A DESCRIPTION OF MODAL LOGICS WHICH ENJOY THE CO-COVER PROPERTY

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ABSTRACT. Here we use admissible rules to characterize modal logics with weak co-cover property. We prove that an arbitrary logic λ over $S4$ satisfies such a property iff all the rules in some family are admissible in λ.

Keywords: modal logic, inference rule, Kripke frame and model, admissible rule.

1. INTRODUCTION

In modern applications of logic in Computer Science and Artificial Intelligence, one often needs languages capable of representing some knowledge about dynamic systems. Various non-standard logics (e.g. modal and temporal logics, logic for multi-agent reasoning) are efficiently used for these applications. They describe statements by some formulas specific to considered models in general and do not take into consideration variable conditions and changing assumptions. These conditions and assumptions can be modelled by distinct variations of the notion of logical consequence. The problem of such an adequate modeling is one of the most important problems originated in mathematical logic and mathematics in general. The most important component of our approach consists of the fact that we study logical consequence in terms of inference rules and clauses, not only of formulas or statements. The formalism concerning the description of properties by formulas is well–developed, widely spreaded, and well represented in scientific literature. It is a basis for representing human reasoning. But formulas describe only stable, static events only; statements only fix the facts and they are unable to handle changing conditions.

Therefore, the study of (structural) inference rules (or sequents), i.e. expressions which have premises (collections of assumptions) and conclusions, brings us more flexibility and more expressive power to model the human reasoning and computing. Premises give us the current information in the form of assumptions and conclusions representing the knowledge, i.e. the facts which we can obtain from our assumptions. The study of reasoning (as a part of AI) again requires the understanding of what are consistent consequences of observable facts. Within Computer Science, these aspects are involved in the analysis of correct instructions for computations, verification of programs, and many other areas. Such rules enable us to study a standard series of questions in the study of logical consequence: what can be obtained from certain given assumptions?

Evidently, the notion of inference rule generalizes the notion of formula, because any formula can be viewed as a structural inference rule without premise and

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assumption. But admissible inference rules are actually stronger then the clauses. It follows from Harrop's paper [2] (1960), that even the intuitionistic logic H is not structurally complete: it has admissible inference rules which cannot be represented by formulas, which are not inferred. The same is valid for a broad range of basic modal logics (see the examples of G. Mints [3] and J. Port [4]).

The notion of admissible inference rule goes back to Lorenzen ([5], 1955). For an arbitrary logic, the admissible rules are exactly those rules under which this logic is closed. Clearly, any derivable rule is admissible, but generally not vice versa. Also, directly from the definition we see that the set of all rules admissible in a logic λ is the *greatest class* of inference rules by which we can extend an axiomatic system of the logic λ preserving the theorems of λ. Derivable rules may replace some fragments of fixed length in derivations, thereby linearly shortening them. Admissible rules which are not derivable basically may reduce derivations even more drastically.

The history of the study of admissible rules could be started since H. Fridman's question ([6], 1975) about the existence of an algorithm which could distinguish rules admissible in intuitionistic logic H. In the middle of 70-th G. Mints [3] found the strong sufficient conditions for derivability in H of admissible rules in special form. The Fridman's question about the existence of an algorithm recognizing admissibility of inference rules was answered affirmatively by V. Rybakov (1984, cf. [7]) for the intuitionistic logic H and a broad class of modal logics (e.g. $K4$, $S4$, GL, cf. [1]).

In 2000-2010 a few results on describing of explicit bases for admissible inference rules for nonstandard logics (S4, K4, H etc.) appeared (see for example [9]). The key condition in these results was the weak co-cover property. We hope that the weak co-cover property, the disjunction property, and FMP enable us to describe such a basis. In addition to the improvement of deductive power in logic, admissible rules are able to describe some semantic properties of a given logic. One of the first attempts was the description of intuitionistic logic H by the set of admissible inference rules (cf. R. Iemhoff [8]). That is why this property is in the focus of this article. Here we use admissible rules to characterize modal logics with weak co-cover property. We prove that an arbitrary logic λ over $S4$ with finite model property satisfies such a property iff all the rules in some family are admissible in λ.

2. Notations, preliminary Facts

We assume the reader to be aware of the algebraic and Kripke semantics for modal logics and to have some initial knowledge concerning the basic facts on inference rules and their admissibility (anyway, we briefly recall all the necessary facts below). As a good entry point to the subject we would recommend among modern literature the monograph by Rybakov [1] for a general techniques as well as for advanced techniques concerning modal logics and inference rules. According to modern trends, by a *logic* we mean the set of all theorems provable in a given axiomatic system or the set of valid formulas for some class of Kripke frames. In particular, a normal modal logic λ is a set of modal formulas which is closed under substitution, modus ponens, and necessitation rule $A \ / \ \Box A$ and including all the theorems of the minimal propositional modal logic K. In the following definitions and results we deal with algebraic propositional logics extending logic $S4$ with finite model property by a modal logic (cf. [1]).

A frame $\mathcal{F} := \langle F, R\rangle$ is a pair, where F is a nonempty set and R is a binary relation on F. For simplicity, the basis set of a frame and a frame itself are often denoted by the same letter. Further on we consider only frames in which R is transitive and reflexive.

