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PARTIALLY COMMUTATIVE GROUPS AND LIE ALGEBRAS

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Abstract. This is a survey of results on partially commutative groups and partially commutative Lie algebras.

Keywords: group, Lie algebra, partially commutative group, partially commutative Lie algebra, varieties of group, varieties of Lie algebras, group automorphism, basis, centralizer, centralizer dimension, decomposition of a group, nilpotent group, nilpotent Lie algebra, metabelian group, metabelian Lie algebra, pro-p-group, universal theory, elementary theory.

Introduction

Many algebraic structures are defined by graphs. Partially commutative algebraic structures are some of them. Let $\mathfrak{M}$ be a variety of algebraic structures of a functional signature Σ containing a binary operation $\circ$. We assume $\circ$ is non-commutative, as otherwise it makes no sense to consider partially commutative laws. The case of a commutative operation is trivial so suppose that this is not so.

In this survey, by a graph we mean an undirected graph without loops and multiple edges. Graphs will be denoted by greek letters.

Let $\Delta = (X, E)$ be a graph (possibly infinite), with the set of vertices $X = \{x_1, x_2, \ldots\}$ and the set of edges $E = \{(x_i, x_j)\}$. For a variety $\mathfrak{M}$ define the *partially commutative structure* $C(\mathfrak{M}, \Delta)$ on this variety as follows

$$C(\mathfrak{M}, \Delta) = \langle X; x_i \circ x_j = x_j \circ x_i, \ \text{ if } (x_i, x_j) \in E, \mathfrak{M}\rangle. \tag{1}$$

Partially commutative structures appear in different areas of mathematics, for example, in computer science and robotics. By now, the most results in partially

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commutative structures have been obtained for so called free partially commutative groups. A free partially commutative group is a partially commutative structure in the variety of all groups such that it is defined by an undirected graph without loops. In recent years, much attention has been paid to researches into partially commutative groups in soluble and nilpotent varieties. Partially commutative associative and Lie algebras are studied as well. In this survey, papers on partially commutative groups and Lie algebras are discussed. There are so many results in free partially commutative groups that a specific survey is needed for them. So, the results on free partially commutative groups are not included in this survey. Some information on these results can be found in [7, 11].

There are two sections in the survey. In Sec. 1, results for partially commutative groups of some varieties are discussed. This section contains four subsections. In Subsec. 1.1, algebraic properties of partially commutative metabelian groups are observed. The results on the structure of the groups, their centralizers, annihilators, bases, subgroups, inclusions into matrix groups, automorphism groups, and centralizer dimensions are described. At the end of the subsection, a decomposition of a group into a direct product is discussed. Such decompositions are considered not only for free partially commutative groups but also for partially commutative groups of some varieties containing the variety of nilpotent groups of degree ≤ 2.

Let C be a structure. The set $Th(C)$ of all first-order sentences of a signature Σ which are true on C is called the *elementary theory* of C. Structures C_1 and C_2 are elementarily equivalent if $Th(C_1) = Th(C_2)$.

The universal theory or the $\forall$-theory of a structure C is a subset of $Th(C)$ consisting of all $\forall$-formulas which are true on C. Structures C_1 and C_2 are existentially equivalent if their existential theories coincide.

Results on the elementary and universal theories theories of partially commutative metabelian groups are considered in Subsec. 1.2. Information on varieties and prevarieties generated by partially commutative metabelian groups and on equations in one variable is also presented in Subsec. 1.2. The most attention is paid to universal theories, in particular, conditions of coincidence of two theories. In Subsec. 1.3, results on the structure and the universal theory of a partially commutative metabelean nilpotent group are considered. In Subsec. 1.4, theorems on centralizers and annihilators in partially commutative pro-p-groups are discussed.

Sec. 2 mainly presents results on partially commutative Lie algebras in some varieties. This section consists of two subsections.

In Subsec. 2.1 algebraic results on partially commutative Lie algebras are discussed. In this subsection the results on isomorphisms, bases, annihilators and centralizers of partially commutative Lie algebras of some varieties are collected.

In Subsec. 2.2, logical questions for partially commutative Lie algebras are discussed. Those are questions on universal and elementary theories of partially commutative Lie algebras.

In Sec. 2, there is a parallel presentation of results for partially commutative and partially commutative metabelian Lie algebras. Moreover, a description of a linear basis is also considered in the case of partially commutative nilpotent Lie algebras.

Results on partially commutative groups defined by infinite graphs are also discussed in Sec. 2, since there are analogous theorems for partially commutative Lie algebras in this section.

Sec. 1 is written by the second author while Sec. 2 is written by the first one.

1. Partially commutative groups

This section is a survey of results for partially commutative groups in soluble varieties.

Research into groups defined by generators and defining relations forms a large field of algebra. This field is called combinatorial group theory. It has a specific collection of problems and methods. A lot of these methods have analogues in algebraic topology. Studies in combinatorial group theory have been intensive since the second half of the 20th century. One of the classes of objects studied in combinatorial group theory consists of groups whose generators are the vertices of a graph.

Let $\Delta = (X, E)$ be a graph. For any variety $\mathfrak{M}$ and any graph Δ the partially commutative group $G(\mathfrak{M}, \Delta)$ in the variety $\mathfrak{M}$ has a representation

$$G(\mathfrak{M}, \Delta) = \langle X \mid x_i x_j = x_j x_i, \text{ if } (x_i, x_j) \in E; \mathfrak{M} \rangle.$$

Although generally problems considered for free partially commutative groups and for partially commutative groups in soluble varieties are same, research methods differ significantly. Methods effectively used for researches into algebraic properties in partially commutative groups of soluble varieties are those using modules over group rings, splitting extensions, Fox derivatives, etc.

Let G be a group, $g, h \in G$. Then we use the following notation. $[g, h] = g^{-1}h^{-1}gh$, $G' = [G, G]$. The subgroup G' is called the commutant of G. The variety of metabelian groups $\mathfrak{A}^2$ is given by the identity $[[y_1, y_2], [y_3, y_4]] = 1$. It means that this variety consists of groups G having an abelian normal subgroup A (possibly trivial) such that the quotient group G/A is commutative. Denote a partially commutative group $G(\mathfrak{A}^2, \Delta)$ by M_Δ for short. Let $\mathfrak{N}_c$ be the variety of nilpotent groups of nilpotence degree at most c. This variety consists of all groups satisfying the identity $v_{c+1} = 1$, where $v_2 = [y_1, y_2], v_{c+1} = [v_c, y_{c+1}]$. For a graph Δ denote by $M_{c,\Delta}$ the partially commutative group defined by Δ in the variety $\mathfrak{A}^2 \cap \mathfrak{N}_c$.

Some results on partially commutative metabelian groups can be found in [38].

1.1. **Algebraic properties of partially commutative metabelian groups.**

Torsion. If $\mathfrak{N}_2 \subseteq \mathfrak{M}$ then the quotient group $G(\mathfrak{M}, \Delta)/G'(\mathfrak{M}, \Delta)$ has no elements of finite order.

Note that the periodic part of a group $G(\mathfrak{M}, \Delta)$ can be non-trivial. This is so, for example, if $\mathfrak{M}$ is a variety of centrally metabelian groups and Δ is a completely disconnected graph with at least four vertices (see [12]). The following theorem implies that partially commutative metabelian group has no elements of finite order.

Theorem 1. [36] *A group M_Δ can be approximated by nilpotent torsion-free groups.*

Center. Let $\Delta = (X; E)$ be a graph, Y a non-empty subset of the set X. We use the following notation.

$$Y^\perp = \{x \in X \mid (x, y) \in E \text{ for all } y \in Y\}. \tag{2}$$

Denote by $\langle Y \rangle$ the group generated by Y.

The following theorem describes the center of a partially commutative metabelian group, the quotient group by the center, and the relation of the center and the commutant.

Theorem 2. [36] *Let $\Delta = (X; E)$ be a graph. Then the following statements hold.*
1) If $X^{\perp}$ is non-empty then $\mathcal{Z}(M_\Delta) = \langle X^{\perp} \rangle$, otherwise the center of M_Δ is trivial.
2) If the subgraph Γ of Δ is generated by the set $X \backslash X^{\perp}$ then $M_\Delta / \mathcal{Z}(M_\Delta) \cong M_\Gamma$.
3) The intersection of the center $\mathcal{Z}(M_\Delta)$ and the commutant M'_Δ is trivial.

Centralizers. Let $\Delta = (X; E)$ be a graph. It is interesting to consider partially commutative groups of varieties $\mathfrak{M}$ such that $u, v \in X$ commute in the corresponding group if and only if u and v are adjacent. Suppose that $\mathfrak{M}$ contains $\mathfrak{N}_2$. It turns out that for $u, v \in X$ the commutator $[u, v]$ is equal to the identity in $G(\mathfrak{M}, \Delta)$ if and only if $(u, v) \in E$.

In Sec. 1, we denote by $\overline{G}$ the quotient group G/G' and by $\overline{g}$ the image of $g \in G$ in the group $\overline{G}$ via the natural homomorphism $G \to \overline{G}$.

Let G be a metabelian (non-abelian) group. Its commutant G' is a non-trivial abelian group and G acts on G' by conjugations: $c \mapsto g^{-1}cg$, for $g \in G$ and $c \in G'$. Since the elements in G' act identically G' is a right module on the integral group ring $\mathbb{Z}[\overline{G}]$. Denote the action of $\overline{g}$ on $c \in G'$ by $c^{\overline{g}}$. For elements $\alpha = l_1\overline{g}_1 + \ldots + l_m\overline{g}_m \in \mathbb{Z}[\overline{G}]$ and $c \in G'$ we put

$$c^{\alpha} = (c^{l_1})^{\overline{g}_1} \cdot \ldots \cdot (c^{l_m})^{\overline{g}_m}.$$

The centralizers of elements $x_i \in X$ and the centralizers in the commutant $\mathcal{C}(g) = C(g) \cap M'_\Delta$ of elements $g \in M_\Delta$ are described in the following theorem.

