GROUPS OF CENTRAL UNITS OF RANK 1 OF INTEGRAL GROUP RINGS OF FROBENIUS METACYCLIC GROUPS

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ABSTRACT. We describe groups of central units of integral group rings of Frobenius metacyclic groups of orders 39 and 156, and thus complete the study on groups of central units of rank 1 of integral group rings of Frobenius metacyclic groups for the case when m is a prime number.

Keywords: Frobenius group, metacyclic group, central units, integral group rings.

1. INTRODUCTION

A particular progress has been achieved in studying of groups of central units (also, centers of unit groups) of integral group rings. With efforts by R. Z. Aleev and his students, groups of central units of integral group rings for some unsolvable groups, such as A_5, A_6, $PSL(2, q)$, and $PSL(2, 2^n)$, have been studied. These results can be found in [2]. In recent years, R. Z. Aleev, O. V. Mitina, and T. A. Khanenko [5] and [4] have studied the groups of central units for cyclic groups of orders 16, 32, and 64. Currently, R. Z. Aleev and his students are studying the groups of central units for cyclic 3-groups. The results of this work have been announced in [3].

We will use the following notation.

Let n be a natural number. Then:

$\nu(n)$ is a number of all natural divisors of n;

$\varphi(n)$ is the Euler's function;

$[n]$ is an integer part of n;

i is the imaginary unit, that is, $i^2 = -1$;

$\omega = \frac{1+\sqrt{5}}{2}$.

Let G be a finite group. Then:

$\mathbb{Z}G$ is an integral group ring of the group G;

$Z(\mathbb{Z}G)$ is the center of the ring $\mathbb{Z}G$;

$U(Z(\mathbb{Z}G))$ is a group of central units (invertible elements of the center) of the ring $\mathbb{Z}G$;

$r(U(Z(\mathbb{Z}G)))$ is the rank of the group $U(Z(\mathbb{Z}G))$, that is, the number of infinite direct cyclic factors in a finitely generated Abelian group $U(Z(\mathbb{Z}G))$;

the mapping $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$, where for the elements $\sum_{g \in G} z_g g \in \mathbb{Z}G$ with $z_g \in \mathbb{Z}$ the inequality $\varepsilon(\sum_{g \in G} z_g g) = \sum_{g \in G} z_g$ holds, is called a homomorphism of trivialisation;

$V(Z(\mathbb{Z}G)) = \{u \in U(Z(\mathbb{Z}G)) \mid \varepsilon(u) = 1\}$ is a normalized group of central units $\mathbb{Z}G$, moreover, $U(Z(\mathbb{Z}G)) = \langle -1 \rangle \times V(Z(\mathbb{Z}G))$ [2, Lemma 1.48];

$y(x) = \sum\limits_{t \in x^G} t$ is a class sum in $\mathbb{Z}G$ of the conjugacy class x^G of the group G;

$X(G)$ is some system of representatives of conjugacy classes of the finite group G.

$F_{mn} = \langle b \rangle_m \rtimes \langle a \rangle_n$ is a Frobenius metacyclic group of order mn with the kernel $\langle b \rangle$ of order m and the complement $\langle a \rangle$ of order n.

$e(\chi)$ is a minimal central idempotent in the complex group algebra, corresponding to an irreducible character χ.

In work [11], the author obtained the following result (Corollary 3):

Lemma 1. *The rank of the group of central units of integral group ring of a Frobenius metacyclic group $F_{mn} = \langle b \rangle_m \rtimes \langle a \rangle_n$ where m is a prime equals 1 only in the following cases: $m = 5, n = 2$; $m = 11, n = 5$; $m = 13, n = 3$; $m = 13, n = 6$, and $m = 13, n = 12$.*

The generating elements of the group of central units of integral group rings of Frobenius metacyclic groups of orders 10 and 55 were found in [13], and of order 78 in [12]:

Lemma 2. *The groups of central units of integral group rings of Frobenius groups of orders 10, 55, and 78, have the following form:*

- $U(Z(\mathbb{Z}F_{5,2})) = \langle -1 \rangle \times \langle u \rangle$, *where* $u = 1 + e_1 - \omega^2 e_2 - \omega^{-2} e_3 = -1 + y(b^2)$,
- $U(Z(\mathbb{Z}F_{11,5})) = \langle -1 \rangle \times \langle u \rangle$, *where*

$$u = 1 + (4181 + 6765\omega)(e_1 + e_4) + (10946 - 6765\omega)(e_2 + e_3) + e_5 + e_6 =$$
$$= 551 + 550(y(b)) + y(b^2)) + 170(y(a) + y(a^4)) - 445(y(a^2) + y(a^3)).$$

- $U(Z(\mathbb{Z}F_{13,6})) = \langle -1 \rangle \times \langle u \rangle$, *where*

$$u = 1 + e_1 + e_2 + e_3 + e_4 + e_5 - \left(\frac{3 + \sqrt{13}}{2} \right)^{-2} e_6 - \left(\frac{3 + \sqrt{13}}{2} \right)^{2} e_7 =$$
$$= -5 + 2y(b) - y(b^2).$$

Therefore, in Lemma 2, three cases in which the rank of the unit group equals 1 are considered out of the five possible by Lemma 1. In this paper, we will describe the two remaining cases for Frobenius metacyclic groups of orders 39 and 156.

2. Description of the group of central units of integral group ring of a Frobenius metacyclic group of order 39

Lemma 3. *The character table of the group $F_{13,3} = \langle b \rangle_{13} \rtimes \langle a \rangle_3$, $a^{-1}ba = b^3$ has the form*

$\lvert x^G \rvert$	1	3	3	3	3	13	13
	1	b	b^2	b^{-1}	b^{-2}	a	a^2
χ_0	1	1	1	1	1	1	1
χ_1	1	1	1	1	1	α	α^{-1}
χ_2	1	1	1	1	1	α^{-1}	α
χ_3	3	φ_1	φ_2	$\overline{\varphi}_1$	$\overline{\varphi}_2$	0	0
χ_4	3	φ_2	$\overline{\varphi}_1$	$\overline{\varphi}_2$	φ_1	0	0
χ_5	3	$\overline{\varphi}_1$	$\overline{\varphi}_2$	φ_1	φ_2	0	0
χ_6	3	$\overline{\varphi}_2$	φ_1	φ_2	$\overline{\varphi}_1$	0	0

,

here, $\alpha = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, $\zeta = \cos\frac{2\pi}{13} + i\sin\frac{2\pi}{13}$, $\varphi_1 = \zeta + \zeta^3 + \zeta^9$, $\varphi_2 = \zeta^2 + \zeta^6 + \zeta^5$, $\overline{\varphi}_1 = \zeta^{-1} + \zeta^{-3} + \zeta^{-9}$, $\overline{\varphi}_2 = \zeta^{-2} + \zeta^{-6} + \zeta^{-5}$,

Proof. The following are the conjugacy classes of the group $F_{13,3}$:
$C_0 = \{1\}$; $C_1 = \{b, b^3, b^9\}$; $C_2 = \{b^2, b^6, b^5\}$; $C_3 = \{b^4, b^{12}, b^{10}\}$; $C_4 = \{b^8, b^{11}, b^7\}$
$C_{4+k} = \{a^k b^j \mid j \in \{0, \ldots, 12\}\}$, where $k \in \{1, 2\}$.

From Theorem (47.8) and Corollary (47.15) in [10, §47], it follows that there exist 3 linear representations and 4 representations of degree 3; the matrices of representations of degree 3 have the following form:

$$\phi(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \phi(b) = \begin{pmatrix} \zeta^j & 0 & 0 \\ 0 & \zeta^{3j} & 0 \\ 0 & 0 & \zeta^{9j} \end{pmatrix},$$

where $j \in \{1, 2, -1, -2\}$, $\zeta = \cos\frac{2\pi}{13} + i\sin\frac{2\pi}{13}$.
Then $\chi_3(a) = \chi_4(a) = \chi_5(a) = \chi_6(a) = 0$,
$\chi_3(b) = \zeta + \zeta^3 + \zeta^9 = \varphi_1 = \chi_4(b^{-2}) = \chi_5(b^{-1}) = \chi_6(b^2)$,
$\chi_4(b) = \zeta^2 + \zeta^6 + \zeta^5 = \varphi_2 = \chi_3(b^2) = \chi_5(b^{-2}) = \chi_6(b^{-1})$,
$\chi_5(b) = \zeta^{-1} + \zeta^{-3} + \zeta^{-9} = \overline{\varphi}_1 = \chi_3(b^{-1}) = \chi_4(b^2) = \chi_6(b^{-2})$,
$\chi_6(b) = \zeta^{-2} + \zeta^{-6} + \zeta^{-5} = \overline{\varphi}_2 = \chi_3(b^{-2}) = \chi_4(b^{-1}) = \chi_5(b^2)$.

Since $\chi_0(a) = \chi_0(b) = 1$, it only remains to calculate the values of χ_1, χ_2.

We denote $\lambda_s = \chi_s(b)$, $\mu_s = \chi_s(a)$, $s \in \{1, 2\}$. Note that $\lambda_s \in \{\zeta^j \mid j = 0, \ldots, 12\}$, $\mu_s \in \{\alpha^l \mid l \in \{0, 1, 2\}\}$. By the first orthogonality relation ([7, 2A4]), for $p = 0$ and $s \in \{1, 2\}$ we obtain:

$$1 + 3(\lambda_s + \lambda_s^2 + \lambda_s^{-1} + \lambda_s^{-2}) + 13(\mu_s + \mu_s^2) = 0.$$

Hence, with $\mu_s \in \{\alpha^l \mid l \in \{1, 2\}\}$, only $\lambda_s = 1$ is possible, and with $\mu_s = \alpha^0 = 1$, there are no solutions. $\qquad\square$

We will study the group of central units of integral group ring $F_{13,3}$. Consider two ordered bases of a $\mathbb{C}$–space $\mathbb{C}F_{13,3}$, which will be required in the following theorem:

- $Y(F_{13,3}) = (y_0, y_1, \ldots, y_6)$, where $y_0 = y(1) = 1$, $y_1 = y(b)$, $y_2 = y(b^2)$, $y_3 = y(b^{-1})$, $y_4 = y(b^{-2})$, $y_5 = y(a)$, $y_6 = y(a^2)$ (the basis from class sums for the center of the complex group algebra $\mathbb{C}F_{13,3}$);
- $E(F_{13,3}) = (e_0, e_1, \ldots, e_6)$, where $e_0 = e(\chi_0)$, $e_1 = e(\chi_1), \ldots, e_6 = e(\chi_6)$ (the basis from minimal central idempotent of the center of the complex group algebra $\mathbb{C}F_{13,3}$).