A model is a triple $\mathcal{M} = \langle W,\ R,\ V\rangle$, where $\mathcal{F} := \langle F, R\rangle$ is a frame and V is a valuation of a set of propositional letters P in the frame $\mathcal{F}$, that is $V : P \to 2^W$. The set $Dom(V) = P$ is called the domain of V.

A frame $\mathcal{F} = \langle F,\ R\rangle$ is called an open subframe of the frame $\mathcal{G} = \langle G,\ R\rangle$ (we denote this fact as $\mathcal{F} \sqsubseteq \mathcal{G}$) if $F \subseteq G$ and $\forall a \in F\, \forall b \in G\, (aRb \Longrightarrow b \in F)$ holds. If $\mathcal{M}_1 = \langle W_1,\ R_1,\ V_1\rangle$, $\mathcal{M}_2 = \langle W_2,\ R_2,\ V_2\rangle$ are models then we call $\mathcal{M}_1$ an open submodel of $\mathcal{M}_2$ (and denote this fact as $\mathcal{M}_1 \sqsubseteq \mathcal{M}_2$) if : 1) $\langle W_1,\ R_1\rangle$ is open subframe of $\langle W_2,\ R_2\rangle$; 2) $Dom(V_1) = Dom(V_2)$ and $\forall p \in Dom(V_1)\ V_1(p) = V_2(p) \cap W_1$.

A mapping $f : \langle F,\ R\rangle \to \langle G,\ S\rangle$ is a called p-morphism if 1) $aRb \Longrightarrow f(a)Sf(b)$; 2) $f(x)Sz \Longrightarrow \exists y \in F : f(y) = z\, \&\, xRy$.

We say that a mapping $f : \mathcal{M}_1 = \langle W_1,\ R_1,\ V_1\rangle \to \mathcal{M}_2 = \langle W_2,\ R_2,\ V_2\rangle$ is a p-morfism of the model $\mathcal{M}_1$ into the $\mathcal{M}_2$ if 1) f is a p-morfism of the frame $\mathcal{F}_1 = \langle W_1,\ R_1\rangle$ into the frame $\mathcal{F}_2 = \langle W_2,\ R_2\rangle$; 2) the valuations $V_1,\ V_2$ are defined on the same set of propositional letters; 3) $\forall\, p \in Dom(V_1), \forall a \in W_1 (a \models_{V_1} p \iff f(a) \models_{V_2} p)$.

The property of primary importance of open submodels and p-morfisms consists of the fact that they preserve the truth of formulas:

Proposition 1. *(cf.* [1]*) 1) If $\mathcal{M}_1$ is an open submodel of a model $\mathcal{M}_2$ then for every formula α, $\mathcal{M}_2 \models \alpha$ implies $\mathcal{M}_1 \models \alpha$;*

2) if f is a p-morfism of the model $\mathcal{M}_1 = \langle W_1,\ R_1,\ V_1\rangle$ onto onto model $\mathcal{M}_2 = \langle W_2,\ R_2,\ V_2\rangle$ then for any formula α which is built from letters from the domain $Dom(V_1)$ holds $\forall a \in W_1 (a \models_{V_1} \alpha \iff f(a) \models_{V_2} \alpha)$.

Let $\mathcal{F}_i = \langle W_i,\ R_i\rangle,\ i \in I$ be a family of pairwise disjoint frames, i.e. $W_i \cap W_j = \emptyset$ for $i \neq j \in I$. The disjoin union of this family is the frame $\sqcup_{i\in I}\mathcal{F}_i = \langle W,\ R\rangle$, where $W = \cup_{i\in I} W_i,\ R = \cup_{i\in I} R_i$. Disjoint union of models is defined similarly.

By Lemma 2.5.26 [1], disjoint unions of frames (or models) preserve the truth of formulas: $\sqcup_{i\in I}\mathcal{F}_i \models \alpha \iff \forall i(\mathcal{F}_i \models \alpha)$. The disjoint union of λ-frames is a λ-frame.

Any subset C of a frame F which is either a single irreflexive point or satisfies the properties (1) $\forall x, y \in C(xRy\, \&\, yRx)$ and (2) $\forall x \in C \forall y \in W(xRy\, \&\, yRx \Longrightarrow y \in C)$ is called a *cluster of the frame* F. A cluster is *proper* if $|C| > 1$, otherwise this cluster is called degenerated or singleton. For any element $a \in F$, a cluster generated by a is denoted by $C(a)$. Any set of clusters of F which are non-comparable by R is called an antichain. The antichain $\mathcal{A}$ is nontrivial if it consists at least of two clusters; otherwise we call it trivial.

We call the maximal number of clusters in chains of clusters generated by an element (or a cluster) the *depth of an element (or of a cluster)*. For any transitive frame F (or a transitive Kripke model M), its *n-slice* $S_n(F)$ ($S_n(M)$) is the set of all elements of depth n from F (M respectively). And $S_{\leq n}(F)$ is the set of all elements from F with depth at most n.