Theorem 3. [13, 36] *Let $X = \{x_1, \ldots, x_n\}$ be the set of vertices of the defining graph Δ of a group M_Δ and $\{x_1\}^{\perp} = \{x_2, \ldots, x_m\}$. The following statements hold.*
1) An element $g \in M_\Delta$ lies in the centralizer $C(x_1)$ of the element x_1 if and only if

$$g = x_1^{l_1} \ldots x_m^{l_m} \prod_{2 \le i < j \le m} [x_i, x_j]^{\alpha_{ij}},$$

where $l_1, \ldots, l_m \in \mathbb{Z}$, $\alpha_{ij} \in \mathbb{Z}[\overline{M_\Delta}]$.
2) For any $m \le n$, $1 \le i_1 < \ldots < i_m \le n$ and for any non-zero integers $q_1, \ldots, q_m$ the following equation holds

$$\mathcal{C}(x_{i_1}^{q_1} \ldots x_{i_m}^{q_m}) = \mathcal{C}(x_{i_1}) \cap \ldots \cap \mathcal{C}(x_{i_m}).$$

Let us notice a couple of useful properties of centralizers of elements in groups M_Δ defined by trees or cycles. These properties are used to study the universal theories of partially commutative metabelian groups. In [13], it was shown that the intersection of centralizers $\mathcal{C}(x_i) \cap \mathcal{C}(x_j)$ of two different elements $x_i, x_j \in X$ in M_Δ is trivial if Δ is a tree. If Δ is a cycle of length at least 4 then the intersection of centralizers $\mathcal{C}(x_i) \cap \mathcal{C}(x_j) \cap \mathcal{C}(x_l)$ of three different elements in X is trivial [14].

Annihilators. Let c be an element in M'_Δ. The *annihilator* $Ann(c)$ of c is the ideal of the ring $\mathbb{Z}[\overline{M}_\Delta]$, consisting of elements γ, such that $c^{\gamma} = 1$.

For any two non-adjacent vertices x_i, x_j of Δ define the ideal $\mathcal{A}_{i,j}$ of the ring $\mathbb{Z}[\overline{M}_\Delta]$ as follows. If x_i and x_j lie in different connected components of Δ then put $\mathcal{A}_{i,j} = 0$. Otherwise, let $a_i = x_i M'_\Delta$ and consider all paths $\{x_i, x_{i_1}, \ldots, x_{i_m}, x_j\}$ connecting x_i and x_j. To each path assign the element $(1 - a_{i_1}) \ldots (1 - a_{x_{i_m}}) \in A = \mathbb{Z}[\overline{M}_\Delta] = \mathbb{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}]$. Let $\mathcal{A}_{i,j}$ be the ideal generated by these elements.

Theorem 4. [13] *Let $\Delta = (X; E)$ be a graph with the set of vertices $X = \{x_1, \ldots, x_n\}$. If $(x_i, x_j) \notin E$ then $Ann([x_i, x_j]) = \mathcal{A}_{i,j}$.*

The *trivialization* of an element α in a group ring $\mathbb{Z}[G]$ is the image $\varepsilon(\alpha)$ of this element under the ring homomorphism

$$\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$$

extending the group homomorphism $G \to 1$. Useful properties of annihilators $\mathcal{A}_{ij}$ are given in the following theorem.

Theorem 5. [13] *Let $X = \{x_1, \dots, x_n\}$ be the set of vertices of a graph Δ and $A = \mathbb{Z}[\overline{M}_\Delta] = \mathbb{Z}[a_1^{\pm 1}, \dots, a_n^{\pm 1}]$. Then the following statements hold.*
1) If $n \geq 3$, $a \in A$, and $a(1-a_3)^2 \in \mathcal{A}_{1,2}$ then $a(1-a_3) \in \mathcal{A}_{1,2}$.
2) If $n \geq 2$, x_1, x_2 are non-adjacent vertices of Δ and $a, \gamma \in A$ are such that $\varepsilon(\gamma) \neq 0$ and $a\gamma \in \mathcal{A}_{1,2}$ then $a \in \mathcal{A}_{1,2}$.
3) if $n \geq 2$, $a \in A$, $q, q_1, \dots, q_m$ are non-zero integers, $1 < i_1 < \dots < i_m \leq n$, and

$$a(1 - a_1^q a_{i_1}^{q_1} \dots a_{i_m}^{q_m}) \in \mathcal{A}_{1,2}$$

then $a \in \mathcal{A}_{1,2}$.
4) if $n \geq 2$, $1 \leq m \leq n$, $q_1, \dots, q_m$ are nonzero integers, $1 \leq i_1 < \dots < i_m \leq n$, and

$$a(1 - a_{i_1}^{q_1} \dots a_{i_m}^{q_m}) \in \mathcal{A}_{1,2}$$

for an element $a \in A$ then all elements $a(1-a_{i_j})$ for $j = 1, \dots, m$ are in $\mathcal{A}_{1,2}$.

Let us recall the definition of an associator. Consider a right module L over a commutative ring A. A simple ideal P of A is associated with L if there exists an element $0 \neq x \in L$ such that the annihilator of this element

$$Ann(x) = \{a \in A \,|\, xa = 0\}$$

coincides with P. The set of ideals associated with the module L is the *associator* of L. In [14], for a partially commutative group M_Δ the associator $\mathbb{Z}[\overline{M}_\Delta]$ of the module M'_Δ is described.

Basis and canonical representation of elements. The authors of [13] provided a theorem on canonical representation of elements of a partially commutative metabelian group. However, the proof of this theorem had a mistake and this was noticed in the paper [36] of the second author of [13]. Later on, in [14, 36] theorems on a canonical representation of some elements in the commutant of M_Δ were proved. The presentation found there enabled to study the universal theory of a group M_Δ defined by a tree Δ. However, a complete proof of the theorem on a canonical representation of elements in partially commutative metabelian group was given by the second author of this survey and [13] only in 2020.

The following theorem describes a basis of the commutant of a partially commutative metabelian group. It implies a canonical representation of elements of a the group.

Theorem 6. [46] *Let the set $X = \{x_1, \dots, x_r\}$ of vertices of a graph Δ be ordered as follows $x_1 < x_2 < \dots < x_r$. Then a basis of the commutant M'_Δ is the set we denote by $\mathcal{B}'(M_\Delta)$ consisting of all elements v of the form*

$$v = [x_i, x_j]^{x_{j_1}^{t_1} \dots x_{j_m}^{t_m}}, \quad \{t_1, \dots, t_m\} \subset \mathbb{Z} \setminus \{0\}$$

such that the following conditions are satisfied:
(1) $j \leq j_1 < j_2 \dots < j_m \leq r$, $1 \leq j < i \leq r$;
(2) the vertices x_i and x_j are in different connected components of the graph Δ_v,

which is generated by all vertices of the set $\{x_i, x_j, x_{j_1}, \ldots, x_{j_m}\}$*;*
(3) $x_i = \max(\Delta_{v,x_i})$*, where* Δ_{v,x_i} *the connected component of the graph* Δ_v *containing* x_i*.*

Corollary 1. *Let* $\mathcal{B}'(M_\Delta)$ *be linearly ordered. Then any element* g *of the group* M_Δ *can be uniquely written in the form*

$$g = x_1^{\alpha_1} \ldots x_r^{\alpha_r} v_1^{\beta_1} \ldots v_m^{\beta_m},$$

where $\alpha_i, \beta_j \in \mathbb{Z}$ *and* $v_1 < \ldots < v_m,\ v_j \in \mathcal{B}'(M_\Delta)$*.*

Centralizer dimensions. The notion of the centralizer dimension was introduced by A. Myasnikov and P. Shumyatsky [21] for comparison of universal theories of groups. Suppose that a sequence

$$A_1 \subset A_2 \subset \ldots \subset A_n$$

of subsets of a group G is such that the chain of centralizers of these subsets

$$C(A_1) > C(A_2) \ldots > C(A_n)$$

is strictly descending. The *centralizer dimension* of a group G is the greatest n for which such subsets $A_1, A_2, \ldots, A_n$ of G exist if there is such an n. The centralizer dimension is denoted by $Cdim(G)$. If the greatest n does not exist then we write $Cdim(G) = \infty$.

In [2], it was shown that centralizer dimension of a free partially commutative nilpotent group of class 2 coincides with centralizer dimension of the free partially commutative group defined by the same graph.

It is known [21] that the centralizer dimension of a finitely generated metabelian group is finite. In papers [41, 42], properties of centralizer dimensions of partially commutative groups were studied and the exact value of $Cdim(M_\Delta)$, where Δ is a tree or a cycle, was found.

Theorem 7. [41] *Let* Δ *be a tree with at least 3 vertices. If* Δ *is a star then* $Cdim(M_\Delta) = 3$*. Otherwise,* $Cdim(M_\Delta) = 5$*.*

Let $g_1, \ldots, g_m \in M_\Delta$ be a finite system of elements and $\overline{g}_1, \ldots, \overline{g}_m$ the images of $g_1, \ldots, g_m$ in the free abelian group $\overline{M}_\Delta$ via the natural homomorphism $M_\Delta \to \overline{M}_\Delta$. The rank of the system $g_1, \ldots, g_m$ is the rank of the subgroup generated by $\overline{g}_1, \ldots, \overline{g}_m$.

Let M_Δ be a non-abelian group. Define a parameter $\alpha(M_\Delta)$ for M_Δ as follows. Assume $\mathcal{Z}(M_\Delta) = 1$. In [42], the following statement has been proved. Let the defining graph Δ of a partially commutative metabelian group G have n vertices and $\mathcal{Z}(G) = 1$. Then if rank of a system $\{g_1, \ldots, g_m\}$ is equal to n then the centralizer $C(g_1, \ldots, g_m)$ is trivial. Put $\alpha(M_\Delta) = a$, where a is the largest integer such that for any system of elements $g_1, \ldots, g_m \in M_\Delta$ of rank at least a the centralizer $C(g_1, \ldots, g_m)$ is trivial. If the center of M_Δ is non-trivial then $M_\Delta / \mathcal{Z}(M_\Delta)$ is a partially commutative group with no center. In this case, put

$$\alpha(M_\Delta) = \alpha(M_\Delta / \mathcal{Z}(M_\Delta)).$$

Define a parameter β for a group M_Δ as follows. Let b be the least natural number such that for any distinct vertices $x_{i_1}, \ldots, x_{i_b}$ of Δ the intersection

$$\mathcal{C}(x_{i_1}) \cap \ldots \cap \mathcal{C}(x_{i_b})$$

is trivial. Then put $\beta(M_\Delta) = b$.

Theorem 8. [42] *Let M_Δ be a non-abelian group. Then*

$$Cdim(M_\Delta) \leq \alpha(M_\Delta) + \beta(M_\Delta) + 1.$$

Corollary 2. [42] *For any group M_Δ the following equation holds:*

$$Cdim(M_\Delta) \leq 2n + 1,$$

where n is quantity of vertices of Δ.

The following theorem shows that the value of centralizer dimension is not bounded by a function of rank of a group and this value can be arbitrarily large even in the case of a two-generated metabelian group.

Theorem 9. [42] *For any $n \in \mathbb{N}$ there exists a two-generated untwisted metabelian group of centralizer dimension at least n.*

By Theorem 7, centralizer dimensions of partially commutative metabelian groups defined by trees are bounded as well as centralizer dimensions of partially commutative groups defined by cycles.