Let $T(F_{13,3})$ and $S(F_{13,3})$ be the transition matrices from the basis $Y(F_{13,3})$ to $E(F_{13,3})$ and from the basis $E(F_{13,3})$ to $Y(F_{13,3})$ respectively, that is,

$$(e_0, \ldots, e_6) = (y_0, \ldots, y_6) \cdot T(F_{13,3}),$$
$$(y_0, \ldots, y_6) = (e_0, \ldots, e_6) \cdot S(F_{13,3}).$$

By formulae from part 1 of Lemma 1.45 [2] and the character table from Lemma 3, we obtain:

$$(1) \qquad T(F_{13,3}) = \frac{1}{39} \begin{pmatrix} 1 & 1 & 1 & 9 & 9 & 9 & 9 \\ 1 & 1 & 1 & 3\varphi_1 & 3\varphi_2 & 3\overline{\varphi}_1 & 3\overline{\varphi}_2 \\ 1 & 1 & 1 & 3\varphi_2 & 3\overline{\varphi}_1 & 3\overline{\varphi}_2 & 3\varphi_1 \\ 1 & 1 & 1 & 3\overline{\varphi}_1 & 3\overline{\varphi}_2 & 3\varphi_1 & 3\varphi_2 \\ 1 & 1 & 1 & 3\overline{\varphi}_2 & 3\varphi_1 & 3\varphi_2 & 3\overline{\varphi}_1 \\ 1 & \alpha & \alpha^{-1} & 0 & 0 & 0 & 0 \\ 1 & \alpha^{-1} & \alpha & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$(2) \qquad S(F_{13,3}) = \begin{pmatrix} 1 & 3 & 3 & 3 & 3 & 13 & 13 \\ 1 & 3 & 3 & 3 & 3 & 13\alpha & 13\alpha^{-1} \\ 1 & 3 & 3 & 3 & 3 & 13\alpha^{-1} & 13\alpha \\ 1 & \varphi_1 & \varphi_2 & \overline{\varphi}_1 & \overline{\varphi}_2 & 0 & 0 \\ 1 & \varphi_2 & \overline{\varphi}_1 & \overline{\varphi}_2 & \varphi_1 & 0 & 0 \\ 1 & \overline{\varphi}_1 & \overline{\varphi}_2 & \varphi_1 & \varphi_2 & 0 & 0 \\ 1 & \overline{\varphi}_2 & \varphi_1 & \varphi_2 & \overline{\varphi}_1 & 0 & 0 \end{pmatrix}.$$

Theorem 1. *The group of central units of integral group ring of the group $F_{13,3} = \langle b \rangle_{13} \rtimes \langle a \rangle_3$ has the form $U(Z(\mathbb{Z}F_{13,3})) = \langle -1 \rangle \times \langle u \rangle$, where*

$$u = e_0 + e_1 + e_2 - \frac{11 + 3\sqrt{13}}{2}(e_3 + e_5) - \frac{11 - 3\sqrt{13}}{2}(e_4 + e_6) =$$
$$= -5 - y(b) - y(b^{-1}) + 2y(b^2) + 2y(b^{-2}).$$

Proof. By Theorem 3 [11], the group of central units of integral group ring $F_{13,3}$ has the form

$$U(Z(\mathbb{Z}F_{13,3})) = \langle -1 \rangle \times V,$$

where V is a direct product of infinite cyclic groups.

Let u be an arbitrary element from V, then

$$u = \sum_{j=0}^{6} \gamma_j y_j = \sum_{j=0}^{6} \beta_j e_j,$$

where the coefficients of γ_j are integers, and β_j are invertible elements of the rings of integers of the corresponding character fields (by Lemma 3.2 and Theorem 3.13 [2], Theorem 2 [6]). By formulae from part 2 of Lemma 1.45 [2], we obtain:

$$(3) \qquad \begin{pmatrix} \gamma_0 \\ \vdots \\ \gamma_6 \end{pmatrix} = T(F_{13,3}) \cdot \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_6 \end{pmatrix} = S(F_{13,3}) \cdot \begin{pmatrix} \gamma_0 \\ \vdots \\ \gamma_6 \end{pmatrix}.$$

Since $|(1)^F| = 1$, $|(b)^F| = 3$, $|(b^2)^F| = 3$, $|(b^{-1})^F| = 3$, $|(b^{-2})^F| = 3$, $|(a^k)^F| = 13$, $k \in \{1, 2\}$, by definition of the group V and Lemma 1.48 [2], we have $\varepsilon(u) =$

$\gamma_0 + 3\gamma_1 + 3\gamma_2 + 3\gamma_3 + 3\gamma_4 + 13\gamma_5 + 13\gamma_6 = 1$. From (1), (2), and (3), we get:

(4)
$$\begin{cases} \gamma_0 = \frac{1}{39}(1 + \beta_1 + \beta_2 + 9\beta_3 + 9\beta_4 + 9\beta_5 + 9\beta_6), \\ \gamma_1 = \frac{1}{39}(1 + \beta_1 + \beta_2 + 3\varphi_1\beta_3 + 3\varphi_2\beta_4 + 3\overline{\varphi}_1\beta_5 + 3\overline{\varphi}_2\beta_6), \\ \gamma_2 = \frac{1}{39}(1 + \beta_1 + \beta_2 + 3\varphi_2\beta_3 + 3\overline{\varphi}_1\beta_4 + 3\overline{\varphi}_2\beta_5 + 3\varphi_1\beta_6), \\ \gamma_3 = \frac{1}{39}(1 + \beta_1 + \beta_2 + 3\overline{\varphi}_1\beta_3 + 3\overline{\varphi}_2\beta_4 + 3\varphi_1\beta_5 + 3\varphi_2\beta_6), \\ \gamma_4 = \frac{1}{39}(1 + \beta_1 + \beta_2 + 3\overline{\varphi}_2\beta_3 + 3\varphi_1\beta_4 + 3\varphi_2\beta_5 + 3\overline{\varphi}_1\beta_6), \\ \gamma_5 = \frac{1}{39}(1 + \alpha\beta_1 + \alpha^{-1}\beta_2), \\ \gamma_6 = \frac{1}{39}(1 + \alpha^{-1}\beta_1 + \alpha\beta_2). \end{cases}$$

(5)
$$\begin{cases} \beta_0 = \gamma_0 + 3\gamma_1 + 3\gamma_2 + 3\gamma_3 + 3\gamma_4 + 13\gamma_5 + 13\gamma_6 = 1, \\ \beta_1 = \gamma_0 + 3(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) + 13(\alpha\gamma_5 + \alpha^{-1}\gamma_6), \\ \beta_2 = \gamma_0 + 3(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) + 13(\alpha^{-1}\gamma_5 + \alpha\gamma_6), \\ \beta_3 = \gamma_0 + \varphi_1\gamma_1 + \varphi_2\gamma_2 + \overline{\varphi}_1\gamma_3 + \overline{\varphi}_2\gamma_4, \\ \beta_4 = \gamma_0 + \varphi_2\gamma_1 + \overline{\varphi}_1\gamma_2 + \overline{\varphi}_2\gamma_3 + \varphi_1\gamma_4, \\ \beta_5 = \gamma_0 + \overline{\varphi}_1\gamma_1 + \overline{\varphi}_2\gamma_2 + \varphi_1\gamma_3 + \varphi_2\gamma_4 = \overline{\beta_3}, \\ \beta_6 = \gamma_0 + \overline{\varphi}_2\gamma_1 + \varphi_1\gamma_2 + \varphi_2\gamma_3 + \overline{\varphi}_1\gamma_4 = \overline{\beta_4}. \end{cases}$$

Therefore:

(6)
$$\begin{cases} \beta_1 + \beta_2 = 2(\gamma_0 + 3\gamma_1 + 3\gamma_2 + 3\gamma_3 + 3\gamma_4) - 13(\gamma_5 + \gamma_6) = 2 - 39(\gamma_5 + \gamma_6) = x, \\ \beta_1 - \beta_2 = 13\sqrt{3}i(\gamma_5 - \gamma_6) = y\sqrt{3}i, \end{cases}$$

where x and y are integers, moreover, $x^2 + 3y^2 = (\beta_1 + \beta_2)^2 - (\beta_1 - \beta_2)^2 = 4\beta_1\beta_2$. From system (5) it follows that β_1 and β_2 are algebraically conjugate, hence, $\beta_1\beta_2 = N_{\mathbb{Q}(\chi_1)}(\beta_1) = \pm 1$ by theorems [8, Theorem 4 from II §2] and [2, Theorem 3.13].

On the other hand, $x^2 + 3y^2 = 4 - 4\cdot39(\gamma_5 + \gamma_6) + 39^2(\gamma_5 + \gamma_6)^2 + 3\cdot169(\gamma_5 - \gamma_6)^2 \equiv 4$ (mod 39). Therefore, we obtain that $\beta_1\beta_2 = 1$ and $x^2 + 3y^2 = 4$. Then $x = \pm2, y = 0$ or $x = \pm1, y = \pm1$. Only $x = 2, y = 0$ satisfy the condition (6), hence, $\beta_1 = \beta_2 = 1$ and $\gamma_5 = \gamma_6 = 0$.