We say that a frame F is a λ–frame for a logic λ if all the theorems of λ are valid at F. And we call $\lambda(F)$ — the set of all formulas valid in F — the logic generated by F. A frame F is rooted if there exists an $a \in F$ such that for all $b \in F$ holds aRb. We call $C(a)$ the root of F.

We put $b^R := \{x | \exists y \in C(b) : yRx\} \cup C(b)$; $b^{<R} := \{x | \exists y \in C(b) : yRx \wedge \neg xRy\}$, where $C(b)$ is a cluster of a frame F containing the element b. For any subset $X \subseteq F$ we let X^R to be $\cup\{x^R | x \in X\}$. That is, the subframe b^R of F is the upper cone generated by b, and X^R is the subframe of F generated by X respectively. An element $b \in F$ or cluster $C(b)$ is said to be *a co-cover* for a set $X \subseteq F$ if $b^R \backslash C(b) = X^R$. We understand a cluster C as a λ-co-cover for X if it generates the λ-frame $C^R := X^R \cup \{C\}$ as a root. We will identify a one-element cluster $C(a)$ and a reflexive element a (generating this cluster) as the co-cover, since they have the same set of R-accessible elements.

A subset $\mathcal{X}$ of a given model $\mathcal{M}$ is definable if there is a formula α such $\forall x \in \mathcal{M}\, [\, x \in \mathcal{X} \iff x \models_V \alpha]$. And a valuation V is definable in a model $\mathcal{M}$ if for any letter p, the set $V(p)$ is definable.

Let $\alpha_1, ..., \alpha_n, \beta$ to be some formulas. We understand the figure r, where

$$r := \frac{\alpha_1(x_1, \ldots, x_n), ..., \alpha_n(x_1, \ldots, x_n)}{\beta(x_1, \ldots, x_n)},$$

as the (structural) inference rule which derives $s(\beta)$ from $s(\alpha_1), ..., s(\alpha_n)$ for every substitution s. We say r is *derivable* in a logic λ if there is a derivation β in λ from the set of assumptions $\{\alpha_1, ..., \alpha_n\}$.

An *inference rule* $r = \{\alpha_1(x_1, \ldots, x_n), \ldots, \alpha_k(x_1, \ldots, x_n)/\beta(x_1, \ldots, x_n)\}$ is called *admissible in a logic* λ, if for any formulas $\delta_1, \ldots, \delta_n$ holds

$$((\alpha_1(\delta_1, \ldots, \delta_n) \in \lambda) \cdots (\alpha_k(\delta_1, \ldots, \delta_n) \in \lambda) \implies \beta(\delta_1, \ldots, \delta_n) \in \lambda).$$

Admissible inference rules have the following algebraic description:

Proposition 2. [1] *An inference rule $r = \{\alpha_1, \ldots, \alpha_k/\beta\}$ is admissible in a logic λ iff the quasi-identity $r^* = \{\alpha_1 = 1 \,\&\ldots\&\, \alpha_k = 1 \Longrightarrow \beta = 1\}$ is true on free algebra of countable rank $\mathcal{F}_w(\lambda)$ from the variety $Var(\lambda)$ generated by λ.*

Any derivable rule is admissible but the converse is not always true. Also directly from the definition we see that the set of all rules admissible in a logic λ is the *greatest* class of inference rules by which we can extend axiomatic system of the logic λ preserving the set of theorems of λ. Derivable rules may replace some fragments of fixed length in derivations, thereby shortening them linearly. Admissible rules which are not derivable in principle may reduce derivations even more drastically.

For a given frame $\mathcal{F}$, valuation V, and inference rule $r := \alpha_1, ..., \alpha_n/\beta$, we say that r *is valid on* $\mathcal{F}$ *with respect to* V and write $\mathcal{F} \models_V r$ if as soon as $\forall x \in \mathcal{F}$ and $\forall i\ (x \models_V \alpha_i)$ holds, we have $\forall x \in \mathcal{F}(x \models_V \beta)$. A rule r *is valid on a frame* $\mathcal{F}$ if r is valid at $\mathcal{F}$ under any valuation; we write then $\mathcal{F} \models r$.

We say that a logic λ has finite model property (FMP for short) if for any $\alpha \notin \lambda$ there exists a finite λ-model refuting α.

A Kripke Model $\langle F, R, V\rangle$, where $V : P_n \to 2^F$, $P_n = \{p_1, p_2, \ldots, p_n\}$, is *$n$-characteristic for a logic* λ iff for any formula α which is built from $p_1, \ldots, p_n$ holds $\alpha \in \lambda$ iff $\langle F, R, V\rangle \models \alpha$.

The admissibility of inference rules in modal logic λ over $S4$ can be described via their validness in certain special n-characteristic Kripke models. The description of such models $Ch_n(\lambda)$ and criteria for recognizing admissibility in λ by means of them can be found, for instance, [1]. As we will esseentially use these techniques in the sequel, we briefly recall the construction of $Ch_n(\lambda)$ for FMP logic λ over $S4$ and the semantic criterion for recognizing admissibility.