Theorem 10. [42] *If M_Δ is a partially commutative group defined by a cycle of length at least 5 then $Cdim(M_\Delta) = 7$.*

By analogy with $Cdim(G)$ the centralizer dimension in commutant $\mathcal{C}dim(G)$ is defined. To define $\mathcal{C}dim(G)$ centralizers in commutant $\mathcal{C}(A_i) = C(A_i) \cap G'$ are considered instead of centralizers $C(A_i)$. Centralizer dimensions in commutant can also be used for a comparison of universal theories.

In [41], the centralizer dimensions $\mathcal{C}dim$ are defined for partially commutative groups defined by trees and cycles.

Inclusions, subgroups, retracts. For any variety $\mathfrak{M}$ and any graph Δ the following statement holds. Let Δ be a graph and Γ its subgraph generated by a set of vertices $Y \subseteq V(\Delta)$. Then there exists a retraction of $G(\mathfrak{M}, \Delta)$ onto the group $G(\mathfrak{M}, \Gamma)$ such that this retraction is identical on Y and takes all other elements in $V(\Delta)$ to the identity.

Let us describe a couple of embeddings of partially commutative metabelian groups into a group of matrices. They allied to the Magnus embedding. This embedding is very important in theory of soluble groups. Recall the definition of the Magnus embedding for a free metabelian group M_n of rank n for $n \geq 2$. Let $X = \{x_1, \ldots, x_n\}$ be a basis of M_n, A_n a free abelian group with a basis $\{a_1, \ldots, a_n\}$, $B = \mathbb{Z}[A_n]$, and F a free right B-module with a basis $\{f_1, \ldots, f_n\}$. Consider the matrix group

$$W_n = \begin{pmatrix} A_n & 0 \\ F & 1 \end{pmatrix}.$$

The map

$$\mu : x_i \mapsto \begin{pmatrix} a_i & 0 \\ f_i & 1 \end{pmatrix}, \text{ for } i = 1, \ldots, n,$$

is extended up to the Magnus embedding μ of the group M_n to the group W_n.

The embedding μ takes a commutator $[x_i, x_j]$ to the matrix

$$\begin{pmatrix} 1 & 0 \\ \tau_{ij} & 1 \end{pmatrix}, \text{ where } \tau_{ij} = f_i(a_j - 1) + f_j(1 - a_i).$$

Let $\Delta = (X; E)$ be a graph and R_Δ a normal subgroup generated by all commutators $[x_p, x_q]$ such that $(x_p, x_q) \in E$. Then μ maps R_Δ to the submodule L of the module F such that L is generated by all τ_{pq} for which $(x_p, x_q) \in E$. Let $T = F/L$. The Magnus embedding μ of M_n to W_n induces an embedding μ_Δ of the group M_Δ to the group of matrices

$$W_\Delta = \begin{pmatrix} A_n & 0 \\ T & 1 \end{pmatrix}. \tag{3}$$

In [43] the existence of one more embedding of a group M_Δ to a group of matrices was shown.

Theorem 11. [43] *Let Δ be a connected graph, $\{x_1, \dots, x_n\}$ the set of vertices of Δ and a basis of the free metabelian group M_n, a_i the image of x_i under the natural homomorphism $M_n \to A_n = M_n/M_n'$, and $\delta = (a_1 - 1) \cdot \ldots \cdot (a_n - 1) \in B$. Then the group M_Δ is embeddable to the group of matrices*

$$\overline{W}_\Delta = \begin{pmatrix} A_n & 0 \\ T/T\delta & 1 \end{pmatrix}, \tag{4}$$

where the ring B and the module $T = F/L$ are defined above.

Groups W_Δ and $\overline{W}_\Delta$ are splittable. For this reason, they are preferable for a study of universal theories. This will be discussed in Subsection 1.2.

Let us present some theorems on subgroups. In [36] it was shown that if a group M_Δ is nilpotent then it is abelian. The following theorem states even more.

Theorem 12. [43] *Any nilpotent subgroup of M_Δ is abelian.*

The *Fitting subgroup $Fit(G)$* of a group G is the product of all nilpotent normal subgroups of G.

Theorem 13. [43] *The Fitting subgroup of a group M_Δ is equal to the direct product of the center and the commutant of this group.*

Let $G = (X \mid R, \mathfrak{A}^2)$ and $H = (Y \mid S, \mathfrak{A}^2)$ be represented in the variety of metabelian groups by generators and defining relations. If $X \cap Y = \varnothing$ then the group $T = (X \sqcup Y \mid R \sqcup S, \mathfrak{A}^2)$ is called the metabelian product of G and H. Let us denote the metabelian product of metabelian groups $G_1, \dots, G_n$ by $M(G_1, \dots, G_n)$.

Theorem 14. [39] *Any partially commutative metabelian group is a subgroup of a direct product of some finite (possibly empty) set of free abelian groups A_i and some finite (possibly empty) set of metabelian products $M_j = M(B_1^{(j)}, \dots, B_{r_j}^{(j)})$ of free abelian groups $B_1^{(j)}, \dots, B_{r_j}^{(j)}$.*

Automorphisms. A vertex x of a graph Δ is called an end-point if its degree is equal to 1.

An automorphism α of a group G is called an IA-automorphism if this automorphism acts identically on the quotient group $\overline{G}$. The group of IA-automorphisms is denoted by $IAut(G)$.

Theorem 15. [45] *Suppose that a graph Δ has no cycles. If an IA-automorphism α of the group M_Δ fixes all end-points and isolated vertices of Δ then α is the identical automorphism.*

Both requirements are essential. If there is a cycle in Δ or α is not identical on the quotient group by the commutant then Theorem 15 does not hold.

Each automorphism α of the group M_Δ induces an automorphism $\overline{\alpha}$ of the free abelian group $\overline{M}_\Delta = M_\Delta/M'_\Delta$. The group of induced automorphisms is called the group of quotient automorphisms of the group $\overline{M}_\Delta$ and is denoted by $\mathcal{F}_\Delta$. Clearly, $\mathcal{F}_\Delta \cong Aut(M_\Delta)/IAut(M_\Delta)$.

In [45], a description of a group of matrices $\mathcal{M}_\Delta$ is given. This description uses the order on the set X of the vertices of Δ. This order is defined in the same work and it depends not only on X but also on the structure of Δ. It induces the order on the basis of the group $\overline{M}_\Delta$. An automorphism $\overline{\alpha}$ of the group $\overline{M}_\Delta$ is called a matrix automorphism if its matrix $[\overline{\alpha}]$ in the chosen basis belongs to the group of matrices $\mathcal{M}_\Delta$. The following statement holds.

Theorem 16. [45] *Let Δ be a graph with no cycles. Then each quotient group automorphism of the group M_Δ can be written as a product of an automorphism of the graph Δ and a matrix automorphism.*

Groups G and H are called *commensurable* if there exist subgroups G_1 and H_1 of finite index in the groups G and H respectively such that $G_1 \cong H_1$.

Let a linear group A be $\mathbb{Q}$-definable. This means that $A \leq GL(n,\mathbb{C})$ and its basic set is defined by a system of equations with coefficients in $\mathbb{Q}$. A subgroup $B \leq A \cap GL(n,\mathbb{Q}) = A_\mathbb{Q}$ is called an arithmetic group or an arithmetic subgroup of A if it is commensurable with $A_\mathbb{Z} = A \cap GL(n,\mathbb{Z})$.

Corollary 3. [45]. *Let Δ be a graph with no cycles. Then the group $\mathcal{F}_\Delta$ of $\overline{M}_\Delta$ is arithmetic.*

Let us give some more information on automorphisms of partially commutative metabelian groups (see [40] for details). Let $\Delta = (X; E)$, be a graph with the set of vertices $X = \{x_1, \ldots, x_n\}$. In [19], Laurence defined four sets of automorphisms generating the group $Aut(F_\Delta)$.

(1) The set of graph automorphisms, namely the set of elements in $Aut(F_\Delta)$ such that these elements are induced by automorphisms $\pi : \Delta \to \Delta$ of the graph Δ.

(2) The set of inverting automorphisms $\alpha \in Aut(F_\Delta)$. These are automorphisms taking one of the vertices $x_i \in X$ to x_i^{-1} and fixing all other vertices.

(3) Consider two distinct vertices x_i, x_j, such that $(x_j, x) \in E$ implies $(x_i, x) \in E$ for any $x \in X$, $x \neq x_i$. The third set consists of transvections taking x_j to $x_j x_i^{\pm 1}$ or to $x_i^{\pm 1} x_j$ and fixing all other vertices.

(4) The fourth set consist of locally interior automorphisms defined as follows. Let $x_i \in X$. Consider the subgraph Γ obtained by deleting x_i, all vertices adjacent to x_i, and all edges incident to deleted vertices. Let Λ be a union of some connected components of Γ. Then define $\beta \in Aut(F_\Delta)$ setting $\beta(x_j) = x_i^{-1} x_j x_i$ for $x_j \in \Lambda$ and $\beta(x_j) = x_j$ for $x_j \notin \Lambda$.

It follows from [47] that if Δ has at least three connected components and each of these components is a complete graph then the group $Aut(M_\Delta)$ is not generated by automorphisms induced by the Laurence automorphisms.

In [40], a stronger result was obtained. Namely, if Δ is a connected graph or even a tree then the group $Aut(M_\Delta)$ can contain automorphisms not induced by automorphisms of the partially commutative group F_Δ. In the same paper, a monoid $\mathcal{P}$ of matrices over a ring $\mathbb{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}]$ of integer Laurent polynomials and a congruence $\approx$ on this monoid are defined in such a way that the group of automorphisms acting identically on the quotient group $\overline{M}_\Delta$ is isomorphic to the quotient monoid of $\mathcal{P}$ by $\approx$.

The structure of the group of automorphisms of partially commutative class two nilpotent group was studied in [32].

Direct decompositions. A group G is *decomposable into a direct product* if $G = A \times B$ for some groups $A \neq 1$ and $B \neq 1$. In [33], the question on existence of direct decomposition for partially commutative groups in varieties containing $\mathfrak{N}_2$ was studied. Two theorems were proved.

Theorem 17. [33] *Let $\mathfrak{M}$ be a variety of groups such that this variety contains $\mathfrak{N}_2$. Suppose that a group $G = G(\mathfrak{M}, \Gamma)$ decomposes into a direct product $H \times A$, where A is an abelian group. Then there exists a subgraph Δ of the graph Γ such that the set of vertices of Δ contains $X \backslash X^\perp$ and $H \cong G(\mathfrak{M}, \Delta)$.*

Theorem 18. [33]. *Let $\mathfrak{M}$ be a variety of soluble groups such that this variety contains $\mathfrak{N}_2$. If a graph Δ is not connected then the group $G(\mathfrak{M}, \Delta)$ is not decomposable into a direct product.*

1.2. **Logical properties of partially commutative metabelian groups.**

Elementary equivalence and isomorphism. In [16], it was shown that two partially commutative associative algebras are isomorphic if and only if so are their defining graphs. Using this result, C. Droms [8] proved an analogous one for partially commutative groups in the variety of all groups. He established the following fact.

