

СИБИРСКИЕ ЭЛЕКТРОННЫЕ МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

<http://semr.math.nsc.ru>

Том 18, №1, стр. 647–667 (2021)

УДК 517.95

DOI 10.33048/semi.2021.18.047

MSC 35J70, 35J60, 35J30

EXISTENCE RESULTS FOR A CLASS OF NONLINEAR DEGENERATE NAVIER PROBLEMS

A.C. CAVALHEIRO

ABSTRACT. In this paper we are interested in the existence of solutions for Navier problem associated with the degenerate nonlinear elliptic equations

$$\begin{aligned} & \Delta[\omega_1(x)|\Delta u|^{p-2}\Delta u + \omega_2(x)|\Delta u|^{q-2}\Delta u] - \sum_{j=1}^n D_j[\omega_3(x)\mathcal{A}_j(x, u, \nabla u)] \\ & = f_0(x) - \sum_{j=1}^n D_j f_j(x), \quad \text{in } \Omega \end{aligned}$$

in the setting of the weighted Sobolev spaces.

Keywords: degenerate nonlinear elliptic equations, weighted Sobolev spaces.

1. INTRODUCTION

In this paper we prove the existence of (weak) solutions in the weighted Sobolev space $X = W^{2,p}(\Omega, \omega_1) \cap W_0^{1,p}(\Omega, \omega_3)$ (see Definition 2.3 and Definition 2.4) for the Navier problem

$$(P) \begin{cases} Lu(x) = f_0(x) - \sum_{j=1}^n D_j f_j(x) & \text{in } \Omega, \\ u(x) = \Delta u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where L is the partial differential operator

CAVALHEIRO, A.C., EXISTENCE RESULTS FOR A CLASS OF NONLINEAR DEGENERATE NAVIER PROBLEMS.

© 2021 CAVALHEIRO A.C.

Received January, 6, 2021, published June, 4, 2021.

$$\begin{aligned} Lu(x) &= \Delta[\omega_1(x)|\Delta u|^{p-2}\Delta u + \omega_2(x)|\Delta u|^{q-2}\Delta u] \\ &\quad - \sum_{j=1}^n D_j[\omega_3(x)\mathcal{A}_j(x, u(x), \nabla u(x))], \end{aligned}$$

where $D_j = \partial/\partial x_j$, Ω is a bounded open set in \mathbb{R}^n , ω_1 , ω_2 and ω_3 are three weight functions, Δ is the usual Laplacian operator, $2 \leq q < p < \infty$ and the functions $\mathcal{A}_j : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ ($j = 1, \dots, n$) satisfying the following conditions:

(H1) $x \mapsto \mathcal{A}_j(x, \eta, \xi)$ is measurable on Ω for all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$,

$(\eta, \xi) \mapsto \mathcal{A}_j(x, \eta, \xi)$ is continuous on $\mathbb{R} \times \mathbb{R}^n$ for almost all $x \in \Omega$.

(H2) there exist a constant $\theta_1 > 0$ such that

$$[\mathcal{A}(x, \eta, \xi) - \mathcal{A}(x, \eta', \xi')].(\xi - \xi') \geq \theta_1 |\xi - \xi'|^p,$$

whenever $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$, where $\mathcal{A}(x, \eta, \xi) = (\mathcal{A}_1(x, \eta, \xi), \dots, \mathcal{A}_n(x, \eta, \xi))$ (where a dot denote here the Euclidean scalar product in \mathbb{R}^n).

(H3) $\mathcal{A}(x, \eta, \xi). \xi \geq \lambda_1 |\xi|^p$, where λ_1 is a positive constant.

(H4) $|\mathcal{A}(x, \eta, \xi)| \leq K_1(x) + h_1(x)|\eta|^{p/p'} + h_2(x)|\xi|^{p/p'}$, where K_1, h_1 and h_2 are positive functions, with h_1 and $h_2 \in L^\infty(\Omega)$, and $K_1 \in L^{p'}(\Omega, \omega_3)$ (with $1/p + 1/p' = 1$).

Let Ω be an open set in \mathbb{R}^n . By the symbol $\mathcal{W}(\Omega)$ we denote the set of all measurable a.e. in Ω positive and finite functions $\omega = \omega(x)$, $x \in \Omega$. Elements of $\mathcal{W}(\Omega)$ will be called *weight functions*. Every weight ω gives rise to a measure on the measurable subsets of \mathbb{R}^n through integration. This measure will be denoted by μ_ω . Thus, $\mu_\omega(E) = \int_E \omega(x) dx$ for measurable sets $E \subset \mathbb{R}^n$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. In the particular case where $p = q = 2$ and $\omega_1 = \omega_2 = \omega_3 \equiv 1$ we have the equation

$$\Delta^2 u - \sum_{j=1}^n D_j \mathcal{A}_j(x, u, \nabla u) = f,$$

where $\Delta^2 u$ is the biharmonic operator. If $p = q$, $\omega_1 = \omega_2 = \omega_3 \equiv 1$ and $\mathcal{A}(x, \eta, \xi) = |\xi|^{p-2} \xi$ we have the equation

$$\Delta(|\Delta|^{p-2} \Delta u) - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = f.$$

Biharmonic equations appear in the study of mathematical model in several real-life processes as, among others, radar imaging (see [1]) or incompressible flows (see [18]).

For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [3], [4], [5], [8], [9], [15] and [20]). In various applications, we can meet boundary value problems for elliptic equations whose ellipticity is disturbed in the sense that some degeneration or singularity appears. There are several very concrete problems from practice which lead to such differential equations, e.g. from glaciology, non-Newtonian fluid mechanics, flows through porous media, differential geometry, celestial mechanics, climatology, petroleum extraction and reaction-diffusion problems (see some examples of applications of degenerate elliptic equations in [2] and [7]).

A class of weights, which is particularly well understood, is the class of A_p -weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [19]). These classes have found many useful applications in harmonic analysis (see [22]). Another reason for studying A_p -weights is the fact that powers of distance to submanifolds of \mathbb{R}^n often belong to A_p (see [16]). There are, in fact, many interesting examples of weights (see [14] for p -admissible weights).

In the non-degenerate case (i.e. with $\omega(x) \equiv 1$), for all $f \in L^p(\Omega)$, the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x) \text{ in } \Omega, \\ u(x) = 0 \text{ on } \partial\Omega, \end{cases}$$

is uniquely solvable in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [13]), and the nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x), \text{ in } \Omega \\ u(x) = 0, \text{ on } \partial\Omega \end{cases}$$

is uniquely solvable in $W_0^{1,p}(\Omega)$ (see [6]), where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian operator. In the degenerate case, the weighted p -Biharmonic operator has been studied by many authors (see [21] and the references therein), and the degenerated p -Laplacian was studied in [9].

The following theorem will be proved in section 3.

Theorem 1.1. *Let $2 \leq q < p < \infty$ and assume (H1)-(H4). If*

(H5) $\omega_1, \omega_3 \in A_p$, $\omega_2 \in \mathcal{W}(\Omega)$ and $\frac{\omega_2}{\omega_1} \in L^r(\Omega, \omega_1)$, where $r = p/(p - q)$;

(H6) $\frac{f_j}{\omega_3} \in L^{p'}(\Omega, \omega_3)$ ($j = 0, 1, \dots, n$).

Then the problem (P) has a unique solution $u \in X = W^{2,p}(\Omega, \omega_1) \cap W_0^{1,p}(\Omega, \omega_3)$. Moreover, we have

$$\|u\|_X \leq \frac{1}{\gamma^{p'/p}} \left(C_\Omega \|f_0/\omega_3\|_{L^{p'}(\Omega, \omega_3)} + \sum_{j=1}^n \|f_j/\omega_3\|_{L^{p'}(\Omega, \omega_3)} \right)^{p'/p},$$

where $\gamma = \min\{\lambda_1, 1\}$ and C_Ω is the constant in Theorem 2.2.

2. DEFINITIONS AND BASIC RESULTS

Let ω be a locally integrable nonnegative function in \mathbb{R}^n and assume that $0 < \omega < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p , $1 < p < \infty$, or that ω is an A_p -weight, if there is a constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|} \int_B \omega \, dx \right) \left(\frac{1}{|B|} \int_B \omega^{1/(1-p)} \, dx \right)^{p-1} \leq C,$$

for all balls $B \subset \mathbb{R}^n$, where $|\cdot|$ denotes the n -dimensional Lebesgue measure in \mathbb{R}^n . If $1 < q \leq p$, then $A_q \subset A_p$ (see [12],[14] or [22] for more information about A_p -weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that

$$\mu(B(x; 2r)) \leq C \mu(B(x; r)),$$

for every ball $B = B(x; r) \subset \mathbb{R}^n$, where $\mu(B) = \int_B \omega(x) \, dx$. If $\omega \in A_p$, then μ is doubling (see Corollary 15.7 in [14]).

As an example of A_p -weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [22]).

If $\omega \in A_p$, then

$$\left(\frac{|E|}{|B|}\right)^p \leq C \frac{\mu(E)}{\mu(B)},$$

whenever B is a ball in \mathbb{R}^n and E is a measurable subset of B (see 15.5 *strong doubling property* in [14]). Therefore, if $\mu(E) = 0$ then $|E| = 0$. The measure μ and the Lebesgue measure $|\cdot|$ are mutually absolutely continuous, i.e., they have the same zero sets ($\mu(E) = 0$ if and only if $|E| = 0$); so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e..

Definition 2.1. Let ω be a weight, and let $\Omega \subset \mathbb{R}^n$ be open. For $0 < p < \infty$ we define $L^p(\Omega, \omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega, \omega)} = \left(\int_{\Omega} |f|^p \omega \, dx \right)^{1/p} < \infty.$$

If $\omega \in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega, \omega) \subset L^1_{\text{loc}}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [23]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega, \omega)$.