Given a FMP logic λ over $S4$ and a set $P_n := \{p_1, ..., p_n\}$ of propositional letters, we construct the first slice $S_1(Ch_n(\lambda))$ as follows. It consists of the collection of all clusters with all possible valuations V of letters from P_n which have no doubling (elements in the same cluster with the same valuation and clusters which are isomorphic as Kripke models).

Assume that $S_{\leq m}(Ch_n(\lambda))$ is already defined. We define $S_{m+1}(Ch_n(\lambda))$ as follows. We take an arbitrary antichain $\mathcal{X}$ of clusters from $S_{\leq m}(Ch_n(\lambda))$ having at least one cluster of depth m and put all clusters C from $S_1(Ch_n(\lambda))$ into $S_{m+1}(Ch_n(\lambda))$, where C is an immediate predecessor for all elements from $\mathcal{X}$ (co-cover for $\mathcal{X}$), such that

(i) $C^R = \mathcal{X}^R \cup C$ is a λ–frame and
(ii) if $\mathcal{X} := \{C_1\}$ then C is not a Kripke submodel of C_1.

Iterating this procedure we obtain a model $Ch_n(\lambda)$ as a result. Recall that a model $\mathcal{M}$ is n-characteristic for a logic λ if for any formula α which is built from letters from P_n holds $\alpha \in \lambda$ iff $\mathcal{M} \models \alpha$. We need the following facts:

Theorem 3. *(cf.* [1]*) For any FMP logic λ over $S4$ the model $Ch_n(\lambda)$ is n-characteristic for λ.*

Theorem 4. *(cf.* [1]*) For any inference rule r, r is admissible in FMP logic λ over $S4$ iff r is valid in the frame of $Ch_n(\lambda)$ with respect to any definable valuation for any given n.*

3. Main results

We say that a logic λ, extending logic $S4$, has *weak co-cover property* (WCP for short) whenever for every finite rooted λ-frame $\mathcal{F}$ and an arbitrary nontrivial antichain $\mathcal{X}$ of clusters from $\mathcal{F}$, the frame $\mathcal{F}_1$ which is result of adjoining a singleton reflexive co-covering as the root to the frame $\bigcup_{c \in \mathcal{X}^R} c^R$ is a λ-frame as well.

Given $n \in N$ with $n > 1$, define the formulas:

$$\pi_i := p_i \wedge \bigwedge_{j \neq i} \neg p_j; \;\; 1 \leq i, j \leq n, \quad A_n := \bigwedge_{1 \leq i \leq n} \Diamond \pi_i;$$

$$A_{n,1} := \Box\Big[\bigwedge_{1 \leq i \leq n} (p_i \rightarrow \neg \Diamond q) \Big]; \quad B := q \vee \neg \Diamond q.$$

Also we define the rules:

$$\mathcal{R}_n := \frac{\Box\big(A_{n,1} \wedge \neg(A_n \wedge B)\big)}{\Box \neg A_n}; \; n = 2, 3, \ldots$$

Note that these rules are special case of those from [9] which gives an explicit basis for admissible rules of logic S4. Next theorem is almost the same as the statement of Lemma 3.1 [9]. The WCP of logic is a key condition in the proof of this statement. Let us reproduce the common part of that proof.

Theorem 1. *The rules $\mathcal{R}_n$, $n > 1$, are admissible in every FMP logic λ over $S4$ that enjoys the weak co-cover property.*

Proof Assume that the statement is false. Suppose that for some n the rule

$$\mathcal{R}_n := \frac{\Box\big(A_{n,1} \wedge \neg(A_n \wedge B)\big)}{\Box\neg A_n}$$

is not admissible in λ. Hence there is a definable valuation V of variables from $\mathcal{R}_n$ in some constructive k–characteristic model $Ch_\lambda(k)$. Therefore

$$Ch_\lambda(k) \models_V \Box\big(A_{n,1} \wedge \neg(A_n \wedge B)\big) \ \&\ Ch_\lambda(k) \not\models_V \Box\neg A_n. \quad (1)$$

Consequently, there exists an element $a \in Ch_\lambda(k)$ such that $a \not\models_V \Box\neg A_n$. Then there are elements $b_1, \ldots, b_n \in Ch_\lambda(k)$ such that aRb_i & $b_i \models_V p_i$. Note that, if all the elements $b_1, \ldots, b_n \in Ch_\lambda(k)$ belong to the same cluster (say $C(b_1)$ for example) then $b_1 \models_V A_n$ holds and by (1) we have $b_1 \models_V A_{n,1}$. Therefore $b_1 \models_V \neg\Diamond q$, e.g. we obtain $b_1 \models_V A_n \wedge B$ which contradicts the statement $b_1 \models_V \Box\neg(A_n \wedge B)$ by the assumption (1). So, the elements $b_1, \ldots, b_n \in Ch_\lambda(k)$ generate a nontrivial antichain.