Theorem 19. [8] *If $\mathfrak{M}$ contains the variety $\mathfrak{N}_2$ then $G(\mathfrak{M}, \Gamma)$ and $G(\mathfrak{M}, \Delta)$ are isomorphic if and only if so are the graphs Γ and Δ.*

The following theorem provides a criterion of coincidence of elementary theories of a partially commutative group in a nilpotent variety containing $\mathfrak{N}_2$ and an arbitrary group.

Theorem 20. [33] *Let a variety of nilpotent groups $\mathfrak{M}$ contain $\mathfrak{N}_2$. Then if a finitely generated group H has the same elementary theory as $G(\mathfrak{M}, \Delta)$ then $G(\mathfrak{M}, \Delta) \cong H$.*

Theorem 20 implies the following result.

Corollary 4. [33] *Let $G = G(\mathfrak{M}, \Gamma)$ and $H = G(\mathfrak{M}, \Delta)$ be groups in a variety $\mathfrak{M}$ of nilpotent groups such that $\mathfrak{M}$ contains $\mathfrak{N}_2$. Then the elementary theories of the groups G and H coincide if and only if $\Gamma \cong \Delta$.*

An elementary theory $Th(G)$ is called *soluble* if there is an effective procedure checking for any sentence Φ if this sentence belongs to $Th(G)$.

In [22], G. A. Noskov proved that the elementary theory of an almost soluble group is soluble if and only if the group is almost abelian. So, if a variety $\mathfrak{M}$ is soluble and $\mathfrak{N}_2 \subseteq \mathfrak{M}$ then the elementary theory of a group $G(\mathfrak{M}, \Delta)$ is not soluble.

Universal theories. One of the reasons for research into centralizer dimensions $Cdim$ and $\mathcal{C}dim$ is the coincidence of universal theories of groups implies the equality of their centralizer dimensions.

In Subsection 1.1, inclusions (3) and (4) of a group M_Δ into the groups of matrices W_Δ and $\overline{W}_\Delta$ were defined. In [13], it was shown that the universal theories of these groups of matrices are soluble. But the universal theory of M_Δ coincides with no universal theories of groups of matrices. This result was obtained in [42].

Therefore we can only say that a group M_Δ is embeddable into a metabelian group with a soluble universal theory. The problem on the solubility of the universal theory of a group M_Δ has not been solved yet. It is included in the Kourovka Notebook [18]. It is known that the universal theory of a free abelian group is soluble. In [5, 6], O. Chapius proved that the universal theory of a free metabelian group is also soluble. Obviously, groups with soluble universal theories can be obtained from free abelian and free metabelian groups by using the direct product of groups. For instance, if Δ_4 is the 4-cycle then the universal theory of the group G_{Δ_4} is soluble, since it is isomorphic to the direct product of two free 2-generated metabelian groups. One can find a non-trivial example of a partially commutative metabelian group having a soluble universal theory. So, in [41], it was shown that if Γ_4 is the linear graph on four vertices then the universal theory of partially commutative metabelian group M_{Γ_4} is soluble. The proof follows from the coincidence of universal theories of groups M_{Δ_4} and M_{Γ_4}.

In [3], the problem on the universal equivalence of partially commutative metabelian groups with acyclic defining graphs was considered. The following theorem was proved.

Theorem 21. [3] *Suppose that the graph Δ^* is obtained from a graph Δ by deleting all end-points and the edges incident to the end-points. Let $\mathfrak{A}_p$ be the variety of abelian groups of exponent p, where p is a prime number or 0. If Δ and Γ are graphs with no cycles such that each connected component of these graphs has at least three vertices then the groups $G(\mathfrak{A}_p\mathfrak{A}, \Gamma)$ and $G(\mathfrak{A}_p\mathfrak{A}, \Delta)$ are universally equivalent if and only if $\Gamma^* \cong \Delta^*$.*

If any connected component of Δ of Γ contains less then three vertices then the corresponding connected component in Δ^* (correspondingly in Γ^*) is empty and the statement of Theorem 21 does not hold.

To prove Theorem 21, generalizations many algebraic properties of a group M_Δ to groups $G(\mathfrak{A}_p\mathfrak{A}, \Delta)$ were used. These generalizations were obtained in [3].

The study of partially commutative metabelian group was continued in [14]. In this paper, an equivalence relation on the set of vertices of a graph Δ is defined. Then, an adjacency relation is determined on the set of equivalence classes. So, a new graph $\Delta^{\maltese}$, appears. This graph is called the *compression* of the initial one and it is usually simpler then Δ. Let us give a strict definition of the compression of a graph. We say that two vertices x and y of a graph Γ are equivalent and write $x \sim y$ if $x^\perp \cup \{x\} = y^\perp \cup \{y\}$ ($x^\perp$ is defined in (2)). Note that equivalent vertices are always adjacent. Then the compression of Δ is the quotient graph $\Delta^{\maltese} = \Delta/\sim$.

An end-point z in [15] is called *bad* if there exists a vertex x adjacent to z and adjacent to at least to two vertices y and v, where y is also an end-point.

Denote by Δ' the graph obtained by deleting one-by-one all bad vertices and the edges incident to them.

Theorem 22. [14, 15] *For any graph Δ the universal theories of the groups M_Δ and $M_{\Delta^\star}$ coincide as well as the universal theories of M_Δ and $M_{\Delta'}$.*

The paper [14] gives an example of graph Δ such that this graph is not a tree while its compression $\Delta^\star$ is. For this reason, the universal theory of a group defined by a tree can coincide with the universal theory of a group defined by a graph with cycles. Theorem 22 implies that the condition of acyclicity of the defining graph in Theorem 21 is essential even for partially commutative groups in the variety $\mathfrak{A}^2$.

It follows from Theorem 22 that the universal theory of metabelian product of free abelian groups coincides with one of a free metabelian group.

For partially commutative metabelian groups defined by cycles the following theorem holds.

Theorem 23. [14]. *If $n, m \geq 3$ then groups M_{Δ_n} and M_{Δ_m} defined by cycles of lengths n and m respectively are universally equivalent if and only if $n = m$.*

Let Δ be a graph with the set of vertices $X = \{x_1, \dots, x_n\}$. Denote by $\varphi(\Delta)$ the following sentence.

$$\exists v_1 \dots v_n (\bigwedge_{(x_i,x_j)\in\Delta} [v_i, v_j] = 1 \wedge \bigwedge_{(x_i,x_j)\notin\Delta} [v_i, v_j] \neq 1 \wedge \bigwedge_{i\neq j} v_i \neq v_j \wedge \bigwedge_{i=1,n} v_i \neq 1).$$

V. N. Remeslennikov formulated the following conjecture.

Let $\mathfrak{M}$ be a variety of groups. If the universal theories of the groups $F(\mathfrak{M}, \Gamma_1)$ and $F(\mathfrak{M}, \Gamma_2)$ are distinct then there exist a graph Δ such that the sentence $\varphi(\Delta)$ is true on one of these groups and is false on the other one.

In [20], the affirmative solution of this conjecture was obtained for partially commutative nilpotent R-groups of class 2, where R is a binomial ring. Let G and H be two partially commutative nilpotent R-groups of class 2 and Δ and Γ defining graphs of G and H respectively. It turns out that if G and H are not universally equivalent then their universal theories differ in $\varphi(\Gamma)$ or $\varphi(\Delta)$.

However, for partially commutative metabelian groups the analogous result does not hold. A counterexample was obtained in [41]. It is not known if the conjecture holds for the variety of metabelian groups. Nevertheless, if only formulas of the form $\varphi(\Delta)$, where Δ is a tree, are considered then the corresponding result is not true. In [41], the second author of this survey has found two groups M_{Γ_1} and M_{Γ_2} such that these groups have distinct universal theories while for any tree T the corresponding formula $\varphi(T)$ is true on one of these groups if and only if it is true on the other one.

Quasi-varieties. A sentence of the type

$$\forall z_1 \dots, z_m((w_1(z_1, \dots, z_m) = 1 \wedge \dots \wedge w_r(z_1, \dots, z_m) = 1) \longrightarrow w(z_1, \dots, z_m) = 1),$$

where w and w_i are group words is called a *quasi-identity.* A class of groups satisfying a collection of quasi-identities is called a *quasi-variety.* A non-empty class of groups form a quasi-variety if and only if this class is closed with respect to taking subgroups, cartesian products, and ultra-products. Denote by $qvar(G)$ the quasi-variety generated by a group G. A class of groups closed with respect to taking subgroups and cartesian products is called a pre-variety.

In [39], it was shown that there exist free partially commutative groups F_{Δ_i} such that

$$qvar(F_{\Delta_1}) \subset qvar(F_{\Delta_2}) \subset \dots \subset qvar(F_{\Delta_n}) \subset \dots,$$

and all inclusions in this infinite chain are strict. This is not so for partially commutative metabelian groups. For them, the following theorem holds.

Theorem 24. [39] *Any two non-abelian partially commutative metabelian groups generate equal quasi-varieties*

The same result takes place for pre-varieties.

Theorem 25. [39] *Any two non-abelian partially commutative metabelian groups generate equal pre-varieties.*

The *positive universal theory* of a group G is the set of all sentences Φ of the form

$$\Phi = \forall z_1 \dots z_m \left(\bigvee_{i \in I} \bigwedge_{j_i \in J_i} w_{j_i}(z_1, \dots, z_m) = 1 \right),$$

such that these sentences are true on G, where $w_{j_i}(z_1, \dots, z_m)$ are group words.

Let us denote the positive universal theory of a group G by $Th^+_\forall(G)$.

The following theorem shows that not only quasi-varieties and pre-varieties generated by partially commutative metabelian groups coincide but also positive universal theories of such groups do.

Theorem 26. [39] *Let M_Γ and M_Δ be non-abelian groups. Then $Th^+_\forall(M_\Gamma) = Th^+_\forall(M_\Delta)$.*

Equations. In Subsection 1.1, the inclusion of a group M_Δ into the corresponding group of matrices W_Δ was defined. In [44], it was shown that, in general, the universal theories of groups M_Δ and W_Δ are distinct. This result was obtained by comparing centralizer dimensions of these groups. However, the groups M_Δ and W_Δ have some common properties allied to their universal theories. Namely, the following theorem was proven.