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $1 < p < \infty$, k be a nonnegative integer and $\omega \in A_p$. We shall denote by $W^{k,p}(\Omega, \omega)$, the weighted Sobolev spaces, the set of all functions $u \in L^p(\Omega, \omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega, \omega)$, $1 \leq |\alpha| \leq k$. The norm in the space $W^{k,p}(\Omega, \omega)$ is defined by

$$(2.1) \quad \|u\|_{W^{k,p}(\Omega, \omega)} = \left(\int_{\Omega} |u|^p \omega \, dx + \sum_{1 \leq |\alpha| \leq k} \int_{\Omega} |D^\alpha u|^p \omega \, dx \right)^{1/p}.$$

If $\omega \in A_p$, then $W^{1,p}(\Omega, \omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (2.1) (see Theorem 2.1.4 in [23]). The spaces $W^{1,p}(\Omega, \omega)$ are Banach spaces.

The space $W_0^{1,p}(\Omega, \omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.1). Equipped with this norm, $W_0^{1,p}(\Omega, \omega)$ is a reflexive Banach space (see [17] for more information about the spaces $W^{1,p}(\Omega, \omega)$). The dual of space $W_0^{1,p}(\Omega, \omega)$ is the space

$$[W_0^{1,p}(\Omega, \omega)]^* = \{T = f_0 - \operatorname{div}(F), F = (f_1, \dots, f_n) : \frac{f_j}{\omega} \in L^{p'}(\Omega, \omega), j = 0, 1, \dots, n\}.$$

It is evident that a weight function ω which satisfies $0 < c_1 \leq \omega(x) \leq c_2$ for $x \in \Omega$ (where c_1 and c_2 are constants), give nothing new (the space $W_0^{1,p}(\Omega, \omega)$ is then identical with the classical Sobolev space $W_0^{1,p}(\Omega)$). Consequently, we shall be interested above all in such weight functions ω which either vanish somewhere in Ω or increase to infinity (or both).

In this paper we use the following results.

Theorem 2.1. Let $\omega \in A_p$, $1 < p < \infty$, and let Ω be a bounded open set in \mathbb{R}^n . If $u_m \rightarrow u$ in $L^p(\Omega, \omega)$ then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi \in L^p(\Omega, \omega)$ such that

(i) $u_{m_k}(x) \rightarrow u(x)$, $m_k \rightarrow \infty$ a.e. on Ω ;

(ii) $|u_{m_k}(x)| \leq \Phi(x)$ a.e. on Ω .

Proof. The proof of this theorem follows the lines of Theorem 2.8.1 in [11]. \square

Theorem 2.2. (The weighted Sobolev inequality) *Let Ω be an open bounded set in \mathbb{R}^n and $\omega \in A_p$ ($1 < p < \infty$). There exist constants C_Ω and δ positive such that for all $u \in W_0^{1,p}(\Omega, \omega)$ and all k satisfying $1 \leq k \leq n/(n-1) + \delta$,*

$$(2.2) \quad \|u\|_{L^{kp}(\Omega, \omega)} \leq C_\Omega \|\nabla u\|_{L^p(\Omega, \omega)}.$$

Proof. It suffices to prove the inequality for functions $u \in C_0^\infty(\Omega)$ (see Theorem 1.3 in [10]). To extend the estimates (2.2) to arbitrary $u \in W_0^{1,p}(\Omega, \omega)$, we let $\{u_m\}$ be a sequence of $C_0^\infty(\Omega)$ functions tending to u in $W_0^{1,p}(\Omega, \omega)$. Applying the estimates (2.2) to differences $u_{m_1} - u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^{kp}(\Omega, \omega)$. Consequently the limit function u will lie in the desired spaces and satisfy (2.2). \square

Lemma 2.1. *Let $1 < p < \infty$.*

(a) *There exists a constant $\alpha_p > 0$ such that*

$$\left| |x|^{p-2}x - |y|^{p-2}y \right| \leq \alpha_p |x - y|(|x| + |y|)^{p-2},$$

for all $x, y \in \mathbb{R}^n$;

(b) *There exist two positive constants β_p, γ_p such that for every $x, y \in \mathbb{R}^n$*

$$\beta_p (|x| + |y|)^{p-2} |x - y|^2 \leq (|x|^{p-2}x - |y|^{p-2}y) \cdot (x - y) \leq \gamma_p (|x| + |y|)^{p-2} |x - y|^2.$$

Proof. See [6], Proposition 17.2 and Proposition 17.3. \square

Definition 2.3. *We denote by $X = W^{2,p}(\Omega, \omega_1) \cap W_0^{1,p}(\Omega, \omega_3)$ with the norm*

$$\|u\|_X = \left(\int_\Omega |\nabla u|^p \omega_3 dx + \int_\Omega |\Delta u|^p \omega_1 dx \right)^{1/p}.$$

Definition 2.4. *We say that an element $u \in X = W^{2,p}(\Omega, \omega_1) \cap W_0^{1,p}(\Omega, \omega_3)$ is a (weak) solution of problem (P) if*

$$\begin{aligned} & \int_\Omega |\Delta u|^{p-2} \Delta u \Delta \varphi \omega_1 dx + \int_\Omega |\Delta u|^{q-2} \Delta u \Delta \varphi \omega_2 dx \\ & + \sum_{j=1}^n \int_\Omega \mathcal{A}_j(x, u, \nabla u) D_j \varphi \omega_3 dx \\ & = \int_\Omega f_0 \varphi dx + \sum_{j=1}^n \int_\Omega f_j D_j \varphi dx, \end{aligned}$$

for all $\varphi \in X$.

Remark 2.1. *If $\omega_1 \in A_p$, $\omega_2 \in \mathcal{W}(\Omega)$, $2 \leq q < p < \infty$ and $\frac{\omega_2}{\omega_1} \in L^r(\Omega, \omega)$ (where $r = p/(p-q)$), then there exists a constant $C_{p,q} = \|\omega_2/\omega_1\|_{L^r(\Omega, \omega_1)}^{1/q}$ such that*

$$\|u\|_{L^q(\Omega, \omega_2)} \leq C_{p,q} \|u\|_{L^p(\Omega, \omega_1)}.$$

In fact, by Hölder's inequality ($1/r + q/p = (p-q)/p + q/p = 1$),

$$\|u\|_{L^q(\Omega, \omega_2)}^q = \int_\Omega |u|^q \omega_2 dx = \int_\Omega |u|^q \frac{\omega_2}{\omega_1} \omega_1 dx$$

$$\begin{aligned} &\leq \left(\int_{\Omega} |u|^p \omega_1 dx \right)^{q/p} \left(\int_{\Omega} (\omega_2/\omega_1)^r \omega_1 dx \right)^{1/r} \\ &= \|u\|_{L^p(\Omega, \omega_1)}^q \|\omega_2/\omega_1\|_{L^r(\Omega, \omega_1)}. \end{aligned}$$

Hence, $\|u\|_{L^q(\Omega, \omega_2)} \leq C_{p,q} \|u\|_{L^p(\Omega, \omega_1)}$, with $C_{p,q} = \|\omega_2/\omega_1\|_{L^r(\Omega, \omega_1)}^{1/q}$.

3. PROOF OF THEOREM 1.1

The basic idea is to reduce the problem (P) to an operator equation $Au = T$ and apply the theorem below.

Theorem 3.1. *Let $A : X \rightarrow X^*$ be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X . Then the following assertions hold:*

- (a) *For each $T \in X^*$ the equation $Au = T$ has a solution $u \in X$;*
- (b) *If the operator A is strictly monotone, then equation $Au = T$ is uniquely solvable in X .*

Proof. See Theorem 26.A in [25]. □

To prove Theorem 1.1, we define $B, B_1, B_2, B_3 : X \times X \rightarrow \mathbb{R}$ and $T : X \rightarrow \mathbb{R}$ by

$$\begin{aligned} B(u, \varphi) &= B_1(u, \varphi) + B_2(u, \varphi) + B_3(u, \varphi), \\ B_1(u, \varphi) &= \int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta \varphi \omega_1 dx \\ B_2(u, \varphi) &= \int_{\Omega} |\Delta u|^{q-2} \Delta u \Delta \varphi \omega_2 dx \\ B_3(u, \varphi) &= \sum_{j=1}^n \int_{\Omega} \mathcal{A}_j(x, u, \nabla u) D_j \varphi \omega_3 dx = \int_{\Omega} \mathcal{A}(x, u, \nabla u) \cdot \nabla \varphi \omega_3 dx \\ T(\varphi) &= \int_{\Omega} f_0 \varphi dx + \sum_{j=1}^n \int_{\Omega} f_j D_j \varphi dx. \end{aligned}$$

Then $u \in X$ is a (weak) solution to problem (P) if

$$B(u, \varphi) = B_1(u, \varphi) + B_2(u, \varphi) + B_3(u, \varphi) = T(\varphi),$$

for all $\varphi \in X$.

Step 1. For $j = 1, \dots, n$ we define the operator $F_j : X \rightarrow L^{p'}(\Omega, \omega_3)$ as

$$(F_j u)(x) = \mathcal{A}_j(x, u(x), \nabla u(x)).$$

We now show that the operator F_j is bounded and continuous.

