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ERGODIC THEOREMS IN BANACH IDEALS OF COMPACT OPERATORS

A.N. AZIZOV, V.I. CHILIN

Abstract. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, and let $\mathcal{B}(\mathcal{H})$ ($\mathcal{K}(\mathcal{H})$) be the $C^\star$–algebra of all bounded (compact) linear operators in $\mathcal{H}$. Let $(E, \|\cdot\|_E)$ be a fully symmetric sequence space. If $\{s_n(x)\}_{n=1}^\infty$ are the singular values of $x \in \mathcal{K}(\mathcal{H})$, let $\mathcal{C}_E = \{x \in \mathcal{K}(\mathcal{H}) : \{s_n(x)\} \in E\}$ with $\|x\|_{\mathcal{C}_E} = \|\{s_n(x)\}\|_E$, $x \in \mathcal{C}_E$, be the Banach ideal of compact operators generated by E. We show that the averages $A_n(T)(x) = \frac{1}{n+1}\sum\limits_{k=0}^{n} T^k(x)$ converge uniformly in $\mathcal{C}_E$ for any Dunford-Schwartz operator T and $x \in \mathcal{C}_E$. Besides, if $0 \leq x \in \mathcal{B}(\mathcal{H})\setminus\mathcal{K}(\mathcal{H})$, there exists a Dunford-Schwartz operator T such that the sequence $\{A_n(T)(x)\}$ does not converge uniformly. We also show that the averages $A_n(T)$ converge strongly in $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ if and only if E is separable and $E \neq l^1$ as sets.

Keywords: symmetric sequence space, Banach ideal of compact operators, Dunford-Schwartz operator, individual ergodic theorem, mean ergodic theorem.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the $C^\star$–algebra of all bounded linear operators in a complex Hilbert space $\mathcal{H}$, equipped with the uniform norm $\|\cdot\|_\infty$. The study of noncommutative individual ergodic theorems in the space of measurable operators affiliated with a semifinite von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ equipped with a faithful normal semifinite trace τ was initiated by F. Yeadon. In [23], as a corollary of a noncommutative maximal ergodic inequality in $L^1 = L^1(\mathcal{M}, \tau)$, the following individual ergodic theorem was established.

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Theorem 1. *Let $T : L^1 \to L^1$ be a positive $L^1 - L^\infty$-contraction. Then for any $x \in L^1$ there exists $\widehat{x} \in L^1$ such that the averages*

$$A_n(T)(x) = \frac{1}{n+1} \sum_{k=0}^{n} T^k(x)$$

converge to $\widehat{x}$ bilaterally almost uniformly (in Egorov's sense), that is, given $\varepsilon > 0$, there exists a projection $e \in \mathcal{M}$ such that $\tau(\mathbf{1} - e) < \varepsilon$ and

$$\|e(A_n(T)(x) - \widehat{x})e\|_\infty \to 0 \quad \text{as} \quad n \to \infty,$$

where $\mathbf{1}$ is the unit of $\mathcal{M}$.

The study of individual ergodic theorems beyond $L^1(\mathcal{M}, \tau)$ started much later with another fundamental paper by M. Junge and Q. Xu [13], where, among other results, individual ergodic theorem was extended to the case with a positive Dunford-Schwartz operator acting in the space $L^p(\mathcal{M}, \tau)$, $1 < p < \infty$. In [3] ([4]), utilizing the approach of [16], an individual ergodic theorem was proved for a positive Dunford-Schwartz operator in a noncommutative Lorentz (respectively, Orlicz) space.

Let $\mathcal{H}$ be a complex infinite-dimensional Hilbert space. Let $E \subset c_0$ be a fully symmetric sequence space. Denote by $\mathcal{C}_E$ the Banach ideal of compact operators in $\mathcal{H}$ associated with E. In Section 3 of the article, we obtain the following individual Dunford-Schwartz-type ergodic theorem.