By the weak co-cover property, there exists a reflexive element $b \in Ch_\lambda(k)$ which is a co-cover for the set of R-minimal clusters from the set $\{C(b_1), \ldots, C(b_n)\}$, that is

$$\{b\}^R := \{b\} \cup \bigcup_{1 \le i \le n} (b_i)^R.$$

It follows from (1) that $b \models_V A_{n,1}$ and $b \models_V A_n$. Since b is a co-cover for $\{b_1, \ldots, b_n\}$, it is clear that $b \models_V B$. Indeed $b_i \models_V p_i$ and $b \models_V A_{n,1}$ holds, therefore $\forall i \le n\ b_i \models_V \neg\Diamond q$. From this we conclude that $b \models_V q$ or $b \models_V \neg q$ and hence $b \models_V \neg\Diamond q$. Therefore we obtain $b \models_V A_n \wedge B$ which contradicts the statement $b \models_V \Box\neg(A_n \wedge B)$ by the assumption (1). ■

Theorem 2. *If for all n the rules $\{\mathcal{R}_n, n > 1\}$ are admissible in FMP logic λ, over $S4$ then the logic λ enjoys the weak co-cover property.*

Proof. Suppose all the rules $\mathcal{R}_n,\ n > 1, n \in N$, to be admissible in FMP logic λ over $S4$, but λ doesn't enjoy the weak co-cover property. By definition, there exists a *finite* rooted λ-frame $G = b^R$ and a nontrivial antichain of clusters $\mathcal{X} \subset G$ such that the frame $\varepsilon^R := \bigcup_{c \in \mathcal{X}^R} c^R \cup \{\varepsilon\}$ obtained by adjoining a singleton reflexive co-cover ε as a root to the frame $\bigcup_{c \in \mathcal{X}^R} c^R$ is not a λ-frame. Fix this nontrivial antichain of clusters $\mathcal{X} \subset G$. We will prove that in such a case at least one rule $\mathcal{R}_n$ is not admissible in the logic λ. To do this, we construct a λ-frame $\mathcal{M}$ containing the frame G as an open subframe and refuting $\mathcal{R}_n,\ n > 1$, under some valuation. Then we define a p-morphism from the frame of k-characteristic model $Ch_k(\lambda)$ for some k on $\mathcal{M}$. Transferring this valuation from $\mathcal{M}$ onto $Ch_k(\lambda)$, we refute this rule $\mathcal{R}_n,\ n > 1$, on $Ch_k(\lambda)$ under some valuation, which contradicts the admissibility of $\mathcal{R}_n$.

Let us take a frame $G \sqcup \{e\}$ where $\{e\}$ is a reflexive singleton which is not R-comparable to any element in G. The frame $\{e\}$ is a λ-frame as a p-morfic image of G. So the frame $G \sqcup \{e\}$ is a λ-frame as a disjoint union of λ-frames.

We define the λ-successor $\mathcal{M}$ of $G \sqcup \{e\}$ as follows. Fix a nontrivial antichain $\mathcal{X}$ from G and define a λ-frame $\mathcal{M}_0 = G \sqcup \{e\}$. Then we choose all nontrivial antichains of clusters $\{X_t \subseteq S_1(\mathcal{M}_0)\}$ which don't have singleton co-cover in $\mathcal{M}_0$ and add such co-cover t whenever it generates a λ-frame $t^R = \{t\} \cup X_t^R$ as a root. So we obtain $\mathcal{M}_1$. Notice that by the construction $\mathcal{M}_1$ is a λ-frame (we add elements generating λ-frame as a root to λ-frame $G \sqcup \{e\}$) and $G \sqsubseteq G \sqcup \{e\} \sqsubseteq \mathcal{M}_1$ as we add new elements as co-covers to $G \sqcup \{e\}$.

Assuming that $\mathcal{M}_k$ has already been constructed and $G \sqcup \{e\} \sqsubseteq \mathcal{M}_k$, we obtain $\mathcal{M}_{k+1}$ as follows: we choose in $S_{\leq(k+1)}(\mathcal{M}_k)$ all nontrivial antichains $\{X_t \subseteq S_{\leq(k+1)}(\mathcal{M}_k)\}$ which don't have singleton co-cover and have at least one cluster of depth $k{+}1$. Then we add singleton reflexive element t as co-cover to each chosen antichain whenever it generates the λ-frame $t^R = \{t\} \cup X_t^R$ as a root. Note that by construction $\mathcal{M}_{k+1}$ is λ-frame and $G \sqcup \{e\} \sqsubseteq \mathcal{M}_{k+1}$ as we add new elements as λ-co-covers to $\mathcal{M}_k$ and $G \sqcup \{e\} \sqsubseteq \mathcal{M}_k$.

Continuing this process we obtain the λ-successor $\mathcal{M} = \cup_{k\in N}\mathcal{M}_k$ of $G \sqcup \{e\}$. This frame is potentially infinite.

It follows from the construction that the frame $\mathcal{M}$ has the following properties:

- first slice $S_1(\mathcal{M})$ of $\mathcal{M}$ contains at least one singleton cluster $C(e) = \{e\}$;
- the fixed antichain $\mathcal{X} \subset G$ does not have co-cover in $\mathcal{M}$ as on each step of construction we adjoin only a co-cover generating λ-frame as a root;
- the frame G is an open subframe of $\mathcal{M}$;
- every nontrivial antichain of clusters of $\mathcal{M}$ (different from $\mathcal{X}$) has singleton λ-co-cover in $\mathcal{M}$ (whenever it generates a λ-frame as a root).