(i) Using (H4) we obtain

$$\begin{aligned} &\|F_j u\|_{L^{p'}(\Omega, \omega_3)}^{p'} \\ &= \int_{\Omega} |F_j u(x)|^{p'} \omega_3 dx \\ &= \int_{\Omega} |\mathcal{A}_j(x, u, \nabla u)|^{p'} \omega_3 dx \\ &\leq \int_{\Omega} \left(K_1 + h_1 |u|^{p/p'} + h_2 |\nabla u|^{p/p'} \right)^{p'} \omega_3 dx \end{aligned}$$

$$\begin{aligned}
 &\leq C_p \int_{\Omega} \left[(K_1^{p'} + h_1^{p'} |u|^p + h_2^{p'} |\nabla u|^p) \omega_3 \right] dx \\
 (3.1) \quad &= C_p \left[\int_{\Omega} K_1^{p'} \omega_3 dx + \int_{\Omega} h_1^{p'} |u|^p \omega_3 dx + \int_{\Omega} h_2^{p'} |\nabla u|^p \omega_3 dx \right],
 \end{aligned}$$

where the constant C_p depends only on p . We have, by Theorem 2.2,

$$\begin{aligned}
 \int_{\Omega} h_1^{p'} |u|^p \omega_3 dx &\leq \|h_1\|_{L^\infty(\Omega)}^{p'} \int_{\Omega} |u|^p \omega_3 dx \\
 &\leq C_\Omega^p \|h_1\|_{L^\infty(\Omega)}^{p'} \int_{\Omega} |\nabla u|^p \omega_3 dx \\
 &\leq C_\Omega^p \|h_1\|_{L^\infty(\Omega)}^{p'} \|u\|_X^p,
 \end{aligned}$$

and

$$\int_{\Omega} h_2^{p'} |\nabla u|^p \omega_3 dx \leq \|h_2\|_{L^\infty(\Omega)}^{p'} \int_{\Omega} |\nabla u|^p \omega_3 dx \leq \|h_2\|_{L^\infty(\Omega)}^{p'} \|u\|_X^p.$$

Therefore, in (3.1) we obtain

$$\begin{aligned}
 &\|F_j u\|_{L^{p'}(\Omega, \omega_3)} \\
 &\leq C_p^{1/p'} \left(\|K\|_{L^{p'}(\Omega, \omega_3)} + (C_\Omega^{p/p'}) \|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)} \|u\|_X^{p/p'} \right).
 \end{aligned}$$

(ii) Let $u_m \rightarrow u$ in X as $m \rightarrow \infty$. We need to show that $F_j u_m \rightarrow F_j u$ in $L^{p'}(\Omega, \omega_3)$. We will apply the Lebesgue Dominated Convergence Theorem. If $u_m \rightarrow u$ in X , then $|\nabla u_m| \rightarrow |\nabla u|$ in $L^p(\Omega, \omega_3)$. Using Theorem 2.1, there exist a subsequence $\{u_{m_k}\}$ and a function Φ_1 in $L^p(\Omega, \omega_3)$ such that

$$\begin{aligned}
 D_j u_{m_k}(x) &\rightarrow D_j u(x), \text{ a.e. in } \Omega, \\
 |\nabla u_{m_k}(x)| &\leq \Phi_1(x), \text{ a.e. in } \Omega.
 \end{aligned}$$

By Theorem 2.2 (with $k = 1$), we have

$$\|u_{m_k}\|_{L^p(\Omega, \omega_3)} \leq C_\Omega \| |\nabla u_{m_k}| \|_{L^p(\Omega, \omega_3)} \leq C_\Omega \|\Phi_1\|_{L^p(\Omega, \omega_3)}.$$

Next, applying (H4) we obtain

$$\begin{aligned}
 &\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega, \omega_3)}^{p'} \\
 &= \int_{\Omega} |F_j u_{m_k}(x) - F_j u(x)|^{p'} \omega_3 dx \\
 &= \int_{\Omega} |\mathcal{A}_j(x, u_{m_k}, \nabla u_{m_k}) - \mathcal{A}_j(x, u, \nabla u)|^{p'} \omega_3 dx \\
 &\leq C_p \int_{\Omega} \left(|\mathcal{A}_j(x, u_{m_k}, \nabla u_{m_k})|^{p'} + |\mathcal{A}_j(x, u, \nabla u)|^{p'} \right) \omega_3 dx \\
 &\leq C_p \left[\int_{\Omega} \left(K_1 + h_1 |u_{m_k}|^{p/p'} + h_2 |\nabla u_{m_k}|^{p/p'} \right)^{p'} \omega_3 dx \right. \\
 &\quad \left. + \int_{\Omega} \left(K_1 + h_1 |u|^{p/p'} + h_2 |\nabla u|^{p/p'} \right)^{p'} \omega_3 dx \right] \\
 &\leq C_p \left[\int_{\Omega} K_1^{p'} \omega_3 dx + \|h_1\|_{L^\infty(\Omega)}^{p'} \int_{\Omega} |u_{m_k}|^p \omega_3 dx + \|h_2\|_{L^\infty(\Omega)}^{p'} \int_{\Omega} |\nabla u_{m_k}|^p \omega_3 dx \right]
 \end{aligned}$$

$$\begin{aligned}
& + \int_{\Omega} K_1^{p'} \omega_3 dx + \|h_1\|_{L^\infty(\Omega)}^{p'} \int_{\Omega} |u|^p \omega_3 dx + \|h_2\|_{L^\infty(\Omega)}^{p'} \int_{\Omega} |\nabla u|^p \omega_3 dx \Big] \\
& \leq 2C_p \left[\int_{\Omega} K_1^{p'} \omega_3 dx + C_{\Omega}^p \|h_1\|_{L^\infty(\Omega)}^{p'} \int_{\Omega} \Phi_1^p \omega_3 dx + \|h_2\|_{L^\infty(\Omega)}^{p'} \int_{\Omega} \Phi_1^p \omega_3 dx \right] \\
& = 2C_p \left[\|K_1\|_{L^{p'}(\Omega)}^{p'} + \left(C_{\Omega}^p \|h_1\|_{L^\infty(\Omega)}^{p'} + \|h_2\|_{L^\infty(\Omega)}^{p'} \right) \|\Phi_1\|_{L^p(\Omega, \omega_3)}^p \right].
\end{aligned}$$

By condition (H1), we have

$$F_j u_{m_k}(x) = \mathcal{A}_j(x, u_{m_k}(x), \nabla u_{m_k}(x)) \rightarrow \mathcal{A}_j(x, u(x), \nabla u(x)) = F_j u(x),$$

as $m_k \rightarrow +\infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain $\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega, \omega_3)} \rightarrow 0$, that is, $F_j u_{m_k} \rightarrow F_j u$ in $L^{p'}(\Omega, \omega_3)$. We conclude from the Convergence Principle in Banach spaces (see Proposition 10.13 in [24]) that

$$(3.2) \quad F_j u_m \rightarrow F_j u \text{ in } L^{p'}(\Omega, \omega_3).$$

Step 2. We define the operator $G_1 : X \rightarrow L^{p'}(\Omega, \omega_1)$ by

$$(G_1 u)(x) = |\Delta u(x)|^{p-2} \Delta u(x).$$

This operator is continuous and bounded. In fact,

(i) We have

$$\begin{aligned}
\|G_1 u\|_{L^{p'}(\Omega, \omega_1)}^{p'} &= \int_{\Omega} |\Delta u|^{p-2} \Delta u \Big|^{p'} \omega_1 dx \\
&= \int_{\Omega} |\Delta u|^{(p-1)p'} \omega_1 dx \\
&= \int_{\Omega} |\Delta u|^p \omega_1 dx \\
&\leq \|u\|_X^p.
\end{aligned}$$

Hence, $\|G_1 u\|_{L^{p'}(\Omega, \omega_1)} \leq \|u\|_X^{p-1}$.

(ii) If $u_m \rightarrow u$ in X then $\Delta u_m \rightarrow \Delta u$ in $L^p(\Omega, \omega_1)$. By Theorem 2.1, there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi_2 \in L^p(\Omega, \omega_1)$ such that

$$(3.3) \quad \Delta u_{m_k}(x) \rightarrow \Delta u(x), \text{ a.e. in } \Omega,$$

$$(3.4) \quad |\Delta u_{m_k}(x)| \leq \Phi_2(x), \text{ a.e. in } \Omega.$$

Hence, using Lemma 2.1 (a), $\theta = \frac{p}{p'} = p-1$ and $\theta' = \frac{(p-1)}{(p-2)}$, we obtain (since $2 \leq q < p < \infty$)

$$\begin{aligned}
\|G_1 u_{m_k} - G_1 u\|_{L^{p'}(\Omega, \omega_1)}^{p'} &= \int_{\Omega} |G_1 u_{m_k} - G_1 u|^{p'} \omega_1 dx \\
&= \int_{\Omega} \left| |\Delta u_{m_k}|^{p-2} \Delta u_{m_k} - |\Delta u|^{p-2} \Delta u \right|^{p'} \omega_1 dx \\
&\leq \int_{\Omega} \left[\alpha_p |\Delta u_{m_k} - \Delta u| (|\Delta u_{m_k}| + |\Delta u|)^{(p-2)} \right]^{p'} \omega_1 dx \\
&\leq \alpha_p^{p'} \int_{\Omega} |\Delta u_{m_k} - \Delta u|^{p'} (2\Phi_2)^{(p-2)p'} \omega_1 dx
\end{aligned}$$

$$\begin{aligned} &\leq 2^{(p-2)p'} \alpha_p^{p'} \left(\int_{\Omega} |\Delta u_{m_k} - \Delta u|^{p'\theta} \omega_1 dx \right)^{1/\theta} \left(\int_{\Omega} \Phi_2^{(p-2)p'\theta'} \omega_1 dx \right)^{1/\theta'} \\ &\leq \alpha_p^{p'} 2^{(p-2)p'} \left(\int_{\Omega} |\Delta u_{m_k} - \Delta u|^p \omega_1 dx \right)^{p'/p} \left(\int_{\Omega} \Phi_2^p \omega_1 dx \right)^{(p-2)/(p-1)} \\ &\leq \alpha_p^{p'} 2^{(p-2)p'} \|u_{m_k} - u\|_X^{p'} \|\Phi_2\|_{L^p(\Omega, \omega_1)}^{(p-2)p'}, \end{aligned}$$

since $(p-2)p'\theta' = (p-2) \frac{p}{(p-1)} \frac{(p-1)}{(p-2)} = p$ if $p \neq 2$. Then

$$\|G_1 u_{m_k} - G_1 u\|_{L^{p'}(\Omega, \omega_1)} \leq 2^{(p-2)} \alpha_p \|\Phi_2\|_{L^p(\Omega, \omega_1)}^{p-2} \|u_{m_k} - u\|_X.$$

Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain (when $m_k \rightarrow \infty$) $\|G_1 u_{m_k} - G_1 u\|_{L^{p'}(\Omega, \omega_1)} \rightarrow 0$, that is, $G_1 u_{m_k} \rightarrow G_1 u$ in $L^{p'}(\Omega, \omega_1)$. By the Convergence Principle in Banach spaces, we have

$$(3.5) \quad G_1 u_m \rightarrow G_1 u \text{ in } L^{p'}(\Omega, \omega_1).$$

Step 3. We define the operator $G_2 : X \rightarrow L^s(\Omega, \omega_1)$, where $s = p/(q-1)$, by

$$(G_2 u)(x) = |\Delta u(x)|^{q-2} \Delta u(x).$$

We also have that the operator G_2 is continuous and bounded. In fact,

(i) We have that

$$\begin{aligned} \|G_2 u\|_{L^s(\Omega, \omega_1)}^s &= \int_{\Omega} \left| |\Delta u|^{q-2} \Delta u \right|^s \omega_1 dx \\ &= \int_{\Omega} |\Delta u|^{(q-1)s} \omega_1 dx \\ &= \int_{\Omega} |\Delta u|^p \omega_1 dx \\ &\leq \|u\|_X^p, \end{aligned}$$

and $\|G_2 u\|_{L^s(\Omega, \omega_1)} \leq \|u\|_X^{q-1}$.