Theorem 2. (i). *Given a Dunford-Schwartz operator $T : \mathcal{C}_E \to \mathcal{C}_E$ and $x \in \mathcal{C}_E$, there exists $\widehat{x} \in \mathcal{C}_E$ such that $\|A_n(T)(x) - \widehat{x}\|_\infty \to 0$ as $n \to \infty$;*

(ii). *If $0 \leq x \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H})$, then there exists a Dunford-Schwartz operator $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that the averages $A_n(T)(x)$ do not converge uniformly.*

Noncommutative mean ergodic theorem can be stated as follows: if T is an $L^1 - L^\infty$-contraction and $1 < p < \infty$, then the averages $A_n(T)$ converge strongly in $L^p = L^p(\mathcal{M}, \tau)$, that is, given $x \in L^p$, there exists $\widehat{x} \in L^p$ such that $\|A_n(T)(x) - \widehat{x}\|_p \to 0$ as $n \to \infty$. If $p = 1$ and $\tau(\mathbf{1}) = \infty$, this is not true in general. As a consequence, if $\tau(\mathbf{1}) = \infty$, mean ergodic theorem may not hold in some noncommutative symmetric spaces. In Yeadon's paper [24], the following mean ergodic theorem was established.

Theorem 3. *Let $E = (E(\mathcal{M}, \tau), \|\cdot\|_E)$ be a noncommutative fully symmetric space such that*

(i) *$L^1(\mathcal{M}, \tau) \cap \mathcal{M}$ is dense in E;*

(ii) *$\|e_n\|_E \to 0$ for any sequence of projections $\{e_n\} \subset L^1(\mathcal{M}, \tau) \cap \mathcal{M}$ with $e_n \downarrow 0$;*

(iii) *$\|e_n\|_E / \tau(e_n) \to 0$ for any increasing sequence of projections $\{e_n\} \subset \mathcal{M}$, $0 < \tau(e_n) < \infty$, with $\tau(e_n) \to \infty$.*

Then for any $x \in E$ and a positive $L^1 - L^\infty$-contraction $T : E \to E$ there exists $\widehat{x} \in E$ such that $\|A_n(T)(x) - \widehat{x}\|_E \to 0$.

In [3], the mean ergodic theorem was established for a noncommutative symmetric space $E(\mathcal{M}, \tau)$ associated with a fully symmetric function space with nontrivial Boyd indices and order continuous norm.

In Section 4, we give the following criterion for the validity of the mean ergodic theorem in a Banach ideal of compact operators in $\mathcal{H}$.

Theorem 4. *The following conditions are equivalent:*

(i). For any Dunford-Schwartz operator $T : \mathcal{C}_E \to \mathcal{C}_E$ the averages $A_n(T)$ converge strongly in $\mathcal{C}_E$;
(ii). $(E, \|\cdot\|_E)$ is separable and $E \neq l^1$ as sets.

Commutative counterparts of Theorems 2 and 4 were established in [2].

In the end of the article, we give applications of Theorems 2 and 4 to the well-studied Orlicz and Lorentz ideals of compact operators. We note that our noncommutative versions of ergodic theorems are true for any Dunford-Schwarz operators without the assumption that these operators are positive.

2. Preliminaries

2.1. **Symmetric sequence spaces.** Let l^∞ (respectively, c_0) be the Banach space of bounded (respectively, converging to zero) sequences $\{\xi_n\}_{n=1}^\infty$ of complex numbers equipped with the uniform norm $\|\{\xi_n\}\|_\infty = \sup\limits_{n\in\mathbb{N}} |\xi_n|$, where $\mathbb{N}$ is the set of natural numbers. If $2^{\mathbb{N}}$ is the σ-algebra of all subsets of $\mathbb{N}$ and $\mu(\{n\}) = 1$ for each $n \in \mathbb{N}$, then $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ is a σ-finite measure space such that $L^\infty(\mathbb{N}, 2^{\mathbb{N}}, \mu) = l^\infty$ and

$$L^1(\mathbb{N}, 2^{\mathbb{N}}, \mu) = l^1 = \left\{ \{\xi_n\}_{n=1}^\infty \subset \mathbb{C} : \ \|\{\xi_n\}\|_1 = \sum_{n=1}^\infty |\xi_n| < \infty \right\} \subset l^\infty,$$

where $\mathbb{C}$ is the field of complex numbers.