Theorem 27. [44] *An equation*

$$g_1 x^{m_1} \dots g_l x^{m_l} = 1, \; g_i \in M_\Delta,$$

is solvable in M_Δ if and only if it is so in W_Δ.

The analogue of Theorem 27 does not hold for equations of two unknowns. Moreover, this analogue does not hold even for a totally disconnected graph Δ.

1.3. **Partially commutative metabelian nilpotent groups.**

Mal'cev basis. Let $\mathfrak{N}_{2,c}$ be the intersection of the variety of metabelian groups $\mathfrak{A}^2$ with the variety of nilpotent groups $\mathfrak{N}_c$.

We introduce some notation from [37]. In this paper, a basis of a group $M_{c,\Delta} = G(\mathfrak{N}_{2,c}, \Delta)$ is constructed. Let G be a finitely generated nilpotent torsion-free group. As is known, G has a central series

$$G = G_1 > G_2 > \dots > G_{s+1} = 1$$

with infinite cyclic quotient groups. Let us choose elements $a_1, \dots, a_s \in G$ such that $G_i = \langle a_i, \; G_{i+1} \rangle$. The ordered system of elements $(a_1, \dots, a_s)$ is called a *Mal'cev basis* of the group G. Each element $g \in G$ can be written in the form

$$g = a_1^{t_1} \dots a_s^{t_s}, \quad t_i \in \mathbb{Z}$$

uniquely.

Let $\Delta = (X; E)$ be a graph with the set of vertices $X = \{x_1, \dots, x_n\}$ and $v(x_{i_1}, \dots, x_{i_m})$ a representation of an element $v \in M_{c,\Delta}$ via generators in X, where vertices $x_{i_1}, \dots, x_{i_m}$ occur in this representation. Then set $\sigma(v) = \{x_{i_1}, \dots, x_{i_m}\}$. Note that, $\sigma(v)$ depends not only on v but also on a specific representation via generators of the group. Denote by Δ_v the subgraph of Δ generated by the set $\sigma(v)$. The connected component of the graph Δ_v containing a vertex $x \in \sigma(v)$ is denoted by $\Delta_{v,x}$. Let us order the set X as $x_1 < x_2 < \ldots < x_r$. Denote the greatest vertex in a connected component $\Delta_{v,x}$ by $\max(\Delta_{v,x})$. Define the commutator $c_m = [y_1, y_2, \ldots, y_m]$ by induction: $c_2 = [y_1, y_2]$, $c_m = [c_{m-1}, y_m]$. Let $\mathcal{B}'(M_{c,\Delta})$ be the set of commutators of the form

$$v = [x_{j_1}, x_{j_2}, \ldots, x_{j_m}], \quad 2 \leq m \leq c$$

in a group $M_{c,\Delta}$ such that the following conditions are satisfied:
(1) $1 \leq j_2 \leq j_3 \leq \ldots \leq j_m \leq r,\ j_2 < j_1 \leq r$;
(2) the vertices x_{j_1} and x_{j_2} are in different connected components of the graph Δ_v;
(3) $x_{j_1} = \max(\Delta_{v,x_{j_1}})$.

Theorem 28. [37] *The set of elements $\mathcal{B}(M_{c,\Delta}) = X \sqcup \mathcal{B}'(M_{c,\Delta})$ is a Mal'cev basis of the group $M_{c,\Delta}$.*

The canonical representation from Theorem 28 is used in [15] for study of algebraic properties and the universal theory of a group $G = M_{c,\Delta}$. Let us present the main results of this paper. The following theorem is similar to Theorem 5 on annihilators of partially commutative metabelian groups and it uses the ideals $\mathcal{A}_{ij}$ defined in Theorem 5. Let $\mathcal{I}$ be the augmentation ideal of a ring $\mathbb{Z}[\overline{G}]$, i.e. the kernel of the natural homomorphism $\mathbb{Z}[\overline{G}] \to \mathbb{Z}$.

Theorem 29. [15] *Let x_i and x_j be two non-adjacent vertices of a graph Δ. Then the annihilator of the commutator $[x_i, x_j]$ in the group $M_{c,\Delta}$ is equal to $\mathcal{A}_{i,j} + \mathcal{I}^{c-1}$.*

Let us present a theorem on the centralizers of elements of a group $M_{c,\Delta}$. Denote by $C(g)$ the centralizer of g and by $\mathcal{C}(g)$ the centralizer of g in the commutant, namely the set

$$\mathcal{C}(g) = C(g) \cap M'_{c,\Delta}.$$

As usual, let $\gamma_m(G)$ denote the mth element of the lower central series of G.

Theorem 30. [15] *Let $X = \{x_1, \ldots, x_n\}$ be the set of vertices of a graph Δ and $G = M_{c,\Delta}$. Then the following conditions hold.*
1) If x_n is an isolated vertex then

$$C(x_n) = \langle x_n \rangle \times \gamma_c(G).$$

2) If x_n is adjacent to only one vertex (say to x_{n-1}) then

$$C(x_n) = \langle x_{n-1} \rangle \times \langle x_n \rangle \times \gamma_c(G).$$

3) If x_n is adjacent to vertices $x_{r+1}, \ldots, x_{n-1}$, where $r \leq n-3$, then $C(x_n)$ consists of all elements of the form

$$\prod_{p=r+1}^{n} {x_p}^{l_p} \cdot \prod_{r+1 \leq i < j \leq n-1} [x_i,\, x_j]^{f_{i,j}} \cdot \gamma_c(G),$$

where $l_p \in \mathbb{Z},\ f_{i,j} \in \mathbb{Z}[\overline{G}]$.

The following theorem can be used to find the centralizers in commutant for any elements of a group $M_{c,\Delta}$.

Theorem 31. [15] *Let $\Delta = (X; E)$ be a graph with the set of vertices $X = \{x_1, \ldots, x_n\}$ and $G = M_{c,\Delta}$. Then*

$$\mathcal{C}(g x_{i_1}^{l_1} \ldots x_{i_t}^{l_t}) = \bigcap_{j=1}^{t} \mathcal{C}(x_{i_j})$$

for any integers $1 \leq i_1 < \ldots < i_t \leq n$, any non-zero integers $l_1, \ldots, l_t$, and any element g in the commutant G'.

To study the universal theory of a group $M_{c,\Delta}$ the following theorem is useful.

Theorem 32. [15] *Let $\Delta = (X; E)$ be a tree, u, v two distinct vertices of X. Then in the group $G = M_{c,\Delta}$ the following identity holds:*

$$\mathcal{C}(u) \cap \mathcal{C}(v) = \gamma_c(G).$$

Theorem 32 implies a description of the center of a partially commutative nilpotent metabelian group.

Theorem 33. [15] *Let $\Delta = (X; E)$. Then the center of $G = M_{c,\Delta}$ is the direct product of the group $\gamma_c(G)$ and the cyclic groups generated by vertices $x_i \in X^{\perp}$.*

Let us move on to the results on the universal theory of a group $M_{c,\Delta}$. The definition of the graph Δ' is given in Subsec. 1.2 before Theorem 22, the graph Δ^* is defined in Theorem 21.

Theorem 34. [15] *Let Δ be a graph. The groups $M_{c,\Delta}$ and $M_{c,\Delta'}$ have the same universal theories.*

Theorem 35. [15]. *Let Γ_1 and Γ_2 be trees. The groups M_{c,Γ_1} and M_{c,Γ_2} have the same universal theories if and only if the graphs Γ_1^* and Γ_2^* are isomorphic.*

Theorem 35 is an analogue of Theorem 21.

1.4. **Partially commutative metabelian pro-p-groups.** In [1], centralizers of elements and annihilators of commutators in partially commutative metabelian pro-p-groups were studied. The results obtained for partially commutative metabelian pro-p-groups are similar to those for partially commutative metabelian abstract groups in [13]. In this subsection, we are talking about pro-p-groups. So, by a subgroup, a homomorphism, a generating set we mean a closed subgroup, a continuous homomorphism, a generating set in the topological sense, respectively. Denote by P a free metabelian pro-p-group and by P_Δ the partially commutative metabelian pro-p-group defined by a graph $\Delta = (X; E)$. Let $X = \{x_1, \ldots, x_n\}$. The quotient group of P_Δ by its commutant P'_Δ is a free abelian pro-p-group A with a basis $\{a_1, \ldots, a_n\}$, where a_i is an image of x_i via the natural homomorphism $P_\Delta \to P_\Delta/P'_\Delta$. This group is isomorphic to the direct sum of n copies of the additive group of the ring of integer p-adic numbers $\mathbb{Z}_p$. The action of P_Δ on P'_Δ by conjugation

$$x \to x^g = g^{-1}xg$$

defines a structure of a right module on P'_Δ over the augmented group algebra $\mathbb{Z}_p[[A]]$. This algebra is identified with the power series algebra $\mathbb{Z}_p[[y_1, \ldots, y_n]]$, where $y_i = a_i - 1$. Similarly, P' is a module over the algebra $\mathbb{Z}_p[[y_1, \ldots, y_n]]$. For this reason, any element $f \in P$ can be written in the form

$$f = x_1^{l_1} \ldots x_n^{l_n} \prod_{1 \leq i < j \leq n} [x_i, x_j]^{\alpha_{ij}},$$

where $l_i \in \mathbb{Z}_p$, $\alpha_{ij} \in \mathbb{Z}_p[[y_1, \ldots, y_n]]$.

For a graph Δ and any of its vertices x_i and x_j, let us define the ideal $\mathcal{A}_{i,j}$ of the algebra $\mathbb{Z}_p[[y_1, \ldots, y_n]]$ as it was made for a partially commutative metabelian group and for a partially commutative metabelian nilpotent group. Namely, if the vertices x_i and x_j lie in different connected components of Δ then set $\mathcal{A}_{i,j} = 0$. If the vertices x_i and x_j lie in the same connected component then consider each path $(x_i, x_{i_1}, \ldots, x_{i_r}, x_j)$ between these vertices. To each such path assign the product $y_{i_1} \ldots y_{i_r}$ if length of the path is greater than 1 and 1 otherwise. By definition, the ideal $\mathcal{A}_{i,j}$ is generated by all such elements. In particular, if x_i, x_j are adjacent then $\mathcal{A}_{i,j}$ contains 1. So, this ideal coincides with the entire algebra $\mathbb{Z}_p[[y_1, \ldots, y_n]]$.

Let us formulate the main results of paper [1].