(ii) If $u_m \rightarrow u$ in X then $\Delta u_m \rightarrow \Delta u$ in $L^p(\Omega, \omega_1)$. If $2 < q < p < \infty$, by (3.3), (3.4) and Lemma 2.1 (a) we have

$$\begin{aligned} \|G_2 u_{m_k} - G_2 u\|_{L^s(\Omega, \omega_1)}^s &= \int_{\Omega} \left| |\Delta u_{m_k}|^{q-2} \Delta u_{m_k} - |\Delta u|^{q-2} \Delta u \right|^s \omega_1 dx \\ &\leq \int_{\Omega} \left[\alpha_q |\Delta u_{m_k} - \Delta u| \left(|\Delta u_{m_k}| + |\Delta u| \right)^{q-2} \right]^s \omega_1 dx \\ &= \alpha_q^s \int_{\Omega} |\Delta u_{m_k} - \Delta u|^s \left(|\Delta u_{m_k}| + |\Delta u| \right)^{(q-2)s} \omega_1 dx \\ (3.6) \quad &\leq 2^{(q-2)s} \alpha_q^s \int_{\Omega} |\Delta u_{m_k} - \Delta u|^s \Phi_2^{(q-2)s} \omega_1 dx. \end{aligned}$$

For $\delta = q-1$, $\delta' = (q-1)/(q-2)$ and $s = p/(q-1)$ we have in (3.6)

$$\begin{aligned} &\|G_2 u_{m_k} - G_2 u\|_{L^s(\Omega, \omega_1)}^s \\ &\leq 2^{(q-2)s} \alpha_q^s \int_{\Omega} |\Delta u_{m_k} - \Delta u|^s \Phi_2^{(q-2)s} \omega_1 dx \end{aligned}$$

$$\begin{aligned}
&\leq 2^{(q-2)s} \alpha_q^s \left(\int_{\Omega} |\Delta u_{m_k} - \Delta u|^{s\delta} \omega_1 dx \right)^{1/\delta} \left(\int_{\Omega} \Phi_2^{(q-2)s\delta'} \omega_1 dx \right)^{1/\delta'} \\
&= 2^{(q-2)s} \alpha_q^s \left(\int_{\Omega} |\Delta u_{m_k} - \Delta u|^p \omega_1 dx \right)^{1/(q-1)} \left(\int_{\Omega} \Phi_2^p \omega_1 dx \right)^{1/\delta'} \\
&\leq 2^{(q-2)s} \alpha_q^s \|u_{m_k} - u\|_X^{p/(q-1)} \|\Phi_2\|_{L^p(\Omega, \omega_1)}^{p/\delta'}.
\end{aligned}$$

Hence, $\|G_2 u_{m_k} - G_2 u\|_{L^s(\Omega, \omega_1)} \leq 2^{(q-2)} \alpha_q \|u_{m_k} - u\|_X \|\Phi_2\|_{L^p(\Omega, \omega_1)}^{(q-2)}$.

In the case $2 = q < p < \infty$, we have $(G_2 u)(x) = \Delta u(x)$ and $s = p$. Hence,

$$\begin{aligned}
\|G_2 u\|_{L^p(\Omega, \omega_1)} &\leq \|u\|_X, \\
\|G_2 u_{m_k} - G_2 u\|_{L^p(\Omega, \omega_1)} &\leq \|u_{m_k} - u\|_X.
\end{aligned}$$

Therefore, for $2 \leq q < p < \infty$, by the Lebesgue Dominated Convergence Theorem we obtain (when $m_k \rightarrow \infty$)

$$\|G_2 u_{m_k} - G_2 u\|_{L^s(\Omega, \omega_1)} \rightarrow 0,$$

that is, $G_2 u_{m_k} \rightarrow G_2 u$ in $L^s(\Omega, \omega_1)$. By the Convergence Principle in Banach spaces, we have

$$(3.7) \quad G_2 u_{m_k} \rightarrow G_2 u \text{ in } L^s(\Omega, \omega_1).$$

Step 4. Since $\frac{f_j}{\omega_3} \in L^{p'}(\Omega, \omega_3)$ ($j = 0, 1, \dots, n$) then $T \in [W_0^{1,p}(\Omega, \omega_3)]^* \subset X^*$.

Moreover, we have, by Theorem 2.2,

$$\begin{aligned}
|T(\varphi)| &\leq \int_{\Omega} |f_0| |\varphi| dx + \sum_{j=1}^n \int_{\Omega} |f_j| |D_j \varphi| dx \\
&= \int_{\Omega} \frac{|f_0|}{\omega_3} |\varphi| \omega_3 dx + \sum_{j=1}^n \int_{\Omega} \frac{|f_j|}{\omega_3} |D_j \varphi| \omega_3 dx \\
&\leq \|f_0/\omega_3\|_{L^{p'}(\Omega, \omega_3)} \|\varphi\|_{L^p(\Omega, \omega_3)} + \sum_{j=1}^n \|f_j/\omega_3\|_{L^{p'}(\Omega, \omega_3)} \|D_j \varphi\|_{L^p(\Omega, \omega_3)} \\
&\leq C_{\Omega} \|f_0/\omega_3\|_{L^{p'}(\Omega, \omega_3)} \|\nabla \varphi\|_{L^p(\Omega, \omega_3)} \\
&+ \left(\sum_{j=1}^n \|f_j/\omega_3\|_{L^{p'}(\Omega, \omega_3)} \right) \|\nabla \varphi\|_{L^p(\Omega, \omega_3)} \\
&\leq \left(C_{\Omega} \|f_0/\omega_3\|_{L^{p'}(\Omega, \omega_3)} + \sum_{j=1}^n \|f_j/\omega_3\|_{L^{p'}(\Omega, \omega_3)} \right) \|\varphi\|_X.
\end{aligned}$$

Moreover, we also have

$$\begin{aligned}
|B(u, \varphi)| &\leq |B_1(u, \varphi)| + |B_2(u, \varphi)| + |B_3(u, \varphi)| \\
&\leq \sum_{j=1}^n \int_{\Omega} |A_j(x, u, \nabla u)| |D_j \varphi| \omega_3 dx + \int_{\Omega} |\Delta u|^{p-2} |\Delta u| |\Delta \varphi| \omega_1 dx \\
(3.8) \quad &+ \int_{\Omega} |\Delta u|^{q-2} |\Delta u| |\Delta \varphi| \omega_2 dx.
\end{aligned}$$

In (3.8) we have, by (H4),

$$\begin{aligned}
 \int_{\Omega} |\mathcal{A}(x, u, \nabla u)| |\nabla \varphi| \omega_3 dx &\leq \int_{\Omega} \left(K_1 + h_1 |u|^{p/p'} + h_2 |\nabla u|^{p/p'} \right) |\nabla \varphi| \omega_3 dx \\
 &\leq \|K_1\|_{L^{p'}(\Omega, \omega_3)} \|\nabla \varphi\|_{L^p(\Omega, \omega_3)} + \|h_1\|_{L^\infty(\Omega)} \|u\|_{L^p(\Omega, \omega_3)}^{p/p'} \|\nabla \varphi\|_{L^p(\Omega, \omega_3)} \\
 &\quad + \|h_2\|_{L^\infty(\Omega)} \|\nabla u\|_{L^p(\Omega, \omega_3)}^{p/p'} \|\nabla \varphi\|_{L^p(\Omega, \omega_3)} \\
 &\leq \left(\|K_1\|_{L^{p'}(\Omega, \omega_3)} + (C_\Omega^{p/p'}) \|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)} \right) \|u\|_X^{p/p'} \|\varphi\|_X,
 \end{aligned}$$

and

$$\begin{aligned}
 \int_{\Omega} |\Delta u|^{p-2} |\Delta u| |\Delta \varphi| \omega_1 dx &= \int_{\Omega} |\Delta u|^{p-1} |\Delta \varphi| \omega_1 dx \\
 &\leq \left(\int_{\Omega} |\Delta u|^p \omega_1 dx \right)^{1/p'} \left(\int_{\Omega} |\Delta \varphi|^p \omega_1 dx \right)^{1/p} \\
 &\leq \|u\|_X^{p/p'} \|\varphi\|_X,
 \end{aligned}$$

and since $s = p/(q-1)$, $r = p/(p-q)$ and $\frac{1}{s} + \frac{1}{r} + \frac{1}{p} = 1$, by the generalized Hölder inequality, we obtain

$$\begin{aligned}
 \int_{\Omega} |\Delta u|^{q-2} |\Delta u| |\Delta \varphi| \omega_2 dx &= \int_{\Omega} |\Delta u|^{q-1} |\Delta \varphi| \frac{\omega_2}{\omega_1} \omega_1 dx \\
 &\leq \left(\int_{\Omega} |\Delta u|^{(q-1)s} \omega_1 dx \right)^{1/s} \left(\int_{\Omega} |\Delta \varphi|^p \omega_1 dx \right)^{1/p} \left(\int_{\Omega} \left(\frac{\omega_2}{\omega_1} \right)^r \omega_1 dx \right)^{1/r} \\
 &= \left(\int_{\Omega} |\Delta u|^p \omega_1 dx \right)^{(q-1)/p} \left(\int_{\Omega} |\Delta \varphi|^p \omega_1 dx \right)^{1/p} \left(\int_{\Omega} \left(\frac{\omega_2}{\omega_1} \right)^r \omega_1 dx \right)^{1/r} \\
 &\leq \|u\|_X^{(q-1)} \|\varphi\|_X \|\omega_2/\omega_1\|_{L^r(\Omega, \omega_1)}.
 \end{aligned}$$

