll prove that in such case at least one rule $\mathcal{R}_n$ is not admissible in the logic λ. To do this we construct λ-frame $\mathcal{M}$ containing the frame G as open subframe and refuting $\mathcal{R}_n,\ n > 1$, under some valuation. Then we define p-morphism from the frame of k-characteristic model $Ch_k(\lambda)$ for some k on $\mathcal{M}$. Transferring this valuation from $\mathcal{M}$ onto $Ch_k(\lambda)$ we refute this rule $\mathcal{R}_n,\ n > 1$, on $Ch_k(\lambda)$ under some valuation that contradict to admissibility of $\mathcal{R}_n$.

Let's take a frame $G \sqcup \{e\}$ where $\{e\}$ is reflexive singleton which is not R-comparable to any element in G. The frame $\{e\}$ is λ-frame as p-morfic image of G. So the frame $G \sqcup \{e\}$ is λ-frame as disjoint union of λ-frames.

We define λ-successor $\mathcal{M}$ of $G \sqcup \{e\}$ as follows. Let's fix the nontrivial antichaine $\mathcal{X}$ from G and define λ-frame $\mathcal{M}_0 = G \sqcup \{e\}$. Then we choose all nontrivial antichaines of clusters $\{X_t \subseteq S_1(\mathcal{M}_0)\}$ which don't have singleton co-cover in $\mathcal{M}_0$ and add such co-cover t whenever it generates λ-frame $t^R = \{t\} \cup X_t^R$ as a root. So, we obtain $\mathcal{M}_1$. Notice by construction $\mathcal{M}_1$ is λ-frame (we add elements generating λ-frame as a root to λ-frame $G \sqcup \{e\}$) and $G \sqsubseteq G \sqcup \{e\} \sqsubseteq \mathcal{M}_1$ as we add new elements as co-covers to $G \sqcup \{e\}$.

Assuming that $\mathcal{M}_k$ has already been constructed and $G \sqcup \{e\} \sqsubseteq \mathcal{M}_k$, we obtain $\mathcal{M}_{k+1}$ as follows: we choose in $S_{\leq(k+1)}(\mathcal{M}_k)$ all nontrivial antichaines $\{X_t \subseteq S_{\leq(k+1)}(\mathcal{M}_k)\}$ which don't have singleton co-cover and have at least one cluster of depth $k+1$. Then we add singleton reflexive element t as co-cover to each chosen antichain whenever it generates λ-frame $t^R = \{t\} \cup X_t^R$ as a root. Notice by construction $\mathcal{M}_{k+1}$ is λ-frame and $G \sqcup \{e\} \sqsubseteq \mathcal{M}_{k+1}$ as we add new elements as λ-co-covers to $\mathcal{M}_k$ and $G \sqcup \{e\} \sqsubseteq \mathcal{M}_k$.

Continuing this process we obtain λ-successor $\mathcal{M} = \cup_{k\in N}\mathcal{M}_k$ of $G \sqcup \{e\}$. This frame is potentially infinite.

By construction, the frame $\mathcal{M}$ has following properties:

- first slice $S_1(\mathcal{M})$ of $\mathcal{M}$ contains at least one singleton cluster $C(e) = \{e\}$;
- fixed antichain $\mathcal{X} \subset G$ does not have co-cover in $\mathcal{M}$ as on each step of construction we adjoin only co-cover generating λ-frame as a root;
- the frame G is open subframe of $\mathcal{M}$;
- every nontrivial antichain of clusters of $\mathcal{M}$ (differ from $\mathcal{X}$) has singleton λ-co-cover in $\mathcal{M}$ (whenever it generates as a root λ-frame).

Proposition 3. *$\mathcal{M} \not\models_V \mathcal{R}_n$ holds for some n and valuation V.*

Proof. We define a valuation V on $\mathcal{M}$ as follows. Let's suppose that fixed antichain $\mathcal{X} \subset b^R$ consists of clusters $\{C_1,\ C_2,\ \ldots, C_n\}$.