Theorem 36. [1] *Let $x_1, \ldots, x_n$, where $n \geq 2$, be vertices of the defining graph Δ of a partially commutative metabelian pro-p-group P_Δ. Then for $i \neq j$, the annihilator of the commutator $[x_i, x_j]$ in the algebra $\mathbb{Z}_p[[y_1, \ldots, y_n]]$ coincides with the ideal $\mathcal{A}_{i,j}$.*

Theorem 37. [1] *Let $x_1, \ldots, x_n$, where $n \geq 2$, be vertices of the defining graph Δ of a partially commutative metabelian pro-p-group P_Δ and let $x_2, \ldots, x_m$ be all the vertices adjacent to x_1. An element $g \in P_\Delta$ lies in the centralizer of x_1 if and only if it can be written in the form*

$$g = x_1^{l_1} \ldots x_m^{l_m} \prod_{2 \leq i < j \leq m} [x_i, x_j]^{\gamma_{i,j}},$$

where $l_i \in \mathbb{Z}_p,\ \gamma_{i,j} \in \mathbb{Z}_p[[y_1, \ldots, y_n]]$.

The following theorem has not been published yet. It is analogous to Theorem 6 and gives a description of a basis for the commutant of a partially commutative metabelian pro-p-group.

Theorem 38. *Let the set $\{x_1, \ldots, x_n\}$ of vertices of a graph Δ be ordered. Then a basis $\mathcal{B}(P'_\Delta)$ of the commutant P'_Δ over $\mathbb{Z}_p$ is the set of all elements w of the form*

$$w = [x_i, x_j]^{y_{j_1}^{s_1} \ldots y_{j_m}^{s_m}}, \{s_1, \ldots, s_m\} \subset \mathbb{N},$$

such that the following conditions are satisfied:
1) $x_j \leq x_{j_1} < \ldots < x_{j_m},\ x_j < x_i$;
2) the vertices x_i, x_j are in different connected components of graph Δ_w generated by all vertices of the set $\{x_i, x_j, x_{j_1}, \ldots, x_{j_m}\}$;
3) $x_i = \max\{\Delta_{w,x_i}\}$, where Δ_{w,x_i} the connected component of the graph Δ_w containing x_i.

2. Partially commutative Lie algebras

Research into partially commutative Lie algebras only began about 30 years ago and these algebras have not been studied so intensively as partially commutative groups.

Since partially commutative groups and Lie algebras are rather similar objects, many results for groups have analogues for Lie algebras. Moreover, some methods for studying Lie algebras come from those for studying groups. Nevertheless, there are specific methods for researching Lie algebra.

Let $\Gamma = (A; E)$ be a graph with a (finite or infinite) set of vertices $A = \{a_1, a_2, \ldots\}$ and a set of edges E. If a_i and a_j are adjacent in Γ then we write $a_i \leftrightarrow a_j$. Similarly, if $B \subseteq A$ and $a_i \leftrightarrow a_j$ for any $a_j \in B$ then we write $a_i \leftrightarrow B$. Finally, let $B, C \subseteq A$. Then $B \leftrightarrow C$ means $a_i \leftrightarrow a_j$ for any $a_i \in B$ and any $a_j \in C$.

Given $B \subseteq A$ denote by $\Gamma(B)$ the full subgraph of Γ generated by the set of vertices B, namely put $\Gamma(B) = (B; E \cap B^2)$. For a subgraph Δ of a graph $\Gamma = \langle A; E\rangle$ denote by $A(\Delta)$ the set of vertices of Δ.

Let R be a unital commutative ring and $\Gamma = (A; E)$ be an undirected graph without loops. The definition of a partially commutative Lie R-algebra in a variety $\mathfrak{M}$ can be written as follows.

$$L_R(\mathfrak{M}, \Delta) = \langle A \mid [a_i, a_j] = 0 \text{ if } a_i \leftrightarrow a_j; \mathfrak{M}\rangle.$$

Indeed, by (1)

$$[a_i, a_j] = [a_j, a_i] \tag{5}$$

for any i and j such that $a_i \leftrightarrow a_j$. On the other hand, $[f, g] = -[g, f]$ for any elements f and g of any Lie R-algebra. In particular,

$$[a_i, a_j] = -[a_j, a_i]. \tag{6}$$

Combining (5) and (6) we obtain $2[a_i, a_j] = 0$ if $a_i \leftrightarrow a_j$. Therefore, characteristic of R is equal to 2 or $[a_i, a_j] = 0$. In the former case, the corresponding Lie R-algebra is commutative, so the notion of partial commutativity makes no sense. So, in this section we assume that characteristic of the basic ring (or the field) is not equal to 2. We say that the algebra $L_R(\mathfrak{M}, \Delta)$ *is defined* by Δ and that Δ is the *defining graph* of $L_R(\mathfrak{M}, \Delta)$.

In this section, we talk mainly about results in partially commutative and partially commutative metabelian Lie R-algebras. Some results concern partially commutative nilpotent Lie R-algebras. For a domain R and a graph $\Gamma = (A; E)$ with the set of vertices A and the set of edges E denote by $\mathcal{L}_R(A; \Gamma)$, $\mathcal{M}_R(A; \Gamma)$, and $\mathcal{N}_{m,R}(A; \Gamma)$ the partially commutative R-algebra, partially commutative metabelian Lie R-algebra, and partially commutative nilpotent R-algebra of nilpotency degree m respectively defined by Γ.

Let $[u]$ be a Lie monomial in a (finite or infinite) set of generators $A = \{a_1, a_2, \dots\}$. The *multi-degree* of $[u]$ is the vector $\overline{\delta} = (\delta_1, \delta_2, \dots,)$, where δ_i is the number of occurrences of a_i in $[u]$.

For a Lie monomial $[u]$ of multi-degree $(\delta_1, \delta_2, \dots,)$ put $\mathrm{supp}([u]) = \{a_i \mid \delta_i \neq 0\}$. Extend this notation to the set of all Lie polynomials as follows. If $g = \sum_j \alpha_j [u_j]$ is a Lie polynomial then $\mathrm{supp}(g) = \bigcup_j \mathrm{supp}([u_j])$.

2.1. **Algebraic properties of partially commutative Lie algebras.**

Isomorphisms. As far as we know partially commutative algebras have been explored since 80's. We start with the result for partially commutative associative algebras obtained by K. H. Kim, L. Makar-Limanov, J. Neggers, and F. W. Roush [16]. This result has been already mentioned in Subsection 1.2. Nevertheless, we decided to give it in more detail because this is one of the significant results for partially commutative algebras.

For an undirected graph without loops $\Gamma = (A; E)$ and a domain R denote by $\mathcal{A}_R(A; \Gamma)$ the partially commutative associative R-algebra defined by Γ.

Theorem 39. *Let $\Gamma = (A; E)$ and $\Delta = (B; F)$ be undirected graphs without loops and $\mathbb{F}$ a field. The partially commutative associative algebras $\mathcal{A}_{\mathbb{F}}(A; \Gamma)$ and $\mathcal{A}_{\mathbb{F}}(B; \Delta)$ are isomorphic if and only if the graphs Γ and Δ are isomorphic.*

G. Duchamp and D. Krob in [10] generalized this result to the case of associative R-algebras where R is an arbitrary domain. Besides, in the same paper they stated a criterion for existence of an isomorphism between partially commutative Lie algebras. The analogous criterion holds also for partially commutative metabelian Lie algebras [30]. In all cases, two algebras are isomorphic if and only if their defining graphs are isomorphic.

Bases. Finding linear bases is a significant problem because a linear basis is a very important tool for studying algebras.

The first result on bases of partially commutative Lie algebras was obtained by G. Duchamp and D. Krob in [9], but they did not give an explicit description of a basis. Their algorithm was recursive. More precisely, let R be a unital commutative ring and $\Gamma(A;E)$ a graph without loops. The corresponding partially commutative Lie R-algebra $\mathcal{L}_R(A;\Gamma)$ is considered, a totally disconnected set $B \subseteq A$ is chosen, and the problem is reduced to finding a linear basis of the algebra $\mathcal{L}_R(A\backslash B;\Gamma(A\backslash B))$.

An explicit construction for bases of partially commutative Lie algebras was obtained in [23]. To make this description let us first recall the definition of Lyndon–Shirshov words.

Denote by $A^\star$ and $A^\#$ the sets of all associative and non-associative words (associative monomials with no coefficient) in A respectively. We define the empty word by 1.

Let us extend an arbitrary well order on A to the lexicographic order on $A^\star$.

An associative word u is called an *associative Lyndon–Shirshov word* if for any pair of nonempty words v and w such that $u = vw$ we have $wv < u$.

A non-associative word (non-associative monomial with no coefficient) $[u]$ is called a *Lyndon–Shirshov word* if

(1) The word u obtained from $[u]$ by omitting brackets is an associative Lyndon–Shirshov word;
(2) if $[u] = [[u_1],[u_2]]$, then $[u_1]$ and $[u_2]$ are Lyndon–Shirshov words (it follows from (1) that $u_1 > u_2$);
(3) if $[u] = \big[[[u_{11}],[u_{12}]],[u_2]\big]$, then $u_2 \geqslant u_{12}$.

Denote the set of all non-associative Lyndon–Shirshov words in A by $\mathrm{LS}(A)$. It was shown in [34] that the set $\mathrm{LS}(A)$ is a basis of the free Lie R-algebra generated by A.

For a partially commutative Lie algebra $\mathcal{L}_R(A;\Gamma)$ over a domain R define by induction *partially commutative Lyndon–Shirshov words* (*PCLS-words* for short).

(1) All elements of A are PCLS-words.
(2) a Lyndon–Shirshov word $[u]$ such that $\ell([u]) > 1$ is a PCLS-word if $[u] = [[v],[w]]$, where $[v]$ and $[w]$ are PCLS-words and there is an element in $\mathrm{supp}([v])$ such that it is not connected by an edge of Γ to the first letter of $[w]$.
(3) There are no other PCLS-words.

Denote the set of all PCLS-words of a partially commutative Lie R-algebra $\mathcal{L}_R(A;\Gamma)$ by $PCLS(A;\Gamma)$. Using the method of Gröbner–Shirshov bases an explicit description of a bases of a partially commutative Lie algebras was obtained.

Theorem 40. [23] *Let R be a unital commutative ring, Γ a finite undirected graph without loops, and A the set of vertices of Γ. Then the set $PCLS(A;\Gamma)$ is a basis of the partially commutative Lie R-algebra $\mathcal{L}_R(A;\Gamma)$.*

A linear basis for a partially commutative nilpotent algebra can be easily obtained from a linear basis of a partially commutative algebra.

Theorem 41. [24] *Let R be a unital commutative ring, Γ a finite undirected graph without loops, and A the set of vertices of Γ. Then the set of all the elements of $PCLS(A;\Gamma)$ whose lengthes are not greater than m is a basis of the partially commutative nilpotent R-algebra $\mathcal{N}_{R,m}(A;\Gamma)$.*

In [23] and [24] the set A is supposed to be finite, but it is easy to see that this restriction is not essential and so, Theorem 40 and Theorem 41 hold for algebras defined by infinite graphs Γ as well.