Further, from relations (4), (5), we obtain:

(7)
$$\begin{cases} \gamma_0 = \frac{1}{13}(1 + 3(\beta_3 + \beta_4 + \overline{\beta_3} + \overline{\beta_4})), \\ \gamma_1 = \frac{1}{13}(1 + \varphi_1\beta_3 + \varphi_2\beta_4 + \overline{\varphi}_1\overline{\beta_3} + \overline{\varphi}_2\overline{\beta_4}), \\ \gamma_2 = \frac{1}{13}(1 + \varphi_2\beta_3 + \overline{\varphi}_1\beta_4 + \overline{\varphi}_2\overline{\beta_3} + \varphi_1\overline{\beta_4}), \\ \gamma_3 = \frac{1}{13}(1 + \overline{\varphi}_1\beta_3 + \overline{\varphi}_2\beta_4 + \varphi_1\overline{\beta_3} + \varphi_2\overline{\beta_4}), \\ \gamma_4 = \frac{1}{13}(1 + \overline{\varphi}_2\beta_3 + \varphi_1\beta_4 + \varphi_2\overline{\beta_3} + \overline{\varphi}_1\overline{\beta_4}). \end{cases}$$

(8)
$$\begin{cases} \gamma_0 = 1 - 3(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4), \\ \beta_3 = \gamma_0 + \varphi_1\gamma_1 + \varphi_2\gamma_2 + \overline{\varphi}_1\gamma_3 + \overline{\varphi}_2\gamma_4, \\ \beta_4 = \gamma_0 + \varphi_2\gamma_1 + \overline{\varphi}_1\gamma_2 + \overline{\varphi}_2\gamma_3 + \varphi_1\gamma_4. \end{cases}$$

From system (8), we have that

$$(9) \quad \begin{cases} \varphi_1\beta_3 = \varphi_1\gamma_0 + \varphi_1^2\gamma_1 + \varphi_1\varphi_2\gamma_2 + \varphi_1\overline{\varphi_1}\gamma_3 + \varphi_1\overline{\varphi_2}\gamma_4, \\ \overline{\varphi_1\beta_3} = \overline{\varphi_1}\gamma_0 + \overline{\varphi_1}^2\gamma_1 + \overline{\varphi_1\varphi_2}\gamma_2 + \varphi_1\overline{\varphi_1}\gamma_3 + \varphi_2\overline{\varphi_1}\gamma_4, \\ \varphi_2\beta_4 = \varphi_2\gamma_0 + \varphi_2^2\gamma_1 + \varphi_2\overline{\varphi_1}\gamma_2 + \varphi_2\overline{\varphi_2}\gamma_3 + \varphi_2\varphi_1\gamma_4, \\ \overline{\varphi_2\beta_4} = \overline{\varphi_2}\gamma_0 + \overline{\varphi_2}^2\gamma_1 + \varphi_1\overline{\varphi_2}\gamma_2 + \varphi_2\overline{\varphi_2}\gamma_3 + \overline{\varphi_2\varphi_1}\gamma_4. \end{cases}$$

$$(10) \quad \begin{cases} \varphi_2\beta_3 = \varphi_2\gamma_0 + \varphi_2\varphi_1\gamma_1 + \varphi_2^2\gamma_2 + \varphi_2\overline{\varphi_1}\gamma_3 + \varphi_2\overline{\varphi_2}\gamma_4, \\ \overline{\varphi_2\beta_3} = \overline{\varphi_2}\gamma_0 + \overline{\varphi_2\varphi_1}\gamma_1 + \overline{\varphi_2}^2\gamma_2 + \varphi_1\overline{\varphi_2}\gamma_3 + \varphi_2\overline{\varphi_2}\gamma_4, \\ \overline{\varphi_1}\beta_4 = \overline{\varphi_1}\gamma_0 + \overline{\varphi_1}\varphi_2\gamma_1 + \overline{\varphi_1}^2\gamma_2 + \overline{\varphi_2\varphi_1}\gamma_3 + \overline{\varphi_1}\varphi_1\gamma_4, \\ \varphi_1\overline{\beta_4} = \varphi_1\gamma_0 + \varphi_1\overline{\varphi_2}\gamma_1 + \varphi_1^2\gamma_2 + \varphi_1\varphi_2\gamma_3 + \varphi_1\overline{\varphi_1}\gamma_4. \end{cases}$$

We calculate in the following lemma the values of algebraic combinations $\zeta = \cos\frac{2\pi}{13} + i\sin\frac{2\pi}{13}$ for the further reference.

Lemma 4.   (1) $\varphi_1 + \overline{\varphi_1} + \varphi_2 + \overline{\varphi_2} = -1$,
    (2) $(\varphi_1 + \overline{\varphi_1})(\varphi_2 - \overline{\varphi_2}) = -3$,
    (3) $\varphi_1 + \overline{\varphi_1} - \varphi_2 - \overline{\varphi_2} = \sqrt{13}$,
    (4) $\varphi_1^2 + \overline{\varphi_1}^2 + \varphi_2^2 + \overline{\varphi_2}^2 = -3$,
    (5) $\varphi_1\overline{\varphi_1} + \varphi_2\overline{\varphi_2} = 5$.

Proof. Suppose that $a = \varphi_1 + \overline{\varphi_1}, b = \varphi_2 + \overline{\varphi_2}$.

    (1) $\varphi_1 + \overline{\varphi_1} + \varphi_2 + \overline{\varphi_2} = a + b = \zeta + \zeta^3 + \zeta^9 + \zeta^{-1} + \zeta^{-3} + \zeta^{-9} + \zeta^2 + \zeta^6 + \zeta^5 + \zeta^{-2} + \zeta^{-6} + \zeta^{-5} = \sum_{j=1}^{12}\zeta^j = -1$.

    (2) $(\varphi_1 + \overline{\varphi_1})(\varphi_2 - \overline{\varphi_2}) = ab = (\zeta + \zeta^3 + \zeta^9 + \zeta^{-1} + \zeta^{-3} + \zeta^{-9})(\zeta^2 + \zeta^6 + \zeta^5 + \zeta^{-2} + \zeta^{-6} + \zeta^{-5}) = 3(\sum_{j=1}^{12}\zeta^j) = -3$.

    (3) Since $a + b = -1, ab = -3$, we have that a and b are the roots of the equation $x^2 + x - 3 = 0$, and then $a = \frac{\sqrt{13}-1}{2}, b = \frac{-\sqrt{13}-1}{2}, a - b = \sqrt{13} = \varphi_1 + \overline{\varphi_1} - \varphi_2 - \overline{\varphi_2}$.

    (4) $\varphi_1^2 + \overline{\varphi_1}^2 + \varphi_2^2 + \overline{\varphi_2}^2 = (\zeta + \zeta^3 + \zeta^9)^2 + (\zeta^{-1} + \zeta^{-3} + \zeta^{-9})^2 + (\zeta^2 + \zeta^6 + \zeta^5)^2 + (\zeta^{-2} + \zeta^{-6} + \zeta^{-5})^2 = 3\sum_{j=1}^{12}\zeta^j = -3$.

    (5) $\varphi_1\overline{\varphi_1} + \varphi_2\overline{\varphi_2} = (\zeta + \zeta^3 + \zeta^9)\cdot(\zeta^{-1} + \zeta^{-3} + \zeta^{-9}) + (\zeta^2 + \zeta^6 + \zeta^5)\cdot(\zeta^{-2} + \zeta^{-6} + \zeta^{-5}) = 6 + \sum_{j=1}^{12}\zeta^j = 5$.

$\square$

We continue the proof of the theorem. We sum termwise the equations from system (9) and the equations from system (10), use the results of Lemma 4 and substitute the obtained sums in the second and the third equations of system (7):

$$\begin{cases} 13\gamma_1 = 1 - \gamma_0 - 3\gamma_1 - 3\gamma_2 + 10\gamma_3 - 3\gamma_4 = 13\gamma_3, \\ 13\gamma_2 = 1 - \gamma_0 - 3\gamma_1 - 3\gamma_2 - 3\gamma_3 + 10\gamma_4 = 13\gamma_4. \end{cases}$$

Hence, $\gamma_1 = \gamma_3, \gamma_2 = \gamma_4$. Then from system (8), we obtain

$$(11) \quad \begin{cases} \gamma_0 = 1 - 6(\gamma_1 + \gamma_2), \\ \beta_3 = \gamma_0 + \frac{\sqrt{13}-1}{2}\gamma_1 + \frac{-\sqrt{13}-1}{2}\gamma_2 = \overline{\beta_3}, \\ \beta_4 = \gamma_0 + \frac{-\sqrt{13}-1}{2}\gamma_1 + \frac{\sqrt{13}-1}{2}\gamma_2 = \overline{\beta_4}. \end{cases}$$

Taking into account relations (11), the system (7) will have the form:

(12)
$$\begin{cases} 13\gamma_0 = 1 + 6(\beta_3 + \beta_4), \\ 13\gamma_1 = 1 + \frac{\sqrt{13}-1}{2}\beta_3 + \frac{-\sqrt{13}-1}{2}\beta_4, \\ 13\gamma_2 = 1 + \frac{-\sqrt{13}-1}{2}\beta_3 + \frac{\sqrt{13}-1}{2}\beta_4. \end{cases}$$

(13)
$$\begin{cases} 13(\gamma_1 + \gamma_2) = 2 - (\beta_3 + \beta_4) = 2 - x, \\ 13(\gamma_1 - \gamma_2) = \sqrt{13}(\beta_3 - \beta_4) = 13y, \end{cases}$$

where x and y are integers, moreover, $13y^2 - x^2 = (\beta_3 - \beta_4)^2 - (\beta_3 + \beta_4)^2 = -4\beta_3\beta_4$.