Hence, in (3.8) we obtain, for all $u, \varphi \in X$

$$\begin{aligned}
 &|B(u, \varphi)| \\
 &\leq \left[\|K_1\|_{L^{p'}(\Omega, \omega_3)} + C_\Omega^{p/p'} \|h_1\|_{L^\infty(\Omega)} \|u\|_X^{p/p'} + \|h_2\|_{L^\infty(\Omega, \omega)} \|u\|_X^{p/p'} + \|u\|_X^{p/p'} \right. \\
 &\quad \left. + \|\omega_2/\omega_1\|_{L^r(\Omega, \omega_1)} \|u\|_X^{q-1} \right] \|\varphi\|_X.
 \end{aligned}$$

Since $B(u, \cdot)$ is linear, for each $u \in X$, there exists a linear and continuous functional on X denoted by Au such that $\langle Au, \varphi \rangle = B(u, \varphi)$, for all $u, \varphi \in X$ (here $\langle f, x \rangle$ denotes the value of the linear functional f at the point x). Moreover

$$\begin{aligned}
 \|Au\|_* &\leq \|K_1\|_{L^{p'}(\Omega, \omega_3)} + C_\Omega^{p/p'} \|h_1\|_{L^\infty(\Omega)} \|u\|_X^{p/p'} + \|h_2\|_{L^\infty(\Omega)} \|u\|_X^{p/p'} \\
 &\quad + \|u\|_X^{p/p'} + \|\omega_2/\omega_1\|_{L^r(\Omega, \omega_1)} \|u\|_X^{q-1}.
 \end{aligned}$$

where $\|Au\|_* = \sup\{|\langle Au, \varphi \rangle| = |B(u, \varphi)| : \varphi \in X, \|\varphi\|_X = 1\}$ is the norm of the operator Au . Hence, we obtain the operator

$$\begin{aligned}
 A : \quad X &\rightarrow X^* \\
 u &\mapsto Au.
 \end{aligned}$$

Consequently, problem (P) is equivalent to the operator equation

$$Au = T, \quad u \in X.$$

Step 5. Using condition (H2) and Lemma 2.1(b), we have

$$\begin{aligned} & \langle Au_1 - Au_2, u_1 - u_2 \rangle \\ &= B(u_1, u_1 - u_2) - B(u_2, u_1 - u_2) \\ &= \int_{\Omega} \mathcal{A}(x, u_1, \nabla u_1) \cdot \nabla(u_1 - u_2) \omega_3 \, dx + \int_{\Omega} |\Delta u_1|^{p-2} \Delta u_1 \Delta(u_1 - u_2) \omega_1 \, dx \\ &+ \int_{\Omega} |\Delta u_1|^{q-2} \Delta u_1 \Delta(u_1 - u_2) \omega_2 \, dx \\ &- \int_{\Omega} \mathcal{A}(x, u_2, \nabla u_2) \cdot \nabla(u_1 - u_2) \omega_3 \, dx - \int_{\Omega} |\Delta u_2|^{p-2} \Delta u_2 \Delta(u_1 - u_2) \omega_1 \, dx \\ &- \int_{\Omega} |\Delta u_2|^{q-2} \Delta u_2 \Delta(u_1 - u_2) \omega_2 \, dx \\ &= \int_{\Omega} \left(\mathcal{A}(x, u_1, \nabla u_1) - \mathcal{A}(x, u_2, \nabla u_2) \right) \cdot \nabla(u_1 - u_2) \omega_3 \, dx \\ &+ \int_{\Omega} (|\Delta u_1|^{p-2} \Delta u_1 - |\Delta u_2|^{p-2} \Delta u_2) \Delta(u_1 - u_2) \omega_1 \, dx \\ &+ \int_{\Omega} (|\Delta u_1|^{q-2} \Delta u_1 - |\Delta u_2|^{q-2} \Delta u_2) \Delta(u_1 - u_2) \omega_2 \, dx \\ &\geq \theta_1 \int_{\Omega} |\nabla(u_1 - u_2)|^p \omega_3 \, dx + \beta_p \int_{\Omega} (|\Delta u_1| + |\Delta u_2|)^{p-2} |\Delta u_1 - \Delta u_2|^2 \omega_1 \, dx \\ &+ \beta_q \int_{\Omega} (|\Delta u_1| + |\Delta u_2|)^{q-2} |\Delta u_1 - \Delta u_2|^2 \omega_2 \, dx \\ &\geq \theta_1 \int_{\Omega} |\nabla(u_1 - u_2)|^p \omega_3 \, dx + \beta_p \int_{\Omega} (|\Delta u_1 - \Delta u_2|)^{p-2} |\Delta u_1 - \Delta u_2|^2 \omega_1 \, dx \\ &+ \beta_q \int_{\Omega} (|\Delta u_1 - \Delta u_2|)^{q-2} |\Delta u_1 - \Delta u_2|^2 \omega_2 \, dx \\ &= \theta_1 \int_{\Omega} |\nabla(u_1 - u_2)|^p \omega_3 \, dx + \beta_p \int_{\Omega} |\Delta u_1 - \Delta u_2|^p \omega_1 \, dx \\ &+ \beta_q \int_{\Omega} |\Delta u_1 - \Delta u_2|^q \omega_2 \, dx \\ &\geq \theta_1 \int_{\Omega} |\nabla(u_1 - u_2)|^p \omega_3 \, dx + \beta_p \int_{\Omega} |\Delta u_1 - \Delta u_2|^p \omega_2 \, dx \\ &\geq \theta \|u_1 - u_2\|_X^p \end{aligned}$$

where $\theta = \min \{\theta_1, \beta_p\}$. Therefore, the operator A is strongly monotone, and this implies that A is strictly monotone. Moreover, from (H3), we obtain

$$\begin{aligned} \langle Au, u \rangle &= B(u, u) = B_1(u, u) + B_2(u, u) + B_3(u, u) \\ &= \int_{\Omega} \mathcal{A}(x, u, \nabla u) \cdot \nabla u \omega_3 \, dx + \int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta u \omega_1 \, dx \\ &+ \int_{\Omega} |\Delta u|^{q-2} \Delta u \Delta u \omega_2 \, dx \end{aligned}$$

$$\begin{aligned}
 &\geq \int_{\Omega} \lambda_1 |\nabla u|^p \omega_3 dx + \int_{\Omega} |\Delta u|^p \omega_1 dx + \int_{\Omega} |\Delta u|^q \omega_2 dx \\
 &\geq \int_{\Omega} \lambda_1 |\nabla u|^p \omega_3 dx + \int_{\Omega} |\Delta u|^p \omega_1 dx \\
 &\geq \gamma \|u\|_X^p,
 \end{aligned}$$

where $\gamma = \min \{\lambda_1, 1\}$. Hence, since $2 \leq q < p < \infty$, we have

$$\frac{\langle Au, u \rangle}{\|u\|_X} \rightarrow +\infty, \text{ as } \|u\|_X \rightarrow +\infty,$$

that is, A is coercive.

Step 6. We need to show that the operator A is continuous.

Let $u_m \rightarrow u$ in X as $m \rightarrow \infty$. We have,

$$\begin{aligned}
 |B_3(u_m, \varphi) - B_3(u, \varphi)| &\leq \sum_{j=1}^n \int_{\Omega} |\mathcal{A}_j(x, u_m, \nabla u_m) - \mathcal{A}_j(x, u, \nabla u)| |D_j \varphi| \omega_3 dx \\
 &= \sum_{j=1}^n \int_{\Omega} |F_j u_m - F_j u| |D_j \varphi| \omega_3 dx \\
 &\leq \sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega, \omega_3)} \|D_j \varphi\|_{L^p(\Omega, \omega_3)} \\
 &\leq \left(\sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega, \omega_3)} \right) \|\varphi\|_X,
 \end{aligned}$$

and

$$\begin{aligned}
 &|B_1(u_m, \varphi) - B_1(u, \varphi)| \\
 &= \left| \int_{\Omega} |\Delta u_m|^{p-2} \Delta u_m \Delta \varphi \omega_1 dx - \int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta \varphi \omega_1 dx \right| \\
 &\leq \int_{\Omega} \left| |\Delta u_m|^{p-2} \Delta u_m - |\Delta u|^{p-2} \Delta u \right| |\Delta \varphi| \omega_1 dx \\
 &= \int_{\Omega} |G_1 u_m - G_1 u| |\Delta \varphi| \omega_1 dx \\
 &\leq \|G_1 u_m - G_1 u\|_{L^{p'}(\Omega, \omega_1)} \|\varphi\|_X.
 \end{aligned}$$

and since $\frac{1}{s} + \frac{1}{r} + \frac{1}{p} = 1$ (remember that $s = p/(q-1)$ and $r = p/(p-q)$),