For any subset $E \subset l^\infty$ we denote $E_h = \{\{\xi_n\}_{n=1}^\infty \in E : \ \xi_n \in \mathbb{R} \text{ for each } n\}$, where $\mathbb{R}$ is the field of real numbers. It is know that $(l_h^\infty, \|\cdot\|_\infty)$ and $((c_0)_h, \|\cdot\|_\infty)$ are Banach lattices with respect to the natural partial order

$$\{\xi_n\} \le \{\eta_n\} \iff \xi_n \le \eta_n \ \text{ for all } \ n \in \mathbb{N}.$$

If $\xi = \{\xi_n\}_{n=1}^\infty \in l^\infty$, then the *non-increasing rearrangement* $\xi^* : (0, \infty) \to (0, \infty)$ of ξ is defined by

$$\xi^*(t) = \inf\{\lambda : \mu\{|\xi| > \lambda\} \le t\}, \quad t > 0,$$

(see, for example, [1, Ch. 2, Definition 1.5]). As such, the non-increasing rearrangement of a sequence $\{\xi_n\}_{n=1}^\infty \in l^\infty$ can be identified with the sequence $\xi^* = \{\xi_n^*\}_{n=1}^\infty$, where

$$\xi_n^* = \inf\left\{ \sup_{n \notin F} |\xi_n| : F \subset \mathbb{N}, \ |F| < n \right\}.$$

If $\{\xi_n\} \in c_0$, then $\xi_n^* \downarrow 0$; in this case there exists a bijection $\pi : \mathbb{N} \to \mathbb{N}$ such that $|\xi_{\pi(n)}| = \xi_n^*$, $n \in \mathbb{N}$.

Hardy-Littlewood-Polya partial order in the space l^∞ is defined as follows:

$$\xi = \{\xi_n\} \prec\prec \eta = \{\eta_n\} \iff \sum_{n=1}^m \xi_n^* \le \sum_{n=1}^m \eta_n^* \quad \text{for all} \quad m \in \mathbb{N}.$$

A non-zero linear subspace $E \subset l^\infty$ with a Banach norm $\|\cdot\|_E$ is called a *symmetric* (*fully symmetric*) sequence space if

$$\eta \in E, \ \xi \in l^\infty, \ \xi^* \le \eta^* \text{ (resp., } \xi^* \prec\prec \eta^*) \implies \xi \in E \ \text{ and } \ \|\xi\|_E \le \|\eta\|_E.$$

Every fully symmetric sequence space is a symmetric sequence space. The converse is not true in general. At the same time, any separable symmetric sequence space is a fully symmetric space.

If $(E, \|\cdot\|_E)$ is a symmetric sequence space, then

$$\|\xi\|_E = \|\,|\xi|\,\|_E = \|\xi^*\|_E \quad \text{for all} \ \ \xi \in E.$$

Besides, $(E_h, \|\cdot\|_E)$ is a Banach lattice with respect to the partial order induced from l^∞.

Immediate examples of fully symmetric sequence spaces are $(l^\infty, \|\cdot\|_\infty)$, $(c_0, \|\cdot\|_\infty)$ and the Banach spaces

$$l^p = \left\{ \xi = \{\xi_n\}_{n=1}^\infty \in l^\infty : \ \|\xi\|_p = \left(\sum_{n=1}^\infty |\xi_n|^p \right)^{1/p} < \infty \right\}, \ 1 \le p < \infty.$$

For any symmetric sequence space $(E, \|\cdot\|_E)$ the following continuous embeddings hold [1, Ch. 2, § 6, Theorem 6.6]: $(l^1, \|\cdot\|_1) \subset (E, \|\cdot\|_E) \subset (l^\infty, \|\cdot\|_\infty)$. Besides, $\|\xi\|_E \le \|\xi\|_1$ for all $\xi \in l^1$ and $\|\xi\|_\infty \le \|\xi\|_E$ for all $\xi \in E$.

If there is $\xi \in E \setminus c_0$, then $\xi^* \ge \alpha \mathbf{1}$ for some $\alpha > 0$, where $\mathbf{1} = \{1, 1, ...\}$. Consequently, $\mathbf{1} \in E$ and $E = l^\infty$. Therefore, either $E \subset c_0$ or $E = l^\infty$.

2.2. **Symmetric operator spaces.** Now, let $(\mathcal{H}, (\cdot,\cdot))$ be an infinite-dimensional Hilbert space over $\mathbb{C}$, and let $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$ be the $C^\star$–algebra of all bounded linear operators in $\mathcal{H}$. Denote by $\mathcal{K}(\mathcal{H})$ ($\mathcal{F}(\mathcal{H})$) the two-sided ideal of compact (respectively, finite rank) linear operators in $\mathcal{B}(\mathcal{H})$. It is well known that, for any proper two-sided ideal $\mathcal{I} \subset \mathcal{B}(\mathcal{H})$, we have $\mathcal{F}(\mathcal{H}) \subset \mathcal{I}$, and if $\mathcal{H}$ is separable, then $\mathcal{I} \subset \mathcal{K}(\mathcal{H})$ (see, for example, [19, Proposition 2.1]). At the same time, if $\mathcal{H}$ is a non-separable Hilbert space, then there exists a proper two-sided ideal $\mathcal{I} \subset \mathcal{B}(\mathcal{H})$ such that $\mathcal{K}(\mathcal{H}) \subsetneq \mathcal{I}$.