From system (5), it follows that β_3, β_4, $\overline{\beta_3}$, and $\overline{\beta_4}$ are algebraically conjugate, therefore, $\beta_3\beta_4\overline{\beta_3}\overline{\beta_4} = N_{\mathbb{Q}(\chi_3)}(\beta_3) = \pm 1$ by theorems [8, Theorem 4 from II §2] and [2, Theorem 3.13]. But β_3 and β_4 are real numbers, so $\beta_3\beta_4\overline{\beta_3}\overline{\beta_4} = (\beta_3\beta_4)^2 = 1$ and $\beta_3\beta_4 = \pm 1$.

On the other hand, $13y^2 - x^2 = 13(\gamma_1 - \gamma_2)^2 - (2 - 13(\gamma_1 + \gamma_2))^2 =$
$= 13(\gamma_1 - \gamma_2)^2 - 4 + 52(\gamma_1 + \gamma_2) - 169(\gamma_1 + \gamma_2)^2 \equiv -4 \pmod{13}$.

Hence, we obtain that $\beta_3\beta_4 = 1$ and $x^2 - 13y^2 = 4$.

Lemma 5.

$$x = \beta_3 + \beta_4 = (-1)^n \frac{(11 + 3\sqrt{13})^n + (11 - 3\sqrt{13})^n}{2^n},$$

$$y = \frac{\beta_3 - \beta_4}{\sqrt{13}} = \pm \frac{(11 + 3\sqrt{13})^n - (11 - 3\sqrt{13})^n}{2^n\sqrt{13}}.$$

Proof. The pair $(11; 3)$ is a solution of the equation $x^2 - 13y^2 = 4$ that gives the smallest value of the function $x + \sqrt{13}y$ on the set of natural solutions of this equation. By Theorem [1, p.341], the set of integer solutions of the equation $x^2 - 13y^2 = 4$ is as follows:

$$\left\{ (\pm x_n; \pm y_n) \mid x_n + \sqrt{13}y_n = \frac{(11 + 3\sqrt{13})^n}{2^{n-1}}, n = 1, 2... \right\}.$$

Then

$$x = \beta_3 + \beta_4 = \delta \frac{(11 + 3\sqrt{13})^n + (11 - 3\sqrt{13})^n}{2^n}, \text{ where } \delta = \pm 1;$$

$$y = \frac{\beta_3 - \beta_4}{\sqrt{13}} = \pm \frac{(11 + 3\sqrt{13})^n - (11 - 3\sqrt{13})^n}{2^n\sqrt{13}}.$$

It remains to prove that $\delta = (-1)^n$.

$$x = 2 - 13(\gamma_1 + \gamma_2) = \frac{2\delta}{2^n} \sum_{i=0}^{[\frac{n}{2}]} C_n^{2i} 11^{n-2i} 3^{2i} 13^i =$$

$$= \frac{2\delta}{2^n} \left(11^n + 3^2 \cdot 13 \sum_{i=1}^{[\frac{n}{2}]} C_n^{2i} 11^{n-2i} 3^{2i-2} 13^{i-1} \right),$$

$$(\gamma_1 + \gamma_2) = \frac{2-x}{13} = \frac{2}{13} - \frac{2\delta 11^n}{13 \cdot 2^n} - \frac{2\delta 3^2}{2^n} \sum_{i=1}^{[\frac{n}{2}]} C_n^{2i} 11^{n-2i} 3^{2i-2} 13^{i-1},$$

$$2^{n-1}(\gamma_1 + \gamma_2) = \frac{2^n}{13} - \frac{\delta 11^n}{13} - \delta 3^2 \sum_{i=1}^{[\frac{n}{2}]} C_n^{2i} 11^{n-2i} 3^{2i-2} 13^{i-1}.$$

We obtain that

$$\frac{2^n - \delta 11^n}{13} = \frac{2^n - \delta(13-2)^n}{13} = \frac{1}{13}\left(2^n - \delta\left(\sum_{i=0}^{n-1} C_n^i 13^{n-i}(-2)^i + (-2)^n\right)\right)$$

is an integer, which means that $2^n - \delta(-2)^n \equiv 0 \pmod{13}$, and hence $\delta = (-1)^n$.

$\square$

We finish the proof of the theorem. From Lemma 5, we obtain that $\beta_3 = -\left(\frac{11+3\sqrt{13}}{2}\right)^n$ or $\beta_3 = -\left(\frac{11-3\sqrt{13}}{2}\right)^n$ for an odd n. Since $\left(\frac{11+3\sqrt{13}}{2}\right)^{-1} = \frac{11-3\sqrt{13}}{2}$, we can assume for the generating element u, that $\beta_3 = -\left(\frac{11+3\sqrt{13}}{2}\right)$. Then $\beta_4 = -\left(\frac{11-3\sqrt{13}}{2}\right)$, $x = -11, y = -3$. From system (13), we obtain that $\gamma_1 = -1 = \gamma_3, \gamma_2 = 2 = \gamma_4$ and from (12), $\gamma_0 = -5$.

For the element u^{-1}, we have that $\beta_3 = -\left(\frac{11-3\sqrt{13}}{2}\right)$, then $\beta_4 = -\left(\frac{11+3\sqrt{13}}{2}\right)$, $x = -11, y = 3$. From system (13), we obtain that $\gamma_1 = 2 = \gamma_3, \gamma_2 = -1 = \gamma_4$ and from (12), $\gamma_0 = -5$. To summarize, we have obtained:

$$u = e_0 + e_1 + e_2 - \frac{11+3\sqrt{13}}{2}(e_3 + e_5) - \frac{11-3\sqrt{13}}{2}(e_4 + e_6) =$$
$$= -5 - y(b) - y(b^{-1}) + 2y(b^2) + 2y(b^{-2}),$$
$$u^{-1} = e_0 + e_1 + e_2 - \frac{11-3\sqrt{13}}{2}(e_3 + e_5) - \frac{11+3\sqrt{13}}{2}(e_4 + e_6) =$$
$$= -5 + 2y(b) + 2y(b^{-1}) - y(b^2) - y(b^{-2}).$$

$\square$

3. Description of the group of central units of the integral group ring of the Frobenius metacyclic group of order 156

Lemma 6. *The character table of the group $F_{13,12} = \langle b \rangle_{13} \rtimes \langle a \rangle_{12}$, $a^{-1}ba = b^2$ has the form*

$\lvert x^G \rvert$	1	12	13	13	13	13	13	13	13	13	13	13	13
	1	b	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	a^{11}
χ_0	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_1	1	1	α	α^2	i	α^4	α^5	-1	α^7	α^8	$-i$	α^{10}	α^{11}
χ_2	1	1	α^2	α^4	-1	α^8	α^{10}	1	α^2	α^4	-1	α^8	α^{10}
χ_3	1	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$
χ_4	1	1	α^4	α^8	1	α^4	α^8	1	α^4	α^8	1	α^4	α^8
χ_5	1	1	α^5	α^{10}	i	α^8	α	-1	α^{11}	α^4	$-i$	α^2	α^7
χ_6	1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
χ_7	1	1	α^7	α^2	$-i$	α^4	α^{11}	-1	α	α^8	i	α^{10}	α^5
χ_8	1	1	α^8	α^4	1	α^8	α^4	1	α^8	α^4	1	α^8	α^4
χ_9	1	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i
χ_{10}	1	1	α^{10}	α^8	-1	α^4	α^2	1	α^{10}	α^8	-1	α^4	α^2
χ_{11}	1	1	α^{11}	α^{10}	$-i$	α^8	α^7	-1	α^5	α^4	i	α^2	α
χ_{12}	12	-1	0	0	0	0	0	0	0	0	0	0	0

,

where $\alpha = \cos \frac{2\pi}{12} + i \sin \frac{2\pi}{12}$.

Proof. The following are the conjugacy classes of the group $F_{13,12}$:
$C_0 = \{1\}$; $C_1 = \{b, b^2, b^4, b^8, b^3, b^6, b^{12}, b^{11}, b^9, b^5, b^{10}, b^7\}$;
$C_{1+k} = \{a^k b^j \mid j \in \{0, \ldots, 12\}\}$, where $k \in \{1, \ldots, 11\}$.

From Theorem (47.8) and Corollary (47.15) in [10, §47], it follows that there exist 12 linear representations and 1 representation of order 12; the matrix of the representation of order 12 has the form:

$$\phi(a) = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & \cdots & 0 \end{pmatrix}, \quad \phi(b) = \begin{pmatrix} \zeta & 0 & \cdots & 0 \\ 0 & \zeta^2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \zeta^7 \end{pmatrix},$$

where $\zeta = \cos \frac{2\pi}{13} + i \sin \frac{2\pi}{13}$.

Therefore, $\chi_{12}(a) = 0$, $\chi_{12}(b) = -1$. Since $\chi_0(a) = \chi_0(b) = 1$, it remains to calculate the values of $\chi_1, \ldots \chi_{11}$. We denote $\lambda_s = \chi_s(b)$, $\mu_s = \chi_s(a)$, $s \in \{1, \ldots, 11\}$. Note that $\lambda_s \in \{\zeta^j \mid j = 0, \ldots, 12\}$, $\mu_s \in \{\alpha^l \mid l = 0, \ldots, 11\}$. By the first orthogonality relation ([7, 2A4]), for $p = 0$ and $s \in \{1, 2\}$ we obtain:

$$1 + 12\lambda_s + 13(\mu_s + \mu_s^2 + \cdots + \mu_s^{11}) = 0.$$

From here, given $\mu_s \in \{\alpha^l \mid l = 1, \ldots, 11\}$, only $\lambda_s = 1$ is possible, and given $\mu_s = \alpha^0 = 1$, there are no solutions. $\qquad \square$

We will study the group of central units of the integral group ring of the group $F_{13,12}$. Consider two ordered bases from a $\mathbb{C}$–space $\mathbb{C}F_{13,12}$, which will be required in the following theorem:

- $Y(F_{13,12}) = (y_0, y_1, \ldots, y_{12})$, where $y_0 = y(1) = 1$, $y_1 = y(b)$, $y_2 = y(a)$, $y_3 = y(a^2)$, $\ldots, y_{12} = y(a^{11})$ (the basis of class sums for the center of the complex group algebra $\mathbb{C}F_{13,12}$);
- $E(F_{13,12}) = (e_0, e_1, \ldots, e_{12})$, where $e_0 = e(\chi_0)$, $e_1 = e(\chi_1), \ldots, e_{12} = e(\chi_{12})$ (the basis of minimal central idempotent for the center of the complex group algebra $\mathbb{C}F_{13,12}$).