Proposition 3. *$\mathcal{M} \not\models_V \mathcal{R}_n$ holds for some n and a valuation V.*

Proof. We define a valuation V on $\mathcal{M}$ as follows. Suppose that the fixed antichain $\mathcal{X} \subset b^R$ consists of clusters $\{C_1,\ C_2,\ \ldots, C_n\}$.

Now we define $\mathcal{X}^{-R} = \{x\ :\ xRC_1 \& xRC_2 \& \ldots xRC_n\}$ and

$$V(q) := \{y \in \mathcal{M} \setminus \mathcal{X}^R : y \notin \mathcal{X}^{-R}\ \& \exists x \in \mathcal{X}^{-R}(xRy)\},\ V(p_i) := C_i,$$

e.g. $\forall x \in C_i\ x \models_V p_i$. Let us prove that the premise of $\mathcal{R}_n$ is valued on $\mathcal{M}$ under V while the conclusion is not.

By definition of V we obtain:

$$\forall x \in \mathcal{X}^R\ x \models_V \neg\Diamond q; \quad \forall x \in \mathcal{M}\ x \models_V p_i \iff x \in C_i.$$

If some element $z \in \mathcal{M}$ R-sees the fixed antichain $\mathcal{X} = \{C_1,\ C_2,\ \ldots, C_n\}$ and $z \models_V A_n$ holds, then $z \in \mathcal{X}^{-R}$. Therefore $z \models_V \neg q$ holds by definition of $V(q)$. Since the cluster $C(z)$ cannot be a co-cover for the antichain $\mathcal{X}$ by construction of $\mathcal{M}$, there should be some y such that $y \notin \mathcal{X}^{-R}$, $y \notin \mathcal{X}^R$, and zRy, in which case $y \models_V q$ and $z \models_V \neg q \wedge \Diamond q$ holds too, which gives us $z \models_V \neg(A_n \wedge B)$.

Indeed, if the cluster $C(z)$ is an immediate R-predecessor for $\mathcal{X}$, but not a co-cover for $\mathcal{X}$ (that is the depth $d(C(z))$ of $C(z)$ is $\max_{i\in\mathcal{X}} d(i){+}1$), then there should be at least one element y such that zRy and $C(y) \cup \mathcal{X}$ forms an antichain for which $C(z)$ is a co-cover (there can be some elements $y_1, \ldots, y_k$ such that $C(y_1) \cup \cdots \cup C(y_k) \cup \mathcal{X}$ forms an antichain). Then for this y hold $y \notin \mathcal{X}^{-R}$, $y \notin \mathcal{X}^R$, and zRy.

If the cluster $C(z)$ is not an immediate R-predecessor for $\mathcal{X}$, then some cluster $C(z_1)$ which is an immediate R-predecessor for $\mathcal{X}$ is R-accessible from it or some elements $z_1, z_2, \ldots, z_k$ are R-accessible which are immediate R-predecessors for some subsets of $\mathcal{X}$ and $\mathcal{X} \subseteq z_1^R \cup \cdots \cup z_k^R$. In the first case (as before) we obtain some element y such that $y \notin \mathcal{X}^{-R}$, $y \notin \mathcal{X}^R$, and z_1Ry (and hence zRy by transitivity of R). In the second case we can take y to be the element z_1 wich has the desirable property.

So, we conclude that $\forall x \in \mathcal{M}\ x \models_V \neg(A_n \wedge B)$.

By the definition of valuation V, $\forall x \in \mathcal{M}\ x \not\models_V \bigvee_{1\leq i\leq n} p_i \wedge \Diamond q$ is obviously true. In force of

$$\Box A_{n,1} \equiv \Box \bigwedge_{1,n}(p_i \to \neg\Diamond q) \equiv \Box \bigwedge_{1,n}(\neg p_i \vee \neg\Diamond q) \equiv \Box\neg[\bigvee_{1,n}(p_i \wedge \Diamond q)],$$

we have $\forall x \in \mathcal{M}\ x \models_V \Box A_{n,1}$. Consequently, we have proved that the premise of the rule is valued on arbitrary elements of $\mathcal{M}$.

Since b is an R-predecessor for the fixed antichain $\mathcal{X} = \{C_1,\ C_2,\ \dots, C_n\}$ and $\forall x \in C_i\ x \models_V p_i$ holds, we have $b \models_V \bigwedge_{1\leq i\leq n} \Diamond p_i$, e.g. $b \models_V A_n$. From this we infer $b \not\models_V \Box\neg A_n$. ■

Proposition 4. *The rule $\mathcal{R}_n$ is not admissible in logic λ, where $n > 1$ is the number of clusters of the antichain $\mathcal{X} = \{C_1, C_2, \dots C_n\}$.*

Proof. Choose some (the least) k so that $\mathcal{M} \sqsubseteq Ch_k(\lambda)$. We can start with a model on the *finite* frame $\mathcal{M}_0 = G \sqcup \{e\}$ and a valuation S which assigns each element $c_i \in \mathcal{M}_0,\ i \in I, |I| = |\mathcal{M}_0| < w$, an individual variable p_i:

$$\forall i \in I\ c_i \models_S p_i;\ c_i \not\models p_j, i \neq j.$$

Under such a valuation S, all the clusters of $\mathcal{M}_0$ are not isomorphic as models and all the elements of any cluster have different valuations.