Denote $\mathcal{B}_h(\mathcal{H}) = \{x \in \mathcal{B}(\mathcal{H}) : x = x^*\}$, $\mathcal{B}_+(\mathcal{H}) = \{x \in \mathcal{B}_h(\mathcal{H}) : x \ge 0\}$, and let $\tau : \mathcal{B}_+(\mathcal{H}) \to [0, \infty]$ be the *canonical trace* on $\mathcal{B}(\mathcal{H})$, that is,

$$\tau(x) = \sum_{j \in J} (x\varphi_j, \varphi_j), \quad x \in \mathcal{B}_+(\mathcal{H}),$$

where $\{\varphi_j\}_{j\in J}$ is an orthonormal basis in $\mathcal{H}$ (see, for example, [20, Ch. 7, E. 7.5]).

Let $\mathcal{P}(\mathcal{H}) = \{e \in \mathcal{B}(\mathcal{H}) : e = e^2 = e^*\}$ be the lattice of projectors in $\mathcal{B}(\mathcal{H})$. If $\mathbf{1}$ is the identity of $\mathcal{B}(\mathcal{H})$ and $e \in \mathcal{P}(\mathcal{H})$, we will write $e^\perp = \mathbf{1} - e$.

Let $x \in \mathcal{B}(\mathcal{H})$, and let $\{e_\lambda(|x|)\}_{\lambda \ge 0}$ be the spectral family of projections for the absolute value $|x| = (x^* x)^{1/2}$ of x, that is, $e_\lambda(|x|) = \{|x| \le \lambda\}$. If $t > 0$, then the *t-th generalized singular number* of x, or the *non-increasing rearrangement* of x, is defined as

$$\mu_t(x) = \inf\{\lambda > 0 : \ \tau(e_\lambda(|x|)^\perp) \le t\}$$

(see [11]).

A non-zero linear subspace $X \subset \mathcal{B}(\mathcal{H})$ with a Banach norm $\|\cdot\|_X$ is called *symmetric* (*fully symmetric*) if the conditions

$$x \in X, \ y \in \mathcal{B}(\mathcal{H}), \ \mu_t(y) \le \mu_t(x) \quad \text{for all} \ \ t > 0$$

(respectively,

$$x \in X, \ y \in \mathcal{B}(\mathcal{H}), \ \int_0^s \mu_t(y)dt \le \int_0^s \mu_t(x)dt \quad \text{for all} \ \ s > 0 \ \ (\text{writing } y \prec\prec x))$$

imply that $y \in X$ and $\|y\|_X \le \|x\|_X$.

The spaces $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$ and $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty)$ as well as the classical Banach two-sided ideals

$$\mathcal{C}^p = \{x \in \mathcal{K}(\mathcal{H}) : \ \|x\|_p = \tau(|x|^p)^{1/p} < \infty\}, \ 1 \le p < \infty,$$

are examples of fully symmetric spaces.

It should be noted that for every symmetric space $(X, \|\cdot\|_X) \subset \mathcal{B}(\mathcal{H})$ and all $x \in X$, $a, b \in \mathcal{B}(\mathcal{H})$,

$$\|x\|_X = \|\,|x|\,\|_X = \|x^*\|_X, \quad axb \in X, \quad \text{and} \quad \|axb\|_X \leq \|a\|_\infty \|b\|_\infty \|x\|_X.$$

Remark 1. *If $X \subset \mathcal{B}(\mathcal{H})$ is a symmetric space and there exists a projection $e \in \mathcal{P}(\mathcal{H}) \cap X$ such that $\tau(e) = \infty$, that is, $\dim e(\mathcal{H}) = \infty$, then $\mu_t(e) = \mu_t(\mathbf{1}) = 1$ for every $t \in (0, \infty)$. Consequently, $\mathbf{1} \in X$ and $X = \mathcal{B}(\mathcal{H})$. If $X \neq \mathcal{B}(\mathcal{H})$ and $x \in X$, then $e_\lambda(|x|)^\perp = \{|x| > \lambda\}$ is a finite-dimensional projection, that is, $\dim e_\lambda(|x|)^\perp(\mathcal{H}) < \infty$ for all $\lambda > 0$. This means that $x \in \mathcal{K}(\mathcal{H})$, hence $X \subset \mathcal{K}(\mathcal{H})$. Therefore, either $X = \mathcal{B}(\mathcal{H})$ or $X \subset \mathcal{K}(\mathcal{H})$.*