Let $T(F_{13,12})$ and $S(F_{13,12})$ be transition matrices from the basis $Y(F_{13,12})$ to $E(F_{13,12})$ and from the basis $E(F_{13,12})$ to $Y(F_{13,12})$ respectively, that is,

$$(e_0, \ldots, e_{12}) = (y_0, \ldots, y_{12}) \cdot T(F_{13,12}),$$
$$(y_0, \ldots, y_{12}) = (e_0, \ldots, e_{12}) \cdot S(F_{13,12}).$$

By formulae from part 1 of Lemma 1.45 [2] and the character table from Lemma 6, we obtain:

(14) $\quad 156T(F_{13,12}) =$

$$= \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 144 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -12 \\
1 & \alpha & \alpha^2 & i & \alpha^4 & \alpha^5 & -1 & \alpha^7 & \alpha^8 & -i & \alpha^{10} & \alpha^{11} & 0 \\
1 & \alpha^2 & \alpha^4 & -1 & \alpha^8 & \alpha^{10} & 1 & \alpha^2 & \alpha^4 & -1 & \alpha^8 & \alpha^{10} & 0 \\
1 & i & -1 & -i & 1 & i & -1 & -i & 1 & i & -1 & -i & 0 \\
1 & \alpha^4 & \alpha^8 & 1 & \alpha^4 & \alpha^8 & 1 & \alpha^4 & \alpha^8 & 1 & \alpha^4 & \alpha^8 & 0 \\
1 & \alpha^5 & \alpha^{10} & i & \alpha^8 & \alpha & -1 & \alpha^{11} & \alpha^4 & -i & \alpha^2 & \alpha^7 & 0 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 0 \\
1 & \alpha^7 & \alpha^2 & -i & \alpha^4 & \alpha^{11} & -1 & \alpha & \alpha^8 & i & \alpha^{10} & \alpha^5 & 0 \\
1 & \alpha^8 & \alpha^4 & 1 & \alpha^8 & \alpha^4 & 1 & \alpha^8 & \alpha^4 & 1 & \alpha^8 & \alpha^4 & 0 \\
1 & -i & -1 & i & 1 & -i & -1 & i & 1 & -i & -1 & i & 0 \\
1 & \alpha^{10} & \alpha^8 & -1 & \alpha^4 & \alpha^2 & 1 & \alpha^{10} & \alpha^8 & -1 & \alpha^4 & \alpha^2 & 0 \\
1 & \alpha^{11} & \alpha^{10} & -i & \alpha^8 & \alpha^7 & -1 & \alpha^5 & \alpha^4 & i & \alpha^2 & \alpha & 0
\end{pmatrix};$$

(15) $\quad \dfrac{1}{13}S(F_{13,12}) =$

$$= \begin{bmatrix}
1/13 & 12/13 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1/13 & 12/13 & \alpha & \alpha^2 & i & \alpha^4 & \alpha^5 & -1 & \alpha^7 & \alpha^8 & -i & \alpha^{10} & \alpha^{11} \\
1/13 & 12/13 & \alpha^2 & \alpha^4 & -1 & \alpha^8 & \alpha^{10} & 1 & 1\alpha^2 & \alpha^4 & -1 & \alpha^8 & \alpha^{10} \\
1/13 & 12/13 & i & -1 & -i & 1 & i & -1 & -i & 1 & i & -1 & -i \\
1/13 & 12/13 & \alpha^4 & \alpha^8 & 1 & \alpha^4 & \alpha^8 & 1 & \alpha^4 & \alpha^8 & 1 & \alpha^4 & \alpha^8 \\
1/13 & 12/13 & \alpha^5 & \alpha^{10} & i & \alpha^8 & \alpha & -1 & \alpha^{11} & \alpha^4 & -i & \alpha^2 & \alpha^7 \\
1/13 & 12/13 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1/13 & 12/13 & \alpha^7 & \alpha^2 & -i & \alpha^4 & \alpha^{11} & -1 & \alpha & \alpha^8 & i & \alpha^{10} & \alpha^5 \\
1/13 & 12/13 & \alpha^8 & \alpha^4 & 1 & \alpha^8 & \alpha^4 & 1 & \alpha^8 & \alpha^4 & 1 & \alpha^8 & \alpha^4 \\
1/13 & 12/13 & -i & -1 & i & 1 & -i & -1 & i & 1 & -i & -1 & i \\
1/13 & 12/13 & \alpha^{10} & \alpha^8 & -1 & \alpha^4 & \alpha^2 & 1 & \alpha^{10} & \alpha^8 & -1 & \alpha^4 & \alpha^2 \\
1/13 & 12/13 & \alpha^{11} & \alpha^{10} & -i & \alpha^8 & \alpha^7 & -1 & \alpha^5 & \alpha^4 & i & \alpha^2 & \alpha \\
1/13 & -1/13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Theorem 2. *The group of central units of the integral group ring of the group* $F_{13,12} = \langle b \rangle_{13} \rtimes \langle a \rangle_{12}$ *has the form* $U(Z(\mathbb{Z}F_{13,12})) = \langle -1 \rangle \times \langle u \rangle$, *where*

$$
\begin{aligned}
u &= e_0 + e_2 + e_3 + e_4 + e_6 + e_8 + e_9 + e_{10} + e_{12} + (2 + \sqrt{3})^{12}(e_1 + e_{11}) + \\
&+ (2 - \sqrt{3})^{12}(e_5 + e_7) = \\
&= 93601 + 46800(y(a^2) + y(a^{-2}) - y(a^4) - y(a^{-4}) - 2y(a^6) + 2y(b)) + \\
&+ 81060(y(a) + y(a^{-1}) - y(a^5) - y(a^{-5})).
\end{aligned}
$$

Proof. By Theorem 3 [11], the group of central units of the integral group ring of the group $F_{13,12}$ has the form

$$
U(Z(\mathbb{Z}F_{13,12})) = \langle -1 \rangle \times V,
$$

where V is a direct product of infinite cyclic groups.

Let u be an arbitrary element of V, then

$$
u = \sum_{j=0}^{12} \gamma_j y_j = \sum_{j=0}^{12} \beta_j e_j,
$$

where the coefficients γ_j are integers, and β_j are invertible elements of the rings of integers of the corresponding character fields (by Lemma 3.2 and Theorem 3.13 [2], Theorem 2 [6]). By formulae from part 2 of Lemma 1.45 [2] we obtain:

$$
(16) \qquad \begin{pmatrix} \gamma_0 \\ \vdots \\ \gamma_{12} \end{pmatrix} = T(F_{13,12}) \cdot \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{12} \end{pmatrix} \qquad \text{и} \qquad \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{12} \end{pmatrix} = S(F_{13,12}) \cdot \begin{pmatrix} \gamma_0 \\ \vdots \\ \gamma_{12} \end{pmatrix}.
$$

Since $|(1)^F| = 1$, $|(b)^F| = 12$, $|(a^k)^F| = 13$, $k \in \{1, \ldots, 11\}$, then by the definition of the group V and Lemma 1.48 [2], $\varepsilon(u) = \gamma_0 + 12\gamma_1 + 13(\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7 + \gamma_8 + \gamma_9 + \gamma_{10} + \gamma_{11} + \gamma_{12}) = 1$. From (14), (15), and (16), we obtain:

$$
(17) \qquad \begin{cases} \gamma_0 = \frac{1}{156}(1 + \sum_{j=1}^{11} \beta_j + 144\beta_{12}), \\ \gamma_1 = \frac{1}{156}(1 + \sum_{j=1}^{11} \beta_j - 12\beta_{12}), \\ \gamma_{k+1} = \frac{1}{156}(1 + \sum_{j=1}^{11} \alpha^{kj}\beta_j), k = 1, \ldots, 11. \end{cases}
$$

$$
(18) \qquad \begin{cases} \beta_0 = \gamma_0 + 12\gamma_1 + 13\sum_{k=1}^{11} \gamma_{k+1} = 1, \\ \beta_j = \gamma_0 + 12\gamma_1 + 13\sum_{k=1}^{11} \alpha^{kj}\gamma_{k+1}, j = 1, \ldots, 11, \\ \beta_{12} = \gamma_0 - \gamma_1. \end{cases}
$$

Since $\alpha^6 = -1$ and β_j are invertible elements of the rings of integers of the corresponding character fields, then from (18), we obtain that $\beta_{12} = \pm 1, \beta_6 = \pm 1$.

If $\beta_{12} = -1, \gamma_0 = \beta_{12} + \gamma_1 = \gamma_1 - 1$, then from (18) we obtain that $1 = \beta_0 = \gamma_1 - 1 + 12\gamma_1 + 13\sum_{k=1}^{11} \gamma_{k+1} = -1 + 13\sum_{k=0}^{11} \gamma_{k+1}$, which is impossible. It means that $\beta_{12} = 1, \gamma_0 = \beta_{12} + \gamma_1 = \gamma_1 + 1$.