It is easy to see that the model $\langle \mathcal{M}_0, S\rangle$ is an open submodel of $Ch_k(\lambda)$ for $k = |\mathcal{M}_0|$:

(1) For each cluster $C \in S_1(\langle \mathcal{M}_0, S\rangle)$ we can find a cluster $K \in S_1(Ch_k(\lambda))$ which is isomorphic to cluster C as a model. So we have $S_1(\langle \mathcal{M}_0, S\rangle) \sqsubseteq S_1(Ch_k(\lambda))$ as models.

(2) Take some cluster $C_1 \in S_2(\langle \mathcal{M}_0, S\rangle)$ which is a co-cover for antichain $A \subseteq S_1(\langle \mathcal{M}_0, S\rangle)$. By (1), there is an antichain $B \subseteq S_1(Ch_k(\lambda))$ such that $\langle A, S\rangle \cong B$ as models. Since $C_1^R = C_1 \cup A$ is a λ-frame, by construction of $Ch_k(\lambda)$ there is a cluster $K_1 \in S_2(Ch_k(\lambda))$ such that $\langle C_1^R, S\rangle \cong K_1^R \sqsubseteq Ch_k(\lambda)$ as models. So we have $\langle S_{\leq 2}(\mathcal{M}), S\rangle \sqsubseteq Ch_k(\lambda)$.

(3) Continuing this reasoning slice by slice we conclude that $\langle \mathcal{M}_0, S\rangle \sqsubseteq Ch_k(\lambda)$. Therefore the frame $\mathcal{M}_0$ is an open subframe of the frame of $Ch_k(\lambda)$.

It is easy to see that the process of construction of $\mathcal{M}$ is a part of the process of construction of one of the frames of $Ch_k(\lambda)$ and the resulting frame is an open subframe of the frame $Ch_k(\lambda)$.

(a) At the first step of the construction of $\mathcal{M}$ we take a nontrivial antichain $X_t \subseteq S_1(\mathcal{M}_0)$ which does not have singleton co-cover in $\mathcal{M}_0$ and add such a co-cover whenever it generates a λ-frame $t^R = \{t\} \cup X_t$ as a root. Since $\langle \mathcal{M}_0, S\rangle \sqsubseteq Ch_k(\lambda)$, there is an antichain $B \subseteq S_1(Ch_k(\lambda))$ such that $\langle X_t, S\rangle \cong B$ as models. As t^R is a λ-frame, by the construction of the model $Ch_k(\lambda)$ we can find a co-cover for the B — a cluster C_B — such that $t^R \cong C_B^R$ as frames. So we can transfer a valuation from C_B on t and obtain that $\langle t^R, S\rangle \cong C_B^R$ as models. Hence we have $\langle \mathcal{M}_1, S\rangle \sqsubseteq Ch_k(\lambda)$.

(b) Assume that $\langle \mathcal{M}_k, S\rangle \sqsubseteq Ch_k(\lambda)$. By the construction of $\mathcal{M}_{k+1}$, we take a nontrivial antichain $X_t \subseteq S_{\leq(k+1)}(\mathcal{M}_k)$ which does not have singleton co-cover in $\mathcal{M}_k$ and has at least one cluster of depth $k+1$. Then we add such a co-cover whenever it generates a λ-frame $t^R = \{t\} \cup X_t$ as a root. For this antichain X_t there is antichain $B \subseteq S_{k+1}(Ch_k(\lambda))$ such that $\langle X_t^R, S\rangle \cong B^R$ as models. As t^R is a λ-frame, by construction of model $Ch_k(\lambda)$, we can find a co-cover for the antichain B — cluster C_B — such that $t^R \cong C_B^R$ as frames. So we can transfer the valuation from C_B on t and obtain that $\langle t^R, S\rangle \cong C_B^R$ as models. Hence we have $\langle \mathcal{M}_{k+1}, S\rangle \sqsubseteq Ch_k(\lambda)$.

(c) Continuing this reasoning slice by slice we conclude that $\langle \mathcal{M}, S\rangle \sqsubseteq Ch_k(\lambda)$. Therefore the frame $\mathcal{M}$ is an open subframe of the frame of $Ch_k(\lambda)$.