$$\begin{aligned}
 &|B_2(u_m, \varphi) - B_2(u, \varphi)| \\
 &= \left| \int_{\Omega} |\Delta u_m|^{q-2} \Delta u_m \Delta \varphi \omega_2 dx - \int_{\Omega} |\Delta u|^{q-2} \Delta u \Delta \varphi \omega_2 dx \right| \\
 &\leq \int_{\Omega} \left| |\Delta u_m|^{q-2} \Delta u_m - |\Delta u|^{q-2} \Delta u \right| |\Delta \varphi| \omega_2 dx \\
 &= \int_{\Omega} |G_2 u_m - G_2 u| |\Delta \varphi| \frac{\omega_2}{\omega_1} \omega_1 dx \\
 &\leq \left(\int_{\Omega} |G_2 u_m - G_2 u|^s \omega_1 dx \right)^{1/s} \left(\int_{\Omega} |\Delta \varphi|^p \omega_1 dx \right)^{1/p} \left(\int_{\Omega} \left(\frac{\omega_2}{\omega_1} \right)^r \omega_1 dx \right)^{1/r}
 \end{aligned}$$

$$\leq \|G_2 u_m - G_2 u\|_{L^s(\Omega, \omega_1)} \|\varphi\|_X \|\omega_2/\omega_1\|_{L^r(\Omega, \omega_1)},$$

for all $\varphi \in X$. Hence,

$$\begin{aligned} & |B(u_m, \varphi) - B(u, \varphi)| \\ & \leq |B_1(u_m, \varphi) - B_1(u, \varphi)| + |B_2(u_m, \varphi) - B_2(u, \varphi)| + |B_3(u_m, \varphi) - B_3(u, \varphi)| \\ & \leq \left[\sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega, \omega_3)} + \|G_1 u_m - G_1 u\|_{L^{p'}(\Omega, \omega_1)} \right. \\ & \quad \left. + \|G_2 u_m - G_2 u\|_{L^s(\Omega, \omega_1)} \|\omega_2/\omega_1\|_{L^r(\Omega, \omega_1)} \right] \|\varphi\|_X. \end{aligned}$$

Then we obtain

$$\begin{aligned} \|Au_m - Au\|_* & \leq \sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega, \omega_3)} + \|G_1 u_m - G_1 u\|_{L^{p'}(\Omega, \omega_1)} \\ & \quad + \|G_2 u_m - G_2 u\|_{L^s(\Omega, \omega_1)} \|\omega_2/\omega_1\|_{L^r(\Omega, \omega)}. \end{aligned}$$

Therefore, using (3.2), (3.5) and (3.7) we have $\|Au_m - Au\|_* \rightarrow 0$ as $m \rightarrow +\infty$, that is, A is continuous and this implies that A is hemicontinuous.

Therefore, by Theorem 3.1, the operator equation $Au = T$ has a unique solution $u \in X$ and it is the unique solution for problem (P).

Step 7. In particular, by setting $\varphi = u$ in Definition 2.4, we have

$$(3.9) \quad B(u, u) = B_1(u, u) + B_2(u, u) + B_3(u, u) = T(u).$$

Hence, using (H3) and $\gamma = \min\{\lambda_1, 1\}$, we obtain

$$\begin{aligned} & B_1(u, u) + B_2(u, u) + B_3(u, u) \\ & = \int_{\Omega} \mathcal{A}(x, u, \nabla u) \cdot \nabla u \omega_3 \, dx + \int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta u \omega_1 \, dx \\ & \quad + \int_{\Omega} |\Delta u|^{q-2} \Delta u \Delta u \omega_2 \, dx \\ & \geq \int_{\Omega} \lambda_1 |\nabla u|^p \omega_3 \, dx + \int_{\Omega} |\Delta u|^p \omega_1 \, dx + \int_{\Omega} |\Delta u|^q \omega_2 \, dx \\ & \geq \lambda_1 \int_{\Omega} |\nabla u|^p \omega_3 \, dx + \int_{\Omega} |\Delta u|^p \omega_1 \, dx \\ & \geq \gamma \|u\|_X^p \end{aligned}$$

and

$$\begin{aligned} T(u) & = \int_{\Omega} f_0 u \, dx + \sum_{j=1}^n \int_{\Omega} f_j D_j u \, dx \\ & \leq \|f_0/\omega_3\|_{L^{p'}(\Omega, \omega_3)} \|u\|_{L^p(\Omega, \omega_3)} + \sum_{j=1}^n \|f_j/\omega_3\|_{L^{p'}(\Omega, \omega_3)} \|D_j u\|_{L^p(\Omega, \omega_3)} \\ & \leq C_{\Omega} \|f_0/\omega_3\|_{L^{p'}(\Omega, \omega_3)} \|\nabla u\|_{L^p(\Omega, \omega_3)} \\ & \quad + \sum_{j=1}^n \|f_j/\omega_3\|_{L^{p'}(\Omega, \omega_3)} \|\nabla u\|_{L^p(\Omega, \omega_3)} \end{aligned}$$

$$\leq \left(C_\Omega \|f_0/\omega_3\|_{L^{p'}(\Omega,\omega_3)} + \sum_{j=1}^n \|f_j/\omega_3\|_{L^{p'}(\Omega,\omega_3)} \right) \|u\|_X.$$

Therefore, in (3.9), we obtain

$$\gamma \|u\|_X^p \leq \left(C_\Omega \|f_0/\omega_3\|_{L^{p'}(\Omega,\omega_3)} + \sum_{j=1}^n \|f_j/\omega_3\|_{L^{p'}(\Omega,\omega_3)} \right) \|u\|_X,$$

and we obtain

$$\|u\|_X \leq \frac{1}{\gamma^{p'/p}} \left(C_\Omega \|f_0/\omega_3\|_{L^{p'}(\Omega,\omega_3)} + \sum_{j=1}^n \|f_j/\omega_3\|_{L^{p'}(\Omega,\omega_3)} \right)^{p'/p}.$$

Example 1. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, the weight functions $\omega_1(x, y) = (x^2 + y^2)^{-1/4}$, $\omega_3(x, y) = (x^2 + y^2)^{-1/2}$ and $\omega_2(x, y) = (x^2 + y^2)^{-1/3}$ ($\omega_1, \omega_3 \in A_4$, $p = 4$ and $q = 3$), and the function

$$\begin{aligned} \mathcal{A} &: \Omega \times \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \\ \mathcal{A}((x, y), \eta, \xi) &= h_2(x, y) |\xi| \xi, \end{aligned}$$

where $h(x, y) = 2e^{(x^2+y^2)}$. Let us consider the partial differential operator

$$Lu(x, y) = \Delta[\omega_1(x, y) |\Delta u|^2 \Delta u + \omega_2(x, y) |\Delta u| \Delta u] - \operatorname{div} [\omega_3(x, y) \mathcal{A}((x, y), u, \nabla u)].$$

Therefore, by Theorem 1.1, the problem

$$(P) \begin{cases} Lu(x) &= \frac{\cos(xy)}{(x^2 + y^2)} - \frac{\partial}{\partial x} \left(\frac{\sin(xy)}{(x^2 + y^2)} \right) - \frac{\partial}{\partial y} \left(\frac{\sin(xy)}{(x^2 + y^2)} \right), \quad \text{in } \Omega \\ u(x) &= 0, \quad \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in X = W^{2,4}(\Omega, \omega_1) \cap W_0^{1,4}(\Omega, \omega_3)$.

Corollary 3.1. Under the assumptions of Theorem 1.1 with $2 \leq q < p < \infty$. If $u_1, u_2 \in X$ are solutions of

$$(P_1) \begin{cases} Lu_1 = f_0 - \sum_{j=1}^n D_j f_j \quad \text{in } \Omega, \\ u_1(x) = \Delta u_1(x) = 0 \quad \text{on } \partial\Omega, \end{cases}$$

and

$$(P_2) \begin{cases} Lu_2 = g_0 - \sum_{j=1}^n D_j g_j \quad \text{in } \Omega, \\ u_2(x) = \Delta u_2(x) = 0 \quad \text{on } \partial\Omega, \end{cases}$$

then

$$\|u_1 - u_2\|_X \leq \frac{1}{\alpha^{1/(p-1)}} \left(C_\Omega \left\| \frac{f_0 - g_0}{\omega_3} \right\|_{L^{p'}(\Omega,\omega_3)} + \sum_{j=1}^n \left\| \frac{f_j - g_j}{\omega_3} \right\|_{L^{p'}(\Omega,\omega_3)} \right)^{1/(p-1)},$$

where α is a positive constant and C_Ω is the constant in Theorem 2.2

Proof. If u_1 and u_2 are solutions of (P1) and (P2) then for all $\varphi \in X$ we have

$$\int_\Omega |\Delta u_1|^{p-2} \Delta u_1 \Delta \varphi \omega_1 dx + \int_\Omega |\Delta u_1|^{q-2} \Delta u_1 \Delta \varphi \omega_2 dx$$

$$\begin{aligned}
& + \int_{\Omega} \mathcal{A}(x, u_1, \nabla u_1) \cdot \nabla \varphi \omega_3 \, dx \\
& - \left(\int_{\Omega} |\Delta u_2|^{p-2} \Delta u_2 \Delta \varphi \omega_1 \, dx + \int_{\Omega} |\Delta u_2|^{q-2} \Delta u_2 \Delta \varphi \omega_2 \, dx \right. \\
& \left. + \int_{\Omega} \mathcal{A}(x, u_2, \nabla u_2) \cdot \nabla \varphi \omega_3 \, dx \right) \\
(3.10) \quad & = \int_{\Omega} (f_0 - g_0) \varphi \, dx + \sum_{j=1}^n \int_{\Omega} (f_j - g_j) D_j \varphi \, dx.
\end{aligned}$$

In particular, for $\varphi = u_1 - u_2$, we obtain in (3.10):