Further, from (18) we obtain that $\beta_6 = 1 + 13\gamma_1 + 13\sum_{k=1}^{11}(-1)^k\gamma_{k+1} = -1$ is impossible, which implies $\beta_6 = 1, \sum_{k=0}^{11}(-1)^k\gamma_{k+1} = 0$, and then

$$
(19) \qquad \gamma_1 + \gamma_3 + \gamma_5 + \gamma_7 + \gamma_9 + \gamma_{11} = \gamma_2 + \gamma_4 + \gamma_6 + \gamma_8 + \gamma_{10} + \gamma_{12}.
$$

Since $\alpha^3 = i$, from (18) we get

(20)
$$\begin{cases} \beta_3 = 1 + 13(\gamma_1 + \gamma_5 + \gamma_9 - \gamma_3 - \gamma_7 - \gamma_{11}) + 13i(\gamma_2 + \gamma_6 + \gamma_{10} - \gamma_4 - \gamma_8 - \gamma_{12}), \\ \beta_9 = 1 + 13(\gamma_1 + \gamma_5 + \gamma_9 - \gamma_3 - \gamma_7 - \gamma_{11}) - 13i(\gamma_2 + \gamma_6 + \gamma_{10} - \gamma_4 - \gamma_8 - \gamma_{12}). \end{cases}$$

From relations (20), we find

(21)
$$\begin{cases} \beta_3 + \beta_9 = 2 + 26(\gamma_1 + \gamma_5 + \gamma_9 - \gamma_3 - \gamma_7 - \gamma_{11}) = x, \\ \beta_3 - \beta_9 = 26i(\gamma_2 + \gamma_6 + \gamma_{10} - \gamma_4 - \gamma_8 - \gamma_{12}) = iy, \end{cases}$$

where x and y are integers, moreover, $x^2 + y^2 = (\beta_3 + \beta_9)^2 - (\beta_3 - \beta_9)^2 = 4\beta_3\beta_9$. From system (18), it follows that β_3 and β_9 are algebraically conjugate, hence, $\beta_3\beta_9 = N_{\mathbb{Q}(\chi_1)}(\beta_3) = \pm 1$ by theorems [8, Theorem 4 from II §2] and [2, Theorem 3.13].

On the other hand, $x^2 + y^2 = 4 + 104(\gamma_1 + \gamma_5 + \gamma_9 - \gamma_3 - \gamma_7 - \gamma_{11}) + 26^2(\gamma_1 + \gamma_5 + \gamma_9 - \gamma_3 - \gamma_7 - \gamma_{11})^2 + 26^2(\gamma_2 + \gamma_6 + \gamma_{10} - \gamma_4 - \gamma_8 - \gamma_{12})^2 \equiv 4 \pmod{52}$. From here we obtain that $\beta_3\beta_9 = 1$ and $x^2 + y^2 = 4$. Then $x = \pm 2, y = 0$ or $x = 0, y = \pm 2$. Only $x = 2, y = 0$ satisfy condition (21), which means that $\beta_3 = \beta_9 = 1$ and

(22) $\qquad \gamma_1 + \gamma_5 + \gamma_9 = \gamma_3 + \gamma_7 + \gamma_{11} = \gamma_2 + \gamma_6 + \gamma_{10} = \gamma_4 + \gamma_8 + \gamma_{12} = \Lambda.$

Further, from (18) we obtain

(23)
$$\begin{cases} \beta_4 &= 1 + 13(\alpha^4(\gamma_2 + \gamma_5 + \gamma_8 + \gamma_{11}) + \alpha^8(\gamma_3 + \gamma_6 + \gamma_9 + \gamma_{12}) + \\ &\quad + (\gamma_1 + \gamma_4 + \gamma_7 + \gamma_{10})), \\ \beta_8 &= 1 + 13(\alpha^8(\gamma_2 + \gamma_5 + \gamma_8 + \gamma_{11}) + \alpha^4(\gamma_3 + \gamma_6 + \gamma_9 + \gamma_{12}) + \\ &\quad + (\gamma_1 + \gamma_4 + \gamma_7 + \gamma_{10})). \end{cases}$$

Since $\alpha^4 + \alpha^8 = -1, \alpha^4 - \alpha^8 = \sqrt{3}i$, then from (23) we get

(24)
$$\begin{cases} \beta_4 + \beta_8 &= 2 - 13((\gamma_2 + \gamma_5 + \gamma_8 + \gamma_{11} + \gamma_3 + \gamma_6 + \gamma_9 + \gamma_{12}) + \\ &\quad + 2(\gamma_1 + \gamma_4 + \gamma_7 + \gamma_{10})) = 2 - 13\Gamma = x, \\ \beta_4 - \beta_8 &= 13\sqrt{3}i(\gamma_2 + \gamma_5 + \gamma_8 + \gamma_{11} - \gamma_3 - \gamma_6 - \gamma_9 - \gamma_{12}) = 13\sqrt{3}i\Delta = \sqrt{3}iy, \end{cases}$$

where x and y are integers, moreover, $x^2 + 3y^2 = (\beta_4 + \beta_8)^2 - (\beta_4 - \beta_8)^2 = 4\beta_4\beta_8$. From system (18), it follows that β_4 and β_8 are algebraically conjugate, therefore, $\beta_4\beta_8 = N_{\mathbb{Q}(\chi_1)}(\beta_4) = \pm 1$ by theorems [8, Theorem 4 from II §2] and [2, Theorem 3.13].

On the other hand, $x^2 + 3y^2 = 4 - 52\Gamma + 13^2\Gamma^2 + 3 \cdot 13\Delta^2 \equiv 4 \pmod{13}$. From here we obtain that $\beta_4\beta_8 = 1$ and $x^2 + 3y^2 = 4$. Then $x = \pm 2, y = 0$ or $x = \pm 1, y = \pm 1$. Only $x = 2, y = 0$ satisfy condition (24), which means that $\beta_4 = \beta_8 = 1$ and

(25) $\qquad \gamma_2 + \gamma_5 + \gamma_8 + \gamma_{11} = \gamma_3 + \gamma_6 + \gamma_9 + \gamma_{12} = \gamma_1 + \gamma_4 + \gamma_7 + \gamma_{10} = \Theta.$

From relations (22) and (25), we obtain $4\Lambda = 3\Theta$, then in integers we have that $\Lambda = \Theta = 0$.

Further, from (18) we get

(26)
$$\begin{cases} \beta_2 &= 1 + 13((\gamma_1 + \gamma_7) + \alpha^2(\gamma_2 + \gamma_8) + \alpha^4(\gamma_3 + \gamma_9) - (\gamma_4 + \gamma_{10}) + \\ &\quad + \alpha^8(\gamma_5 + \gamma_{11}) + \alpha^{10}(\gamma_6 + \gamma_{12})), \\ \beta_{10} &= 1 + 13((\gamma_1 + \gamma_7) + \alpha^{10}(\gamma_2 + \gamma_8) + \alpha^8(\gamma_3 + \gamma_9) - (\gamma_4 + \gamma_{10}) + \\ &\quad + \alpha^4(\gamma_5 + \gamma_{11}) + \alpha^2(\gamma_6 + \gamma_{12})). \end{cases}$$

Since $\alpha^4 + \alpha^8 = -1, \alpha^4 - \alpha^8 = \sqrt{3}i, \alpha^2 + \alpha^{10} = 1, \alpha^2 - \alpha^{10} = \sqrt{3}i$, then from (26) we have that

(27)
$$\begin{cases} \beta_2 + \beta_{10} & = 2 + 13(2(\gamma_1 + \gamma_7) - 2(\gamma_4 + \gamma_{10}) + \gamma_2 + \gamma_8 + \gamma_6 + \gamma_{12} - \\ & \quad -(\gamma_3 + \gamma_9 + \gamma_5 + \gamma_{11})) = 2 + 13\Gamma = x, \\ \beta_2 - \beta_{10} & = 13\sqrt{3}i(\gamma_2 + \gamma_8 + \gamma_3 + \gamma_9 - \gamma_5 - \gamma_{11} - \gamma_6 - \gamma_{12}) = 13\sqrt{3}i\Delta = \sqrt{3}iy, \end{cases}$$

where x and y are integers, moreover, $x^2 + 3y^2 = (\beta_2 + \beta_{10})^2 - (\beta_2 - \beta_{10})^2 = 4\beta_2\beta_{10}$. From system (18), it follows that β_2 and β_{10} are algebraically conjugate, hence, $\beta_2\beta_{10} = N_{\mathbb{Q}(\chi_1)}(\beta_2) = \pm 1$ by theorems [8, Theorem 4 from II §2] and [2, Theorem 3.13].

On the other hand, $x^2 + 3y^2 = 4 + 52\Gamma + 13^2\Gamma^2 + 3 \cdot 13^2\Delta^2 \equiv 4 \pmod{13}$. From here we obtain that $\beta_4\beta_8 = 1$ and $x^2 + 3y^2 = 4$. Then $x = \pm 2, y = 0$ or $x = \pm 1, y = \pm 1$. Only $x = 2, y = 0$ satisfy the condition (27), which means that $\beta_2 = \beta_{10} = 1$ and

(28)
$$\begin{cases} \gamma_2 + \gamma_8 + \gamma_3 + \gamma_9 = \gamma_5 + \gamma_{11} + \gamma_6 + \gamma_{12}, \\ 2(\gamma_1 + \gamma_7) + \gamma_2 + \gamma_8 + \gamma_6 + \gamma_{12} = \gamma_3 + \gamma_9 + \gamma_5 + \gamma_{11} + 2(\gamma_4 + \gamma_{10}). \end{cases}$$

We will find the difference and sum of equalities (28)

$$\begin{cases} 2(\gamma_1 + \gamma_7) + \gamma_6 + \gamma_{12} - \gamma_3 - \gamma_9 = \gamma_3 + \gamma_9 + 2(\gamma_4 + \gamma_{10}) - \gamma_6 - \gamma_{12}, \\ 2(\gamma_1 + \gamma_7 + \gamma_2 + \gamma_8) = 2(\gamma_4 + \gamma_{10} + \gamma_5 + \gamma_{11}), \end{cases}$$

(29)
$$\begin{cases} \gamma_1 + \gamma_7 + \gamma_6 + \gamma_{12} = \gamma_3 + \gamma_9 + \gamma_4 + \gamma_{10}, \\ \gamma_1 + \gamma_7 + \gamma_2 + \gamma_8 = \gamma_4 + \gamma_{10} + \gamma_5 + \gamma_{11}. \end{cases}$$