Now we define $\mathcal{X}^{-R} = \{x\ :\ xRC_1\&xRC_2\&\ldots xRC_n\}$ and

$$V(q) := \{y \in \mathcal{M} \setminus \mathcal{X}^R : y \notin \mathcal{X}^{-R}\ \&\exists x \in \mathcal{X}^{-R}(xRy)\},\ V(p_i) := C_i,$$

e.g. $\forall x \in C_i\ x \models_V p_i$. Let's prove that premise of $\mathcal{R}_n$ is valued on $\mathcal{M}$ under V while conclusion is not.

By definition of V we obtain:

$$\forall e \in \mathcal{X}^R\ e \models_V \neg\Diamond q; \quad \forall x \in \mathcal{M}\ e \models_V p_i \iff x \in C_i.$$

If some element $z \in \mathcal{M}$ R-sees fixed antichain $\mathcal{X} = \{C_1,\ C_2,\ \ldots, C_n\}$ and $z \models_V A_n$ holds, then the cluster $C(z)$ can't be a co-cover for antichain $\mathcal{X}$ by construction of $\mathcal{M}$, and hence $z \models_V \neg q$ holds by definition of $V(q)$. Hence there should be some y such that $y \notin \mathcal{X}^{-R}$, $y \notin \mathcal{X}^R$ and zRy, in which case $y \models_V q$ and $z \models_V \neg q \wedge \Diamond q$ holds too, that infer $z \models_V \neg(A_n \wedge B)$.

So, we conclude $\forall x \in \mathcal{M}\ x \models_V \neg(A_n \wedge B)$.

By the definition of valuation V, $\forall x \in \mathcal{M}\ x \not\models_V \bigvee_{1\leq i\leq n} p_i \wedge \Diamond q$ is true obviously. In force of

$$\Box A_{n,1} = \Box \bigwedge_{1,n}(p_i \to \neg\Diamond q) = \Box \bigwedge_{1,n}(\neg p_i \vee \neg\Diamond q) = \Box\neg[\bigvee_{1,n}(p_i \wedge \Diamond q)],$$

we have $\forall x \in \mathcal{M}\ x \models_V \Box A_{n,1}$. Consequently we proved that the premise of the rule is valued on arbitrary elements of $\mathcal{M}$.

As element b is R-predecessor for fixed antichain $\mathcal{X} = \{C_1,\ C_2,\ \ldots, C_n\}$ and $\forall x \in C_i\ x \models_V p_i$ holds, we have $b \models_V \bigwedge_{1\leq i\leq n} \Diamond p_i$, e.g. $b \models_V A_n$. From this we infer $b \not\models_V \Box\neg A_n$. ■

Proposition 4. *The rule $\mathcal{R}_n$ is not admissible in logic λ, where $n > 1$ is a number of clusters of antichain $\mathcal{X} = \{C_1, C_2, \ldots C_n\}$.*

Proof. Let's choose the least k so that $\mathcal{M} \sqsubseteq Ch_k(\lambda)$. Indeed, we can start with model on the finite frame $\mathcal{M}_0 = G \sqcup \{e\}$ and valuation S which compare to each different cluster $C_i \in \mathcal{M}_0,\ i \in I$, a different variable p_i. By theorem 3.3.6 [1] this model is open submodel of same k-characterizing model $Ch_k(\lambda)$. Therefore the frame $\mathcal{M}_0$ is open subframe of the frame of $Ch_k(\lambda)$. Than it's easy to see the process of construction of $\mathcal{M}$ is the part of one of the frame of $Ch_k(\lambda)$ for same k and the resulting frame is open subframe of the frame $Ch_k(\lambda)$.

Now we define a p-morphism from the frame of k-characteristic model $Ch_k(\lambda)$ onto frame $\mathcal{M}$ as follows.

- for elements of frame $\mathcal{M} \sqsubseteq C_k(\lambda)$ we define the p-morphism f as identical, e.g $\forall\ x \in \mathcal{M}\ f(x) := x$.
- for all elements $x \in S_1(C_k(\lambda) \setminus \mathcal{M})$ we set $f(x) := e$, where $\{e\}$ is singleton cluster of first slice of $G \sqcup \{e\}$.
- Let's suppose that for all elements $x \in S_{\leq t}(Ch_k(\lambda) \setminus \mathcal{M})$ of the depth no more than t required p-morphism is defined already. Let's choose arbitrary element $y \in S_{t+1}(Ch_k(\lambda) \setminus \mathcal{M})$ which is co-cover for antichain (possible trivial) $\mathcal{A} \subset S_{\leq t}(Ch_k(\lambda))$. By the construction of k-characteristic model $Ch_k(\lambda)$ the frames $y^R = C(y) \cup \mathcal{A}^R$ and $\mathcal{A}^R$ are λ-frames.