(i) By Lemma 2.1(b) and since $2 \leq q < p < \infty$, there exist two positive constants β_p and β_q such that

$$\begin{aligned}
& \int_{\Omega} \left(|\Delta u_1|^{p-2} \Delta u_1 - |\Delta u_2|^{p-2} \Delta u_2 \right) \Delta(u_1 - u_2) \omega_1 \, dx \\
& \geq \beta_p \int_{\Omega} \left(|\Delta u_1| + |\Delta u_2| \right)^{p-2} |\Delta u_1 - \Delta u_2|^2 \omega_1 \, dx \\
& \geq \beta_p \int_{\Omega} |\Delta u_1 - \Delta u_2|^{p-2} |\Delta u_1 - \Delta u_2|^2 \omega_1 \, dx = \beta_p \int_{\Omega} |\Delta(u_1 - u_2)|^p \omega_1 \, dx,
\end{aligned}$$

and analogously

$$\int_{\Omega} \left(|\Delta u_1|^{q-2} \Delta u_1 - |\Delta u_2|^{q-2} \Delta u_2 \right) \Delta(u_1 - u_2) \omega_2 \, dx \geq \beta_q \int_{\Omega} |\Delta(u_1 - u_2)|^q \omega_2 \, dx \geq 0.$$

(ii) By condition (H2)

$$\int_{\Omega} \left(\mathcal{A}(x, u_1, \nabla u_1) - \mathcal{A}(x, u_2, \nabla u_2) \right) \cdot \nabla(u_1 - u_2) \omega_3 \, dx \geq \theta_1 \int_{\Omega} |\nabla u_1 - \nabla u_2|^p \omega_3 \, dx.$$

(iii) We also have by condition (H6) and Theorem 2.2

$$\begin{aligned}
& \left| \int_{\Omega} (f_0 - g_0) (u_1 - u_2) \, dx + \sum_{j=1}^n \int_{\Omega} (f_j - g_j) D_j (u_1 - u_2) \, dx \right| \\
& \leq C_{\Omega} \left\| \frac{f_0 - g_0}{\omega_3} \right\|_{L^{p'}(\Omega, \omega_3)} \|\nabla(u_1 - u_2)\|_{L^p(\Omega, \omega_3)} \\
& + \left(\sum_{j=1}^n \left\| \frac{f_j - g_j}{\omega_3} \right\|_{L^{p'}(\Omega, \omega_3)} \right) \|\nabla(u_1 - u_2)\|_{L^p(\Omega, \omega_3)} \\
& \leq \left(C_{\Omega} \left\| \frac{f_0 - g_0}{\omega_3} \right\|_{L^{p'}(\Omega, \omega_3)} + \sum_{j=1}^n \left\| \frac{f_j - g_j}{\omega_3} \right\|_{L^{p'}(\Omega, \omega_3)} \right) \|u_1 - u_2\|_X.
\end{aligned}$$

Hence, with $\alpha = \min\{\beta_p, \theta_1\}$, we obtain

$$\begin{aligned}
& \alpha \|u_1 - u_2\|_X^p \\
& \leq \beta_p \int_{\Omega} |\Delta(u_1 - u_2)|^p \omega_1 \, dx + \theta_1 \int_{\Omega} |\nabla u_1 - \nabla u_2|^p \omega_3 \, dx \\
& \leq \left(C_{\Omega} \left\| \frac{f_0 - g_0}{\omega_3} \right\|_{L^{p'}(\Omega, \omega_3)} + \sum_{j=1}^n \left\| \frac{f_j - g_j}{\omega_3} \right\|_{L^{p'}(\Omega, \omega_3)} \right) \|u_1 - u_2\|_X.
\end{aligned}$$

Therefore, since $2 \leq q < p < \infty$,

$$\|u_1 - u_2\|_X \leq \frac{1}{\alpha^{1/(p-1)}} \left(C_\Omega \left\| \frac{f_0 - g_0}{\omega_3} \right\|_{L^{p'}(\Omega, \omega_3)} + \sum_{j=1}^n \left\| \frac{f_j - g_j}{\omega_3} \right\|_{L^{p'}(\Omega, \omega_3)} \right)^{1/(p-1)}.$$

□

Corollary 3.2. *Assume $2 \leq q < p < \infty$. Let the assumptions of Theorem 1.1 be fulfilled, and let $\{f_{0m}\}$ and $\{f_{jm}\}$ ($j = 1, \dots, n$) be sequences of functions satisfying $\frac{f_{0m}}{\omega_3} \rightarrow \frac{f_0}{\omega_3}$ in $L^{p'}(\Omega, \omega_3)$ and $\frac{f_{jm}}{\omega_3} \rightarrow \frac{f_j}{\omega_3}$ in $L^{p'}(\Omega, \omega_3)$ as $m \rightarrow \infty$. If $u_m \in X$ is a solution of the problem*

$$(P_m) \begin{cases} Lu_m(x) = f_{0m}(x) - \sum_{j=1}^n D_j f_{jm} & \text{in } \Omega, \\ u_m(x) = \Delta u_m(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

then $u_m \rightarrow u$ in X and u is a solution of problem (P).

Proof. By Corollary 3.1 we have

$$\begin{aligned} & \|u_m - u_k\|_X \\ & \leq \frac{1}{\alpha^{1/(p-1)}} \left(C_\Omega \left\| \frac{f_{0m} - f_{0k}}{\omega_3} \right\|_{L^{p'}(\Omega, \omega_3)} + \sum_{j=1}^n \left\| \frac{f_{jm} - f_{jk}}{\omega_3} \right\|_{L^{p'}(\Omega, \omega_3)} \right)^{1/(p-1)}. \end{aligned}$$

Therefore $\{u_m\}$ is a Cauchy sequence in X . Hence, there is $u \in X$ such that $u_m \rightarrow u$ in X . We have that u is a solution of problem (P). In fact, since u_m is a solution of (P_m) , for all $\varphi \in X$ we have

$$\begin{aligned} & \int_\Omega |\Delta u|^{p-2} \Delta u \Delta \varphi \omega_1 dx + \int_\Omega |\Delta u|^{q-2} \Delta u \Delta \varphi \omega_2 dx \\ & + \int_\Omega \mathcal{A}(x, u, \nabla u) \cdot \nabla \varphi \omega_3 dx \\ & = \int_\Omega \left(|\Delta u|^{p-2} \Delta u - |\Delta u_m|^{p-2} \Delta u_m \right) \Delta \varphi \omega_1 dx \\ & + \int_\Omega \left(|\Delta u|^{q-2} \Delta u - |\Delta u_m|^{q-2} \Delta u_m \right) \Delta \varphi \omega_2 dx \\ & + \int_\Omega \left(\mathcal{A}(x, u, \nabla u) - \mathcal{A}(x, u_m, \nabla u_m) \right) \cdot \nabla \varphi \omega_3 dx \\ & + \int_\Omega |\Delta u_m|^{p-2} \Delta u_m \Delta \varphi \omega_1 dx + \int_\Omega |\Delta u_m|^{q-2} \Delta u_m \Delta \varphi \omega_2 dx \\ & + \int_\Omega \mathcal{A}(x, u_m, \nabla u_m) \cdot \nabla \varphi \omega_3 dx \\ & = I_1 + I_2 + I_3 + \int_\Omega f_{0m} \varphi dx + \sum_{j=1}^n \int_\Omega f_{jm} D_j \varphi dx \\ & = I_1 + I_2 + I_3 + \int_\Omega f_0 \varphi dx + \sum_{j=1}^n \int_\Omega f_j D_j \varphi dx \end{aligned}$$

$$(3.11) \quad + \int_{\Omega} (f_{0m} - f_0) \varphi \, dx + \sum_{j=1}^n \int_{\Omega} (f_{jm} - f_j) D_j \varphi \, dx,$$

where

$$\begin{aligned} I_1 &= \int_{\Omega} \left(|\Delta u|^{p-2} \Delta u - |\Delta u_m|^{p-2} \Delta u_m \right) \Delta \varphi \omega_1 \, dx, \\ I_2 &= \int_{\Omega} \left(|\Delta u|^{q-2} \Delta u - |\Delta u_m|^{q-2} \Delta u_m \right) \Delta \varphi \omega_2 \, dx, \\ I_3 &= \int_{\Omega} \left(\mathcal{A}(x, u, \nabla u) - \mathcal{A}(x, u_m, \nabla u_m) \right) \cdot \nabla \varphi \omega_3 \, dx. \end{aligned}$$

We have that:

(1) By Lemma 2.1(a) there exists $\alpha_p > 0$ such that

$$\begin{aligned} |I_1| &\leq \int_{\Omega} \left| |\Delta u|^{p-2} \Delta u - |\Delta u_m|^{p-2} \Delta u_m \right| |\Delta \varphi| \omega_1 \, dx \\ &\leq \alpha_p \int_{\Omega} |\Delta u - \Delta u_m| (|\Delta u| + |\Delta u_m|)^{p-2} |\Delta \varphi| \omega_1 \, dx. \end{aligned}$$

Let $\delta = p/(p-2)$. Since $\frac{1}{p} + \frac{1}{p} + \frac{1}{\delta} = 1$, by the Generalized Hölder inequality we obtain

$$\begin{aligned} |I_1| &\leq \alpha_p \left(\int_{\Omega} |\Delta u - \Delta u_m|^p \omega_1 \, dx \right)^{1/p} \left(\int_{\Omega} |\Delta \varphi|^p \omega_1 \, dx \right)^{1/p} \\ &\quad \times \left(\int_{\Omega} (|\Delta u| + |\Delta u_m|)^{(p-2)\delta} \omega_1 \, dx \right)^{1/\delta} \\ &\leq \alpha_p \|u - u_m\|_X \|\varphi\|_X \| |\Delta u| + |\Delta u_m| \|_{L^p(\Omega, \omega_1)}^{(p-2)}. \end{aligned}$$

Now, since $u_m \rightarrow u$ in X , then exists a constant $M > 0$ such that $\|u_m\|_X \leq M$. Hence,

$$(3.12) \quad \| |\Delta u| + |\Delta u_m| \|_{L^p(\Omega, \omega_1)} \leq \|u\|_X + \|u_m\|_X \leq 2M.$$

Therefore,

$$\begin{aligned} |I_1| &\leq \alpha_p (2M)^{p-2} \|u - u_m\|_X \|\varphi\|_X \\ &= C_1 \|u - u_m\|_X \|\varphi\|_X, \end{aligned}$$

where $C_1 = \alpha_p (2M)^{p-2}$.