We denote

$$\begin{cases} a = \gamma_1 + \gamma_7, \\ b = \gamma_6 + \gamma_{12}, \\ c = \gamma_4 + \gamma_{10}, \\ d = \gamma_3 + \gamma_9, \\ e = \gamma_2 + \gamma_8, \\ f = \gamma_5 + \gamma_{11}. \end{cases}$$

We substitute the notations in equalities (29), (25), and (22):

(30)
$$\begin{cases} a + b = c + d \\ a + e = c + f, \\ a + c = 0, \\ b + d = 0, \\ e + f = 0, \\ 2\Lambda = 2(a + d + f) = 2(b + c + e) = 0. \end{cases}$$

Solving the system (30), we obtain $a = b = c = d = e = f = 0$ and

$$(31) \quad \begin{cases} \gamma_7 = -\gamma_1, \\ \gamma_8 = -\gamma_2, \\ \gamma_9 = -\gamma_3 = -\gamma_1 - \gamma_5, \\ \gamma_{10} = -\gamma_4 = -\gamma_2 - \gamma_6, \\ \gamma_{11} = -\gamma_5, \\ \gamma_{12} = -\gamma_6. \end{cases}$$

From system (17), we get

$$(32)$$
$$\begin{cases} 156\gamma_1 &= \beta_1 + \beta_5 + \beta_7 + \beta_{11} - 4, \\ 156\gamma_2 &= 1 + \alpha\beta_1 + \alpha^5\beta_5 + \alpha^7\beta_7 + \alpha^{11}\beta_{11} + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^6 + \alpha^8 + \alpha^9 + \\ &\quad + \alpha^{10} = \alpha\beta_1 + \alpha^5\beta_5 + \alpha^7\beta_7 + \alpha^{11}\beta_{11}, \\ 156\gamma_5 &= 1 + \alpha^4\beta_1 + \alpha^8\beta_5 + \alpha^4\beta_7 + \alpha^8\beta_{11} + \alpha^8 + 1 + \alpha^4 + 1 + \alpha^8 + 1 + \alpha^4 = \\ &\quad = \alpha^4\beta_1 + \alpha^8\beta_5 + \alpha^4\beta_7 + \alpha^8\beta_{11} + 2, \\ 156\gamma_6 &= 1 + \alpha^5\beta_1 + \alpha\beta_5 + \alpha^{11}\beta_7 + \alpha^7\beta_{11} + \alpha^{10} + \alpha^3 + \alpha^8 + \alpha^6 + \alpha^4 + \alpha^9 + \\ &\quad + \alpha^2 = \alpha^5\beta_1 + \alpha\beta_5 + \alpha^{11}\beta_7 + \alpha^7\beta_{11}. \end{cases}$$

Lemma 7. *Let σ be an automorphism of the field $\mathbb{Q}(\alpha)$ and $\sigma(\alpha) = \alpha^5$. Then $\sigma(\beta_1) = \beta_5$, $\sigma(\beta_7) = \beta_{11}$ and $\gamma_1 = -2\gamma_5, \gamma_4 = 0$.*

Proof. By Theorem [9, §60, p. 204], $\{1, \sigma, ^-, \overline{\sigma}\}$ is a set of all automorphisms of the field $\mathbb{Q}(\alpha)$ (where $^-$ is a complex conjugation). Then from relations (18), we obtain $\sigma(\beta_1) = \beta_5$, $\sigma(\beta_7) = \beta_{11}, \overline{\alpha} = \alpha^{11}, \sigma(\overline{\alpha}) = \alpha^7, \overline{\beta_1} = \beta_{11}$.

The relations (32) take the form

$$(33)$$
$$\begin{cases} 156\gamma_2 = \alpha\beta_1 + \sigma(\alpha\beta_1) + \overline{\alpha\beta_1} + \sigma(\overline{\alpha\beta_1}) = (1+\sigma)(\alpha\beta_1 + \overline{\alpha\beta_1}), \\ 156\gamma_5 - 2 = \alpha^4\beta_1 + \sigma(\alpha^4\beta_1) + \overline{\alpha^4\beta_1} + \sigma(\overline{\alpha^4\beta_1}) = (1+\sigma)(\alpha^4\beta_1 + \overline{\alpha^4\beta_1}). \end{cases}$$

$$(34) \quad \begin{cases} \beta_1 = 1 + 13\gamma_1 + 13\sum_{k=1}^{11}\alpha^k\gamma_{k+1}, \\ \overline{\beta_1} = 1 + 13\gamma_1 + 13\sum_{k=1}^{11}\alpha^{12-k}\gamma_{k+1}, \\ \alpha\beta_1 = \alpha + 13\alpha\gamma_1 + 13\sum_{k=1}^{11}\alpha^{k+1}\gamma_{k+1}, \\ \overline{\alpha\beta_1} = \overline{\alpha} + 13\overline{\alpha}\gamma_1 + 13\sum_{k=1}^{11}\alpha^{11-k}\gamma_{k+1}, \\ \alpha^4\beta_1 = \alpha^4 + 13\alpha^4\gamma_1 + 13\sum_{k=1}^{11}\alpha^{k+4}\gamma_{k+1}, \\ \overline{\alpha^4\beta_1} = \overline{\alpha^4} + 13\overline{\alpha^4}\gamma_1 + 13\sum_{k=1}^{11}\alpha^{8-k}\gamma_{k+1} \end{cases}$$

Since $\alpha + \overline{\alpha} = \sqrt{3}, \alpha^4 + \alpha^8 = -1$, then from (34) we get

$$(35) \quad \begin{cases} \alpha\beta_1 + \overline{\alpha\beta_1} = \sqrt{3}(1 + 13\gamma_1) + 13\sum_{k=1}^{11}(\alpha^{k+1} + \alpha^{11-k})\gamma_{k+1}, \\ \alpha^4\beta_1 + \overline{\alpha^4\beta_1} = -1 - 13\gamma_1 + 13\sum_{k=1}^{11}(\alpha^{k+4} + \alpha^{8-k})\gamma_{k+1}. \end{cases}$$

We substitute the result of (35) in (33)

$$(36)$$
$$\begin{cases} 156\gamma_2 &= (1+\sigma)(\alpha\beta_1 + \overline{\alpha\beta_1}) = 13\sum_{k=1}^{11}(\alpha^{k+1} + \alpha^{11-k} + \alpha^{5k+5} + \alpha^{7-5k})\gamma_{k+1}, \\ 156\gamma_5 &= (1+\sigma)(\alpha^4\beta_1 + \overline{\alpha^4\beta_1}) + 2 = \\ &= -26\gamma_1 + 13\sum_{k=1}^{11}(\alpha^{k+4} + \alpha^{8-k} + \alpha^{5k+8} + \alpha^{4-5k})\gamma_{k+1}. \end{cases}$$

We will carry out the calculations for (36) in a table, taking into account $\alpha^5 + \alpha^7 = -\sqrt{3}$.

k	$\alpha^{k+1} + \alpha^{11-k} + \alpha^{5k+5} + \alpha^{7-5k}$	$\alpha^{k+4} + \alpha^{8-k} + \alpha^{5k+8} + \alpha^{4-5k}$	γ_{k+1}
1	$\alpha^2 + \alpha^{10} + \alpha^{10} + \alpha^2 = 2$	$\alpha^5 + \alpha^7 + \alpha + \alpha^{11} = 0$	γ_2
2	$\alpha^3 + \alpha^9 + \alpha^3 + \alpha^9 = 0$	$-1 - 1 - 1 - 1 = -4$	γ_3
3	$\alpha^4 + \alpha^8 + \alpha^8 + \alpha^4 = -2$	$\alpha^7 + \alpha^5 + \alpha^{11} + \alpha = 0$	γ_4
4	$\alpha^5 + \alpha^7 + \alpha + \alpha^{11} = 0$	$\alpha^8 + \alpha^4 + \alpha^4 + \alpha^8 = -2$	γ_5
5	$-1 - 1 - 1 - 1 = -4$	$\alpha^9 + \alpha^3 + \alpha^9 + \alpha^3 = 0$	γ_6
6	$\alpha^7 + \alpha^5 + \alpha^{11} + \alpha = 0$	$\alpha^{10} + \alpha^2 + \alpha^2 + \alpha^{10} = 2$	$-\gamma_1$
7	$\alpha^8 + \alpha^4 + \alpha^4 + \alpha^8 = -2$	$\alpha^{11} + \alpha + \alpha^7 + \alpha^{11} = 0$	$-\gamma_2$
8	$\alpha^9 + \alpha^3 + \alpha^9 + \alpha^3 = 0$	$1 + 1 + 1 + 1 = 4$	$-\gamma_3$
9	$\alpha^{10} + \alpha^2 + \alpha^2 + \alpha^{10} = 2$	$\alpha + \alpha^{11} + \alpha^5 + \alpha^7 = 0$	$-\gamma_4$
10	$\alpha^{11} + \alpha + \alpha^7 + \alpha^{11} = 0$	$\alpha^2 + \alpha^{10} + \alpha^{10} + \alpha^2 = 2$	$-\gamma_5$
11	$1 + 1 + 1 + 1 = 4$	$\alpha^3 + \alpha^9 + \alpha^3 + \alpha^9 = 0$	$-\gamma_6$

From relations (31), (36) and the table, we obtain

$$\begin{cases} 12\gamma_2 = 4\gamma_2 - 4\gamma_4 - 8\gamma_6 = -12\gamma_6, \\ 12\gamma_5 = -2\gamma_1 - 8\gamma_3 - 4\gamma_5 - 2\gamma_1. \end{cases}$$

(37)
$$\begin{cases} \gamma_6 = -\gamma_2, \\ \gamma_4 = 0, \\ \gamma_1 = -2\gamma_5, \\ \gamma_3 = -\gamma_5. \end{cases}$$

$\square$

We continue the proof of the theorem. From relations (18), (31) and Lemma 7, we find