The problem of finding a linear basis for a partially commutative metabelian Lie algebra is also rather interesting. An explicit description of such a basis was obtained in [26]. The idea used for constructing a basis for a partially commutative metabelian Lie algebra is similar to one for a partially commutative Lie algebra. Namely, a basis is constructed by choosing some elements from a linear basis of a free metabelian Lie algebra of the corresponding variety.

A basis for a free metabelian Lie algebra was obtained independently by L. A. Bokut [4] and A. L. Shmelkin [35]. Let $A = \{a_1, a_2, \dots, a_n\}$ be the set of generators of free metabelian algebra. Then the set of all the elements of the form

$$[[\dots[a_{i_1}, a_{i_2}], \dots], a_{i_k}],$$

where $a_{i_1} > a_{i_2}$, $a_{i_2} \leqslant a_{i_3} \leqslant \dots \leqslant a_{i_k}$ is a basis of this algebra. Denote this set by $\mathrm{Bas}(A)$.

Fix an arbitrary multi-degree $\overline{\delta} = (\delta_1, \delta_2, \dots, \delta_n)$, where $n = |A|$. Let $N = \sum_{i=1}^{n} \delta_i$, $A_{\overline{\delta}} = \{a_i \in A \mid \delta_i \neq 0\}$, and b the smallest element of $A_{\overline{\delta}}$ with respect to the lexicographic order. Denote the connected components of the graph $\Gamma(A_{\overline{\delta}})$ by $\Delta_0, \Delta_1, \dots, \Delta_k$ in such a way that $b \in A(\Delta_0)$. Let $[u_i] \in \mathrm{Bas}(A)$ be an element of multi-degree $\overline{\delta}$ such that $[u_i] = [[\dots[[a_{j_{i,1}}, b], a_{j_{i,3}}], \dots], a_{j_{i,N}}]$, where $a_{j_{i,1}}$ is the largest element of $A(\Delta_i)$. Denote by $B_{\overline{\delta}}(A;\Gamma)$ the subset $\{[u_1], [u_2], \dots, [u_k]\}$. Finally, put

$$\mathrm{Bas}(A;\Gamma) = \bigcup_{\overline{\delta}} B_{\overline{\delta}}(A;\Gamma),$$

where the union is taken on all multi-degrees.

Theorem 42. [26] *Let R be a unital commutative ring and Γ a finite undirected graph without loops. Then the set $\mathrm{Bas}(A;\Gamma)$ is a basis of the partially commutative metabelian Lie R-algebra $\mathcal{M}_R(A;\Gamma)$.*

Note that Theorem 42 also holds for infinitely generated partially commutative metabelian Lie algebras.

Annihilators. Let R be an infinite integral domain and $R[A]$ be the set of all commutative associative polynomials over R. The derived subalgebra $\mathcal{M}'_R(A;\Gamma)$ of the R-algebra $\mathcal{M}_R(A;\Gamma)$ is an $R[A]$-module with respect to the adjoint representation.

Define the ideal $I^{\Gamma}_{i,j}$ of $R[A]$ as follows. If a_i and a_j are vertices belonging to different connected components in Γ then put $I^{\Gamma}_{i,j} = 0$. Suppose this is not so. Then for each path $(a_i, b_1, b_2, \dots, b_s, a_j)$ connecting these vertices in Γ consider the associative monomial $b_1 b_2 \dots b_s$. Define $I^{\Gamma}_{i,j}$ as the ideal generated by all such monomials.

Theorem 43. [26] *Let R be an infinite domain and $\Gamma = (A; E)$ a finite undirected graph without loops. For $a_i, a_j \in A$ if a_i and a_j are not adjacent in Γ then the annihilator of $[a_i, a_j]$ in $\mathcal{M}_R(A; \Gamma)$ is equal to $I^{\Gamma}_{i,j}$.*

Centralizers. As well as for isomorphisms, the first results for centralizers of partially commutative algebras were obtained for associative ones. For a graph $\Gamma = (A; E)$ denote by Γ^c the complement of Γ, i.e. the graph $(A; A^2 \backslash (E \cup \mathrm{id}_A))$, where $\mathrm{id}_A = \{(a, a) \mid a \in A\}$. In 1980, K. H. Kim and F. W. Roush obtained a description of centralizers of monomials [17].

Theorem 44. *Let R be a unital commutative ring, Γ a finite undirected graph without loops, and A the set of vertices of Γ. Let also u be a monomial of degree > 0 in the partially commutative associative R-algebra $\mathcal{A}_R(A; \Gamma)$ and let v be a monomial of degree > 0 in $\mathcal{A}_R(A; \Gamma)$ such that this monomial commutes with u. Finally, let $\Delta_1, \Delta_2, \ldots, \Delta_p$ be the connected components of $\Gamma^c(\mathrm{supp}(u))$. Write $u = u_1 u_2 \ldots u_p$, where $\mathrm{supp}(u_i) = A(\Delta_i)$. Then v is a product of generators a such that $a \notin \mathrm{supp}(u)$ but $a \leftrightarrow \mathrm{supp}(u)$, and words w such that some power of w equals one of the u_i.*

It seems that the requirement of finiteness of the defining graph can be eliminated.

We use the following notation Let R be a domain and L a Lie R-algebra. For $f, g \in L \backslash \{0\}$ we write $f \backsim g$ if $\alpha f = \beta g$ for some $\alpha, \beta \in R$. For any $g \in L$ the centralizer of g is denoted by $C(g)$. We also put $\mathcal{C}(g) = C(g) \cap L'$.

Unlike the case of partially commutative associative algebras, centralizers of elements of partially commutative Lie algebras over domains have been described completely, i.e. for an arbitrary domain R an explicit description for centralizers of all elements in any R-algebra $\mathcal{L}_R(A; \Gamma)$ has been obtained.

Theorem 45. [25] *Let R be a domain Γ a finite undirected graph without loops, and A the set of vertices of Γ. For an arbitrary element g of the Lie R-algebra $\mathcal{L}_R(A; \Gamma)$ denote by $\Delta_1, \Delta_2, \ldots, \Delta_p$ all connected components of the graph $\Gamma^c(\mathrm{supp}(g))$. Then $g = \sum_{i=1}^{p} g_i$, where $\mathrm{supp}(g_i) = A(\Delta_i)$ for all $i = 1, 2, \ldots p$, and $C(g)$ consists of elements of the form $h = \sum_{i=1}^{p} h_i + h^{(1)}$, where for each $i = 1, 2, \ldots p$ either $h_i = 0$ or $g_i \backsim h_i$. Moreover, $\mathrm{supp}(g) \leftrightarrow \mathrm{supp}(h^{(1)})$.*

For partially commutative metabelian Lie algebras there is no complete description of centralizers. Nevertheless, in [26, 27] some specific results were obtained.

For $f \in \mathcal{M}'_R(A; \Gamma)$ and $g \in R[A]$ denote by $f.g$ the image of f via the adjoint action by g.

Theorem 46. *Let R be a domain, Γ a finite undirected graph without loops, and $A = \{a_1, a_2, \ldots, a_n\}$ the set of vertices of this graph. Then for partially commutative metabelian Lie R-algebra $\mathcal{M}_R(A; \Gamma)$ the following statements hold.*
1) If a_n is an isolated vertex in Γ then $C(a_n)$ consists of the elements v of the form

$$v = \alpha_n a_n,$$

where $\alpha_n \in R$.
2) If the degree of x_n is equal to 1 in G (say, it is adjacent to a_{n-1}) then $C(a_n)$ consists of all elements v of the form

$$v = \alpha_{n-1} a_{n-1} + \alpha_n a_n,$$

where $\alpha_{n-1}, \alpha_n \in R$.
3) If a_n *is adjacent to* $a_{r+1}, \ldots, a_{n-1}$ *in* Γ *(*$r \leqslant n-3$*), then* $C(a_n)$ *consists of all elements* v *of the form*

$$v = \sum_{k=r+1}^{n} \alpha_k a_k + \sum_{r+1 \leqslant i<j \leqslant n-1} [a_i, a_j].f_{ij},$$

where $\alpha_k \in R$, $f_{ij} \in R[A \backslash \{a_n\}]$.

There are some results on "centralizers in the commutant" $\mathcal{C}(g)$ in partially commutative metabelian Lie R-algebras.

Theorem 47. [26] *Let* R *be a domain,* Γ *a finite undirected graph without loops, and* $A = \{a_1, a_2, \ldots, a_n\}$ *the set of vertices of this graph. Then in the partially commutative metabelian Lie* R*-algebra* $\mathcal{M}_R(A;\Gamma)$ *the following equation holds.*

$$\mathcal{C}(\sum_{j=1}^{m} \alpha_{i_j} a_{i_j}) = \bigcap_{j=1}^{m} \mathcal{C}(a_{i_j})$$

for any elements $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$ *and for any* $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_m} \in R \backslash \{0\}$.

Theorem 47 has some corollaries for partially commutative metabelian Lie R-algebras defined by specific graphs.

Corollary 5. [27] *Let* R *be a domain,* C_n *a cycle of length* $n \geqslant 3$*, and* $A = \{a_1, a_2, \ldots, a_n\}$ *the set of vertices of* C_n*. Then the following statements hold in* $\mathcal{M}_R(A; C_n)$.
a) If a_i *and* a_j *are adjacent then* $\mathcal{C}(\alpha a_i + \beta a_n) = 0$ *for any* $\alpha, \beta \in R \backslash \{0\}$.
b) If a_i *and* a_j *are not adjacent then for any* $\alpha, \beta \in R\backslash\{0\}$ *the set* $\mathcal{C}(\alpha a_i + \beta a_j)$ *consists of linear combinations of non-zero Lie monomials* $[u_r]$ *such that* $A(\mathrm{supp}([u_r])) = A \backslash \{a_i, a_j\}$*. Moreover, any element of* $\mathcal{C}(\alpha x_i + \beta x_j)$ *can be represented in the form* $f = [a_{i-1}, a_{i+1}].g$ *for some* $g \in R[A]$.
c) If $m \geqslant 3$*, then* $\mathcal{C}(\sum_{j=1}^{m} \alpha_j x_{i_j}) = 0$ *for any* $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_m \in R \backslash \{0\}$.