Since all the elements of $Ch_k(\lambda)$ are definable and $\langle \mathcal{M}, S\rangle \sqsubseteq Ch_k(\lambda)$, all the elements of $\langle \mathcal{M}, S\rangle$ are also definable (one can find the definition and the structure of this formula β in the proof of Theorem 3.3.7 in [1]):

$$\forall\, x \in \mathcal{M} \exists\, \beta_x : x \models_S \beta_x \;\&\; (y \neq x \Longrightarrow y \not\models_S \beta_x).$$

So we conclude that any (finite) subset $A \subseteq \mathcal{M}$ is also definable under the valuation S: $\beta(A) = \bigvee_{a\in A} \beta_a$. Hence, now we can define

$$\mathbf{V}(\mathbf{p_i}) = \bigvee_{x\in C_i} \beta_x, 1 \leq i \leq n;\; \mathbf{V}(\mathbf{q}) = \{y \in \mathcal{F} \setminus X^R : y \notin X^{-R} \& \exists z \in X^{-R}(zRy)\} =$$

$$= V\Big(\neg \bigwedge_{\beta_j\in\mathcal{X}} \Diamond\beta_j \wedge \neg\Box\big(\bigvee_{\beta_j\in\mathcal{X}^R} \beta_i\big) \wedge \bigvee\{\beta_y : \exists \beta_z(z \models_S \bigwedge_{\beta_j\in\mathcal{X}} \Diamond\beta_j \Longrightarrow z \models_S \Diamond\beta_y\}\Big).$$

It follows that the sets $V(p_i), 1 \leq i \leq n$, and $V(q)$ are definable in $\langle \mathcal{M}, S\rangle$.

Now we define a p-morphism from the frame of k-characteristic model $Ch_k(\lambda)$ onto $\mathcal{M}$ as follows.

- for elements of the frame $\mathcal{M} \sqsubseteq C_k(\lambda)$, we define p-morphism f to be identical, e.g. $\forall\; x \in \mathcal{M}\; f(x) := x$.
- for all $x \in S_1(C_k(\lambda) \setminus \mathcal{M})$ we let $f(x) := e$, where $\{e\}$ is the singleton cluster of the first slice of $G \sqcup \{e\}$.
- Suppose that for all $x \in S_{\leq t}(Ch_k(\lambda)\setminus\mathcal{M})$ of depth at most t the required p-morphism is already defined. Choose an arbitrary element $y \in S_{t+1}(Ch_k(\lambda)\setminus \mathcal{M})$ which is a co-cover for the antichain (possible trivial) $\mathcal{A} \subset S_{\leq t}(Ch_k(\lambda))$. By the construction of the k-characteristic model $Ch_k(\lambda)$, the frames $y^R = C(y) \cup \mathcal{A}^R$ and $\mathcal{A}^R$ are λ-frames.

  By the inductive conjecture, the p-morphism $f(\mathcal{A}^R)$ is already defined. If $f(\mathcal{A}) = C$ is a trivial antichain and is a cluster, we define $f(y) := \varepsilon \in f(\mathcal{A})$.

  Assume $f(\mathcal{A}^R)$ is nontrivial. Since p-morphisms preserve the truth of formulas, the subframe $f(\mathcal{A}^R)$ is a λ-frame. And the result of adding a co-cover (root) to $f(\mathcal{A}^R)$ is a p-morphic image of y^R and by this it is also a λ-frame. From this by the construction of λ-successor $\mathcal{M}$ we infer that the antichain of R-minimal clusters from $f(\mathcal{A}^R)$ has a singleton reflexive co-cover ε in $\mathcal{M}$ and such an element actually exists in $\mathcal{M}$. So we can define $f(y) := \varepsilon$. Since this co-cover is unique in $\mathcal{M}$, all the elements of $C(y)$ will be taken to it.

  Since y was arbitrary, thus we define a p-morphism f on the whole slice $S_{t+1}(Ch_k(\lambda)$.
- Continuing this process, we obtain a p-morphism f from the frame of k-characteristic model $Ch_k(\lambda)$ onto $\mathcal{M}$. Note that the mapping f is defined in a way that preserves the property of being a p-morphism.

Then we transfer the valuation V from $\mathcal{M}$ to $Ch_k(\lambda)$ as follows:

$$\forall x \in Ch_k(\lambda)\; x \models_{f^{-1}(V)} p \iff f(x) \models_V p.$$

This gives us the p-morphism of models:

$$\langle Ch_k(\lambda),\ f^{-1}(V)\rangle \longrightarrow_f \langle \mathcal{M},\ V\rangle.$$

Since the sets $V(p_i), 1 \le i \le n$, and $V(q)$ are definable in the model $\langle \mathcal{M}, S\rangle$ and our p-morphism preserves the truth of formulas, the sets $f^{-1}(V(p_i)), 1 \le i \le n$, and $f^{-1}(V(q))$ are also definable in $Ch_k(\lambda)$.

Since the p-morphism f preserves the truth of formulas and the rule $\mathcal{R}_n$ is refuted on $\langle \mathcal{M},\ V\rangle$ we refute this rule on the k-characteristic model $\langle Ch_k(\lambda),\ f^{-1}(V)\rangle$ under a definable valuation. Consequently, the rule $\mathcal{R}_n$ is not admissible in logic λ. Theorem 2 is complete. ■

So from theorems 1 and 2 we obtain

Theorem 5. *Suppose that an FMP logic λ extends $S4$. Then all the rules $\mathcal{R}_n,\ n > 1,\ n \in N$, are admissible in λ if and only if the logic λ enjoys the weak co-cover property.*

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