  By inductive conjecture p-morphism $f(\mathcal{A}^R)$ is defined already. As p-morphism preserves the truth of formulas the subframe $f(\mathcal{A}^R)$ is λ-frame. And the result of adding co-cover (root) to $f(\mathcal{A}^R)$ would be a p-morphic image of y^R and thus also a λ-frame. From this by construction of λ-successor $\mathcal{M}$ we infer that the antichain of minimal clusters from $f(\mathcal{A}^R)$ has singleton reflexive co-cover ε in $\mathcal{M}$ and such element actually exists in $\mathcal{M}$. So we can define $f(y) := \varepsilon$. As thus co-cover is unique in $\mathcal{M}$ so all elements of $C(y)$ will be mapped to it.

  Accordingly to the arbitrariness of y in such way we define p-morphism f on whole slice $S_{t+1}(Ch_k(\lambda)$.
- Continued this process we obtain p-morphism f from the frame of k-characteristic model $Ch_k(\lambda)$ onto $\mathcal{M}$. Note the mapping f is defined in a way that preserves the property of being a h-morphism.

Then we transfer the valuation V from $\mathcal{M}$ on $Ch_k(\lambda)$ as

$$\forall x \in Ch_k(\lambda)\ x \models_{f^{-1}(V)} p \iff f(x) \models_V p.$$

This gives us the p-morphism of models:

$$\langle Ch_k(\lambda),\ f^{-1}(V)\rangle \longrightarrow_f \langle \mathcal{M},\ V\rangle.$$

As p-morphism f preserves the truth of formulas and the rule $\mathcal{R}_n$ is refuted on $\langle \mathcal{M},\ V\rangle$ we refute this rule on k-characteristic model $\langle Ch_k(\lambda),\ f^{-1}(V)\rangle$. Consequently the rule $\mathcal{R}_n$ is not admissible in logic λ. Theorem 2 is proved. ■

So from theorems 1 and 2 we obtain

Theorem 5. *Let FMP logic λ extends $S4$. All rules $\mathcal{R}_n,\ n > 1,\ n \in N$, are admissible in $\lambda \iff$ logic λ enjoys the weak co-cover property.*

Список литературы

[1] V.V. Rybakov. Admissibility of logical inference rules, *Studies in Logic and the Foundations of Mathematics, Elsevier Sci. Publ.*, New-York – Amsterdam, 136 (1997).

[2] R. Harrop. Concerning Formulas of the Types $A \to B \vee C,\ A \to \exists x B(x)$, Journal of Symbolic Logic, 25:1 (1960), 27 – 32.

[3] G.E. Mints. Derivability of Admissible Rules. J. of Soviet Mathematics, V.6, 1976, No.4, pp. 417 – 421.

[4] J. Port. The deducibilities of S5, J. of Phylosophical Logic, 10:1 (1981), 409 – 422.

[5] P. Lorenzen. *Einfung in Operative Logik und Mathematik*, Berlin – Gottingen – Heidelberg, 1955.

[6] H. Fridman. One hundred and two problems in mathematical logic, Journal of Symbolic Logic, 40:3 (1975), 113 – 130.

[7] V.V. Rybakov. A Critarion for Admissibility of Rules in Modal System S4 and the Intuitionistic Logic. Algebra and Logic, V.23, 1984, No. 5, pp. 369 – 384 (Eng. transletion).

[8] R. Iemhoff. A(nother) characterization of Intuitionistic Propositional Logic, Annals of Pure and Applied Logic, 113:1-3 (2001), 161 – 173.

[9] V.V. Rybakov. An explicit basis for rules admissible in modal system S4? Bulletin of the Section of Logic, 28 (3) : 135–144, 1999.

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