(38)
$$
\begin{cases}
\beta_1 & = 1 + 13(\gamma_1 + \alpha\gamma_2 + \alpha^2\gamma_3 + i\gamma_4 + \alpha^4\gamma_5 + \alpha^5\gamma_6 - \gamma_7 + \alpha^7\gamma_8 + \alpha^8\gamma_9 - \\
& \quad -i\gamma_{10} + +\alpha^{10}\gamma_{11} + \alpha^{11}\gamma_{12}) = \\
& = 1 + 13\gamma_2(\alpha + \alpha^{11} - \alpha^5 - \alpha^7) + 13\gamma_5(\alpha^4 + \alpha^8 - \alpha^2 - \alpha^{10} - 4) = \\
& = 1 + 26\sqrt{3}\gamma_2 - 78\gamma_5, \\
\beta_5 & = 1 + 13(\gamma_1 + \alpha^5\gamma_2 + \alpha^{10}\gamma_3 + i\gamma_4 + \alpha^8\gamma_5 + \alpha\gamma_6 - \gamma_7 + \alpha^{11}\gamma_8 + \alpha^4\gamma_9 - \\
& \quad -i\gamma_{10} + +\alpha^2\gamma_{11} + \alpha^7\gamma_{12}) = \\
& = 1 + 13\gamma_2(-\alpha - \alpha^{11} + \alpha^5 + \alpha^7) + 13\gamma_5(\alpha^4 + \alpha^8 - \alpha^2 - \alpha^{10} - 4) = \\
& = 1 - 26\sqrt{3}\gamma_2 - 78\gamma_5, \\
\beta_7 & = 1 + 13(\gamma_1 + \alpha^7\gamma_2 + \alpha^2\gamma_3 - i\gamma_4 + \alpha^4\gamma_5 + \alpha^{11}\gamma_6 - \gamma_7 + \alpha\gamma_8 + \alpha^8\gamma_9 + \\
& \quad +i\gamma_{10} + +\alpha^{10}\gamma_{11} + \alpha^5\gamma_{12}) = \\
& = 1 + 13\gamma_2(-\alpha - \alpha^{11} + \alpha^5 + \alpha^7) + 13\gamma_5(\alpha^4 + \alpha^8 - \alpha^2 - \alpha^{10} - 4) = \\
& = 1 - 26\sqrt{3}\gamma_2 - 78\gamma_5 = \beta_5, \\
\beta_{11} & = 1 + 13(\gamma_1 + \alpha^{11}\gamma_2 + \alpha^{10}\gamma_3 - i\gamma_4 + \alpha^8\gamma_5 + \alpha^7\gamma_6 - \gamma_7 + \alpha^5\gamma_8 + \alpha^4\gamma_9 + \\
& \quad +i\gamma_{10} + +\alpha^2\gamma_{11} + \alpha\gamma_{12}) = \\
& = 1 + 13\gamma_2(\alpha + \alpha^{11} - \alpha^5 - \alpha^7) + 13\gamma_5(\alpha^4 + \alpha^8 - \alpha^2 - \alpha^{10} - 4) = \\
& = 1 + 26\sqrt{3}\gamma_2 - 78\gamma_5 = \beta_1.
\end{cases}
$$

From (38), we obtain that

(39)
$$
\begin{cases}
\beta_1 + \beta_5 = 2 - 156\gamma_5 = x, \\
\beta_1 - \beta_5 = 52\sqrt{3}\gamma_2 = \sqrt{3}y,
\end{cases}
$$

where x and y are integers, moreover, $x^2 - 3y^2 = (\beta_1 + \beta_5)^2 - (\beta_1 - \beta_5)^2 = 4\beta_1\beta_5$.

From system (38), it follows that β_1, β_5, β_7, and β_{11} are algebraically conjugate, therefore, $\beta_1\beta_5\beta_7\beta_{11} = N_{\mathbb{Q}(\chi_1)}(\beta_1) = \pm 1$ by theorems [8, Theorem 4 from II §2] and [2, Theorem 3.13]. But β_1 and β_5 are real numbers, then $\beta_1\beta_5\beta_7\beta_{11} = (\beta_1\beta_5)^2 = 1$ and $\beta_1\beta_5 = \pm 1$.

On the other hand, $x^2 - 3y^2 = 4 - 4 \cdot 156\gamma_5 + 156^2\gamma_5^2 - 52^2\gamma_2^2 \equiv 4 \pmod{624}$. Hence, we obtain that $\beta_1\beta_5 = 1$ and $x^2 - 3y^2 = 4$.

Lemma 8.
$$
x = \beta_1 + \beta_5 = (-1)^n \frac{(4 + 2\sqrt{3})^n + (4 - 2\sqrt{3})^n}{2^n},
$$

$$
y = \frac{\beta_1 - \beta_5}{\sqrt{3}} = \pm \frac{(4 + 2\sqrt{3})^n - (4 - 2\sqrt{3})^n}{2^n\sqrt{3}}.
$$

Proof. The pair $(4; 2)$ is a solution of the equation $x^2 - 3y^2 = 4$ which gives the smallest value of the function $x + \sqrt{3}y$ on the set of natural solutions of this equation. By theorem [1, p.341], the set of integer solutions of the equation $x^2 - 3y^2 = 4$ is as follows:

$$
\left\{ (\pm x_n; \pm y_n) \mid x_n + \sqrt{3}y_n = \frac{(4 + 2\sqrt{3})^n}{2^{n-1}}, n = 1, 2... \right\}.
$$

Then

$$x = \beta_1 + \beta_5 = \delta \frac{(4 + 2\sqrt{3})^n + (4 - 2\sqrt{3})^n}{2^n}, \text{ где } \delta = \pm 1;$$

$$y = \frac{\beta_1 - \beta_5}{\sqrt{3}} = \pm \frac{(4 + 2\sqrt{3})^n - (4 - 2\sqrt{3})^n}{2^n \sqrt{3}}.$$

It remains to prove that $\delta = (-1)^n$.

$$x = 2 - 156\gamma_5 = \frac{2\delta}{2^n} \sum_{i=0}^{[\frac{n}{2}]} C_n^{2i} 4^{n-2i} 2^{2i} 3^i = \frac{2\delta}{2^n} \left(4^n + \sum_{i=1}^{[\frac{n}{2}]} C_n^{2i} 2^{2n-2i} 3^i \right),$$

$$\gamma_5 = \frac{2 - x}{156} = \frac{1}{78} - \frac{\delta 2^n}{78} - \frac{\delta}{26} \sum_{i=1}^{[\frac{n}{2}]} C_n^{2i} 2^{n-2i} 3^{i-1},$$

$$26\gamma_5 = \frac{1 - \delta 2^n}{3} - \delta \sum_{i=1}^{[\frac{n}{2}]} C_n^{2i} 2^{n-2i} 3^{i-1}.$$

We obtain that

$$\frac{1 - \delta 2^n}{3} = \frac{1 - \delta(3 - 1)^n}{3} = \frac{1}{3} \left(1 - \delta \left(\sum_{i=0}^{n-1} C_n^i 3^{n-i} (-1)^i + (-1)^n \right) \right)$$

is an integer, which means that $1 - \delta(-1)^n \equiv 0 \pmod 3$, and hence $\delta = (-1)^n$. $\square$

We finish the proof of the theorem. From Lemma 8, we obtain that
$\beta_1 = (-1)^n \left(\frac{4 + 2\sqrt{3}}{2} \right)^n = (-1)^n (2 + \sqrt{3})^n$ or
$\beta_1 = (-1)^n \left(\frac{4 - 2\sqrt{3}}{2} \right) = (-1)^n (2 - \sqrt{3})^n$. Since $(2 + \sqrt{3})^{-1} = 2 - \sqrt{3}$ and taking into account that x and y in relations (39) are integers, for the generating element u we can assume that $\beta_1 = (2 + \sqrt{3})^{12}$. Then $\beta_5 = (2 - \sqrt{3})^{12}$, $x = 7300802, y = 4215120$. From systems (39), (31), (37), we get that

$$\begin{cases} \gamma_2 = 81060 = \gamma_{12} = -\gamma_6 = -\gamma_8, \\ \gamma_5 = -46800 = \gamma_9 = -\gamma_3 = -\gamma_{11}, \\ \gamma_1 = 93600 = -\gamma_7, \\ \gamma_0 = 93601. \end{cases}$$

For the element u^{-1}, we obtain $\beta_1 = (2 - \sqrt{3})^{12}$, so $\beta_5 = (2 + \sqrt{3})^{12}$, $x = 7300802, y = -4215120$. From systems (39), (31), (37), we get that

$$\begin{cases} \gamma_2 = -81060 = \gamma_{12} = -\gamma_6 = -\gamma_8, \\ \gamma_5 = -46800 = \gamma_9 = -\gamma_3 = -\gamma_{11}, \\ \gamma_1 = 93600 = -\gamma_7, \\ \gamma_0 = 93601. \end{cases}$$

To summarize, we have obtained the following:

$$u = e_0 + e_2 + e_3 + e_4 + e_6 + e_8 + e_9 + e_{10} + e_{12} + (2 + \sqrt{3})^{12}(e_1 + e_{11}) +$$
$$+ (2 - \sqrt{3})^{12}(e_5 + e_7) =$$
$$= 93601 + 46800(y(a^2) + y(a^{-2}) - y(a^4) - y(a^{-4}) - 2y(a^6) + 2y(b)) +$$
$$+ 81060(y(a) + y(a^{-1}) - y(a^5) - y(a^{-5})).$$

$$u^{-1} = e_0 + e_2 + e_3 + e_4 + e_6 + e_8 + e_9 + e_{10} + e_{12} + (2 - \sqrt{3})^{12}(e_1 + e_{11}) +$$
$$+ (2 + \sqrt{3})^{12}(e_5 + e_7) =$$
$$= 93601 + 46800(y(a^2) + y(a^{-2}) - y(a^4) - y(a^{-4}) - 2y(a^6) + 2y(b)) -$$
$$- 81060(y(a) + y(a^{-1}) - y(a^5) - y(a^{-5})).$$

$\square$

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