(2) By Lemma 2.1(a) there exists a positive constant α_q such that

$$\begin{aligned} |I_2| &\leq \int_{\Omega} \left| |\Delta u|^{q-2} \Delta u - |\Delta u_m|^{q-2} \Delta u_m \right| |\Delta \varphi| \omega_2 \, dx \\ &\leq \alpha_q \int_{\Omega} |\Delta u - \Delta u_m| (|\Delta u| + |\Delta u_m|)^{q-2} |\Delta \varphi| \omega_2 \, dx. \end{aligned}$$

Let $\varepsilon = q/(q-2)$ (if $2 < q < p < \infty$). Since $\frac{1}{q} + \frac{1}{q} + \frac{1}{\varepsilon} = 1$, by the Generalized Hölder inequality we obtain

$$|I_2|$$

$$\begin{aligned} &\leq \alpha_q \left(\int_{\Omega} |\Delta u - \Delta u_m|^q \omega_2 dx \right)^{1/q} \left(\int_{\Omega} |\Delta \varphi|^q \omega_2 dx \right)^{1/q} \\ &\times \left(\int_{\Omega} (|\Delta u| + |\Delta u_m|)^{(q-2)\varepsilon} \omega_2 dx \right)^{1/\varepsilon} \\ &= \alpha_q \|\Delta u - \Delta u_m\|_{L^q(\Omega, \omega_2)} \|\Delta \varphi\|_{L^q(\Omega, \omega_2)} \| |\Delta u| + |\Delta u_m| \|_{L^q(\Omega, \omega_2)}^{q-2}. \end{aligned}$$

Now, by Remark 2.1 and (3.12) we have

$$\begin{aligned} |I_2| &\leq \alpha_q C_{p,q} \|\Delta u - \Delta u_m\|_{L^p(\Omega, \omega_1)} C_{p,q} \|\Delta \varphi\|_{L^p(\Omega, \omega_1)} C_{p,q}^{q-2} \| |\Delta u| + |\Delta u_m| \|_{L^p(\Omega, \omega_1)}^{q-2} \\ &\leq \alpha_q C_{p,q}^q \|u - u_m\|_X \|\varphi\|_X (2M)^{q-2} \\ &= C_2 \|u - u_m\|_X \|\varphi\|_X, \end{aligned}$$

where $C_2 = (2M)^{q-2} \alpha_q C_{p,q}^q$.

In case $q = 2$, we have $I_2 = \int_{\Omega} (\Delta u - \Delta u_m) \Delta \varphi \omega_2 dx$, and by Remark 2.1 we obtain

$$\begin{aligned} |I_2| &\leq \|\Delta u - \Delta u_m\|_{L^2(\Omega, \omega_2)} \|\Delta \varphi\|_{L^2(\Omega, \omega_2)} \\ &\leq C_{p,2}^2 \|\Delta u - \Delta u_m\|_{L^p(\Omega, \omega_1)} \|\Delta \varphi\|_{L^p(\Omega, \omega_1)} \\ &\leq C_{p,2}^2 \|u - u_m\|_X \|\varphi\|_X. \end{aligned}$$

And we also have, by Step 1 (Theorem 1.1),

$$\begin{aligned} |I_3| &\leq \sum_{j=1}^n \int_{\Omega} |\mathcal{A}_j(x, u, \nabla u) - \mathcal{A}_j(x, u_m, \nabla u_m)| |D_j \varphi| \omega_3 dx \\ &= \sum_{j=1}^n \int_{\Omega} |F_j(u) - F_j(u_m)| |D_j \varphi| \omega_3 dx \\ &\leq \left(\sum_{j=1}^n \|F_j(u) - F_j(u_m)\|_{L^{p'}(\Omega, \omega_3)} \right) \|\nabla \varphi\|_{L^p(\Omega, \omega_3)} \\ &\leq \left(\sum_{j=1}^n \|F_j(u) - F_j(u_m)\|_{L^{p'}(\Omega, \omega_3)} \right) \|\varphi\|_X. \end{aligned}$$

Therefore, we have $I_1, I_2, I_3 \rightarrow 0$ when $m \rightarrow \infty$.

(3) We also have

$$\begin{aligned} &\left| \int_{\Omega} (f_{0m} - f_0) \varphi dx + \sum_{j=1}^n \int_{\Omega} (f_{jm} - f_j) D_j \varphi dx \right| \\ &\left(C_{\Omega} \left\| \frac{f_{0m} - f_0}{\omega_3} \right\|_{L^{p'}(\Omega, \omega_3)} + \sum_{j=1}^n \left\| \frac{f_{jm} - f_j}{\omega_3} \right\|_{L^{p'}(\Omega, \omega_3)} \right) \|\varphi\|_X \rightarrow 0, \end{aligned}$$

when $m \rightarrow \infty$.

Therefore, in (3.11), we obtain when $m \rightarrow \infty$ that

$$\int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta \varphi \omega_1 dx + \int_{\Omega} |\Delta u|^{q-2} \Delta u \Delta \varphi \omega_2 dx$$

$$\begin{aligned}
& + \int_{\Omega} \mathcal{A}(x, u, \nabla u) \cdot \nabla \varphi \omega_3 \, dx \\
& = \int_{\Omega} f_0 \varphi \, dx + \sum_{j=1}^n \int_{\Omega} f_j D_j \varphi \, dx,
\end{aligned}$$

i.e., u is a solution of problem (P). \square

REFERENCES

- [1] L.-E. Anderson, T. Elfving, G.H. Golub, *Solution of biharmonic equations with application to radar imaging*, J. Comput. Appl. Math., **94**:2 (1998), 153–180. Zbl 0933.65128
- [2] D. Bresch, J. Lemoine, F. Guillen-Gonzalez, *A note on a degenerate elliptic equations with applications for lakes and seas*, Electron. J. Differ. Equ., (2004), Paper No. 42. Zbl 1063.35131
- [3] A.C. Cavalheiro, *Existence and uniqueness of solutions for some degenerate nonlinear Dirichlet problems*, J. Appl. Anal., **19**:1 (2013), 41–54. Zbl 1278.35086
- [4] A.C. Cavalheiro, *Existence results for Dirichlet problems with degenerate p -Laplacian*, Opusc. Math., **33**:3 (2013), 439–453. Zbl 1278.35110
- [5] A.C. Cavalheiro, *Topics on Degenerate Elliptic Equations*, Lambert Academic Publishing, Germany (2018).
- [6] M. Chipot, *Elliptic Equations. An Introductory Course*, Birkhäuser, Basel, 2009. Zbl 1171.35003
- [7] M. Colombo, *Flows of non-smooth vector fields and degenerate elliptic equations. With applications to the Vlasov-Poisson and semigeostrophic systems*, Publications on the Scuola Normale Superiore Pisa, **22**, Pisa, 2017. Zbl 1417.37010
- [8] M. Colombo, A. Figalli, *Regularity results for very degenerate elliptic equations*, J. Math. Pures Appl., **101**:1 (2014), 94–117. Zbl 1282.35175
- [9] P. Drábek, A. Kufner, F. Nicolosi, *Quasilinear elliptic equations with degenerations and singularities*, Walter de Gruyter, Berlin, 1997. Zbl 0894.35002
- [10] E. Fabes, C. Kenig, R. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Commun. Partial Differ. Equations, **7** (1982), 77–116. Zbl 0498.35042
- [11] A. Kufner, O. John, S. Fučík, *Function spaces. From the Czech*, Noordhoff International Publ., Leyden, 1977. Zbl 0364.46022
- [12] J. Garcia-Cuerva, J.L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, **116**, Amsterdam etc., 1985. Zbl 0578.46046
- [13] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer, Berlin etc., 1983. Zbl 0562.35001
- [14] J. Heinonen, T. Kilpeläinen, O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Clarendon Press, Oxford, 1993. Zbl 0780.31001
- [15] A.A. Kovalevsky, Y.S. Gorban, *Solvability of degenerate anisotropic elliptic second-order equations with L^1 -data*, Electron. J. Differ. Equ., (2013), 2013, Paper No. **167**, 1–17. Zbl 1288.35225
- [16] A. Kufner, *Weighted Sobolev spaces*, John Wiley & Sons, Chichester etc., 1985. Zbl 0567.46009
- [17] A. Kufner, B. Opic, *How to define reasonably weighted Sobolev spaces*, Commentat. Math. Univ. Carol., **25** (1984), 537–554. Zbl 0557.46025
- [18] M.-C. Lai, H.-C. Liu, *Fast direct solver for the biharmonic equations on a disk and its application to incompressible flows*, Appl. Math. Comput., **164**:3 (2005), 679–695. Zbl 1070.65121
- [19] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Am. Math. Soc., **165** (1972), 207–226. Zbl 0236.26016
- [20] J. Petean, *Degenerate solutions of a nonlinear elliptic equation on the sphere*, Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods, **100** (2014), 23–29. Zbl 1287.35026
- [21] M. Talbi, N. Tsouli, *On the spectrum of the weighted p -biharmonic operator with weight*, Mediterr. J. Math., **4**:1 (2007), 73–86. Zbl 1150.35072
- [22] A. Torchinsky, *Real-variable methods in harmonic analysis*, Academic Press, Orlando etc., 1986. Zbl 0621.42001
- [23] B.O. Turesson, *Nonlinear potential theory and weighted Sobolev spaces*, Lecture Notes in Mathematics, **1736**, Springer-Verlag, Berlin, 2000. Zbl 0949.31006

- [24] E. Zeidler, *Nonlinear functional analysis and its applications. I*, Springer-Verlag, New York etc., 1990. Zbl 0583.47050
- [25] E. Zeidler, *Nonlinear functional analysis and its applications. II/B*, Springer-Verlag, New York etc., 1990. Zbl 0684.47029

ALBO CARLOS CAVALHEIRO
DEPARTMENT OF MATHEMATICS,
STATE UNIVERSITY OF LONDRINA
LONDRINA, 86057-970, BRAZIL
Email address: `accava@gmail.com`