Corollary 6. [26] *Let* R *be a domain,* Γ *a finite tree, and* $A = \{a_1, a_2, \ldots, a_n\}$ *the set of vertices of this tree. Suppose that* $g = \sum_{j=1}^{m} \alpha_j a_{i_j}$*, where* $m \geqslant 2$ *and* $\alpha_j \in R \backslash \{0\}$ *for* $j = 1, 2, \ldots, m$*. Then* $\mathcal{C}(g) = 0$ *in* $\mathcal{M}_R(A;\Gamma)$.

2.2. **Logical properties of partially commutative and partially commutative metabelian Lie algebras.**

Universal equivalence. Conditions of universal equivalence of partially commutative Lie R-algebras over domains R were studied in the series of papers [26, 27, 28, 29, 31]. The problem of finding a criteria for universal equivalence on the entire class of partially commutative (metabelian) Lie R-algebras seems to be very complicated. So, this problem is considered on some specific classes of R-algebras. Although the methods used in partially commutative and partially commutative metabelian Lie R-algebras differ, essentially the results turned out to be similar. The first criterion for universal equivalence were obtained for Lie R-algebras defined by cycles and trees.

Theorem 48. [28] *Let* R *be a domain and* $C_n = (A; E)$ *and* $C_m = (B; F)$ *cycle graphs such that* $|A| = n$, $|B| = m$ *with* $n, m \geqslant 3$*. Then the partially commutative*

Lie R-algebras $\mathcal{L}_R(A; C_n)$ and $\mathcal{L}_R(B; C_m)$ are universally equivalent if and only if $n = m$.

The analogous result holds also for partially commutative metabelian Lie R-algebras with some restriction on a domain R.

Theorem 49. [27] *Let R be a domain containing $\mathbb{Z}$ as a subring and $C_n = (A; E)$ and $C_m = (B; F)$ cycle graphs such that $|A| = n$, $|B| = m$ with $n, m \geqslant 3$. Then the partially commutative metabelian Lie R-algebras $\mathcal{M}_R(A; C_n)$ and $\mathcal{M}_R(B; C_m)$ are universally equivalent if and only if $n = m$.*

A criterion of universal equivalence of partially commutative R-algebras, where R is a domain was found in [28]. The analogous result in the metabelian case was obtained in [26]. Despite the fact that this result was obtained only for partially commutative metabelian rings it can be easily generalized to the case of partially commutative metabelian R-algebra where R is a domain containing $\mathbb{Z}$ as a subring.

It turned out that these results can be generalized to the case of algebras defined by graphs with at most countably many vertices.

Let $\Delta = (A; E)$ be a graph. Denote by A^* the set obtained from A by deleting all end-points of Δ and put $\Delta^* = \Delta(A^*)$. Actually, the definition of the graph Δ^* coincides with one given in Theorem 21.

We say that a tree Γ (finite or infinite) is a *tree of finite type* if the tree Γ^* is finite and a *tree of infinite type* if Γ^* is infinite.

Theorem 50. [29] *Let R be a domain, $\Gamma = (A; E)$ a tree of infinite type, and $\Delta = (B; F)$ a tree of finite type, where the sets A and B are at most countable. Then the partially commutative Lie R-algebras $\mathcal{L}_R(A; \Gamma)$ and $\mathcal{L}_R(B; \Delta)$ are not universally equivalent for any domain R.*

The following theorem is an analogue of Theorem 50 for partially commutative metabelian Lie algebras.

Theorem 51. [29] *Let R be a domain containing $\mathbb{Z}$ as a subring, $\Gamma = (A; E)$ a tree of infinite type, and $\Delta = (B; F)$ a tree of finite type, where the sets A and B are at most countable. Then the partially commutative metabelian Lie R-algebras $\mathcal{M}_R(A; \Gamma)$ and $\mathcal{M}_R(B; \Delta)$ are not universally equivalent.*

So, there are separate criteria for Lie algebras defined by graphs of finite type and for Lie algebras defined by graphs of infinite type. The case of finite type graphs is considered in Theorem 52 for partially commutative Lie algebras and in Theorem 53 for partially commutative metabelian Lie algebras.

Theorem 52. [28, 29] *Let R be a domain, $\Gamma = (A; E)$ and $\Delta = (B; F)$ be trees of finite type such that A and B are at most countable and $|A^*| \geqslant 2$, $|B^*| \geqslant 2$. Then partially commutative Lie R-algebras $\mathcal{L}_R(A; \Gamma)$ and $\mathcal{L}_R(B; \Delta)$ are universally equivalent if and only if $\Gamma^* \simeq \Delta^*$.*

Theorem 53. [26, 29] *Let R be a domain containing $\mathbb{Z}$ as a subring, $\Gamma = (A; E)$ and $\Delta = (B; F)$ be trees of finite type such that A and B are at most countable and $|A^*| \geqslant 2$, $|B^*| \geqslant 2$. Then the partially commutative metabelian Lie R-algebras $\mathcal{L}_R(A; \Gamma)$ and $\mathcal{L}_R(B; \Delta)$ are universally equivalent if and only if $\Gamma^* \simeq \Delta^*$.*

Finally, the following two theorems provide criteria for universal equivalence of partially commutative and partially commutative metabelian Lie algebras generated by graphs of infinite type.

Graphs Γ and Δ are *mutually locally embeddable* if any finite subgraph of each graph Γ and Δ is isomorphically embeddable to the other one. The following theorem shows that two partially commutative Lie algebras are generated by graphs of different types then this algebras are not universally equivalent.

Theorem 54. [29] *Let R be a domain, $\Gamma = (A; E)$ and $\Delta = (B; F)$ trees of infinite type with at most countable sets of vertices. Then the partially commutative Lie R-algebras $\mathcal{L}_R(A; \Gamma)$ and $\mathcal{L}_R(B; \Delta)$ are universally equivalent if and only if Γ^* and Δ^* are mutually locally embeddable.*

Theorem 55. [29] *Let R be a domain containing $\mathbb{Z}$ as a subring, $\Gamma = (A; E)$ and $\Delta = (B; F)$ trees of infinite type and these trees with at most countable sets of vertices. Then the partially commutative metabelian Lie R-algebras $\mathcal{M}_R(A; \Gamma)$ and $\mathcal{M}_R(B; \Delta)$ are universally equivalent if and only if Γ^* and Δ^* are mutually locally embeddable.*

Note that the statements similar to ones in Theorems 50–55 were obtained for countably generated partially commutative metabelian groups. The following theorem shows that no partially commutative metabelian group defined by a graph of finite type can be universally equivalent to one defined by a graph of infinite type.

Theorem 56. [29] *Let $\Gamma = (A; E)$ be a tree of infinite type, and $\Delta = (B; F)$ be a tree of finite type, where the sets A and B are at most countable.Then the partially commutative metabelian groups $G = (\mathfrak{A}^2, \Gamma)$ and $H = (\mathfrak{A}^2, \Delta)$ are not universally equivalent.*

The following two theorems establish criteria of universal equivalence of partially commutative metabelian groups defined by graphs of finite (Theorem 57) and infinite (Theorem 58) types.

Theorem 57. [29] *Let $\Gamma = (A; E)$ and $\Delta = (B; F)$ be trees of finite type such that A and B are at most countable, at least one of them is countable, and $|A^*| \geqslant 2$, $|B^*| \geqslant 2$. Then the partially commutative metabelian groups $G = (\mathfrak{A}^2, \Gamma)$ and $H = (\mathfrak{A}^2, \Delta)$ are universally equivalent if and only if $\Gamma^* \simeq \Delta^*$.*

Actually, Theorem 57 generalizes the criterion of universal equivalence of partially commutative metabelian groups defined by finite trees [14].

Theorem 58. [29] *Let R be a domain, $\Gamma = (A; E)$ and $\Delta = (B; F)$ trees of infinite type and these trees have at most countable sets of vertices. Then the partially commutative metabelian groups $G = (\mathfrak{A}^2, \Gamma)$ and $H = (\mathfrak{A}^2, \Delta)$ are universally equivalent if and only if Γ^* and Δ^* are mutually locally embeddable.*

It is rather easy to see that the condition on cardinalities of sets A and B in Theorems 50–58 can be excluded.

The results in [28, 29] for partially commutative Lie algebras were generalized in [31] as follows.

Theorem 59. *Let R be a domain, $\Gamma = (A; E)$ and $\Delta = (B; F)$ be finite undirected graphs without loops, triangles, squares, and isolated vertices. Then the partially commutative Lie R-algebras $\mathcal{L}_R(A; \Gamma)$ and $\mathcal{L}_R(B; \Delta)$ are universally equivalent if and only if $\Gamma^* \simeq \Delta^*$ and the numbers of two-vertex connected components in Γ and Δ are equal.*

In [28], it was shown that the class of partially commutative Lie algebras defined by finite trees is not distinguished in the class of all finitely generated partially commutative algebras by universal theories.

Elementary equivalence. The problem of finding criteria for elementary equivalence for partially commutative and partially commutative metabelian Lie algebras was studied in [30]. This problem can be considered for algebras not only in "classical" signature, but also when Lie algebras are considered as two-sorted algebraic systems. Namely, there are three operations considered: addition and multiplication of elements in the algebra and multiplication of an element in the basic field by an element in the algebra.

Criteria for elementary equivalence of partially commutative and partially commutative metabelian Lie algebras were found in the case when Lie algebras over a field are considered as two-sorted systems and for Lie rings.

Theorem 60. [30] *Let $\mathbb{F}$ be a field and $\Gamma = (A; E)$ and $\Delta = (B; F)$ be finite undirected graphs without loops.*

(1) *The partially commutative Lie algebras $\mathcal{L}_{\mathbb{F}}(A;\Gamma)$ and $\mathcal{L}_{\mathbb{F}}(B;\Delta)$, considered as two-sorted algebraic systems are elementarily equivalent if and only if $\Gamma \simeq \Delta$.*
(2) *The partially commutative metabelian Lie algebras $\mathcal{M}_{\mathbb{F}}(A;\Gamma)$ and $\mathcal{M}_{\mathbb{F}}(B;\Delta)$, considered as two-sorted algebraic systems are elementarily equivalent if and only if $\Gamma \simeq \Delta$.*

Theorem 61. [30] *Let $\Gamma = (A; E)$ and $\Delta = (B; F)$ be finite undirected graphs without loops.*

(1) *The partially commutative Lie rings $\mathcal{L}_{\mathbb{Z}}(A;\Gamma)$ and $\mathcal{L}_{\mathbb{Z}}(B;\Delta)$ are elementarily equivalent if and only if $\Gamma \simeq \Delta$.*
(2) *The partially commutative metabelian Lie rings $\mathcal{M}_{\mathbb{Z}}(A;\Gamma)$ and $\mathcal{M}_{\mathbb{Z}}(B;\Delta)$ are elementarily equivalent if and only if $\Gamma \simeq \Delta$.*

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