Thus, if $\mathcal{H}$ is non-separable, then there exists a proper two-sided ideal $\mathcal{I} \subset \mathcal{B}(\mathcal{H})$ such that $\mathcal{K}(\mathcal{H}) \subsetneq \mathcal{I}$ and $(\mathcal{I}, \|\cdot\|_\infty)$ is a Banach space which is not a symmetric subspace of $\mathcal{B}(\mathcal{H})$.

If $x \in \mathcal{K}(\mathcal{H})$, then $|x| = \sum\limits_{n=1}^{m(x)} s_n(x) p_n$ (if $m(x) = \infty$, the series converges uniformly), where $\{s_n(x)\}_{n=1}^{m(x)}$ is the set of singular values of x, that is, the set of eigenvalues of the compact operator $|x|$ in the decreasing order, and p_n is the projection onto the eigenspace corresponding to $s_n(x)$. Consequently, the non-increasing rearrangement $\mu_t(x)$ of $x \in \mathcal{K}(\mathcal{H})$ can be identified with the sequence $\{s_n(x)\}_{n=1}^\infty$, $s_n(x) \downarrow 0$ (if $m(x) < \infty$, we set $s_n(x) = 0$ for all $n > m(x)$).

2.3. **Duality between symmetric sequence and operator spaces.** Let $(X, \|\cdot\|_X) \subset \mathcal{K}(\mathcal{H})$ be a symmetric space. Fix an orthonormal basis $\{\varphi_j\}_{j \in J}$ in $\mathcal{H}$ and choose a countable subset $\{\varphi_{j_n}\}_{n=1}^\infty$. Let p_n be the one-dimensional projection on the subspace $\mathbb{C} \cdot \varphi_{j_n} \subset \mathcal{H}$. It is clear that the set

$$E(X) = \left\{ \xi = \{\xi_n\}_{n=1}^\infty \in c_0 : \; x_\xi = \sum_{n=1}^\infty \xi_n p_n \in X \right\}$$

(the series converges uniformly), is a symmetric sequence space with respect to the norm $\|\xi\|_{E(X)} = \|x_\xi\|_X$. Consequently, each symmetric subspace $(X, \|\cdot\|_X) \subset \mathcal{K}(\mathcal{H})$ uniquely generates a symmetric sequence space $(E(X), \|\cdot\|_{E(X)}) \subset c_0$. The converse is also true: every symmetric sequence space $(E, \|\cdot\|_E) \subset c_0$ uniquely generates a symmetric space $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E}) \subset \mathcal{K}(\mathcal{H})$ by the following rule (see, for example, [17, Ch. 3, Section 3.5]):

$$\mathcal{C}_E = \{x \in \mathcal{K}(\mathcal{H}) : \{s_n(x)\} \in E\}, \quad \|x\|_{\mathcal{C}_E} = \|\{s_n(x)\}\|_E.$$

In addition,

$$E(\mathcal{C}_E) = E, \; \|\cdot\|_{E(\mathcal{C}_E)} = \|\cdot\|_E, \; \mathcal{C}_{E(\mathcal{C}_E)} = \mathcal{C}_E, \; \|\cdot\|_{\mathcal{C}_{E(\mathcal{C}_E)}} = \|\cdot\|_{\mathcal{C}_E}.$$

We will call the pair $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ a *Banach ideal of compact operators* (cf. [12, Ch. III]). It is known that $(\mathcal{C}^p, \|\cdot\|_p) = (\mathcal{C}_{l^p}, \|\cdot\|_{\mathcal{C}_{l^p}})$ for all $1 \leq p < \infty$ and $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty) = (\mathcal{C}_{c_0}, \|\cdot\|_{\mathcal{C}_{c_0}})$.

Hardy-Littlewood-Polya partial order in the Banach ideal $\mathcal{K}(\mathcal{H})$ is defined by

$$x \prec\prec y, \; x, y \in \mathcal{K}(\mathcal{H}) \iff \{s_n(x)\} \prec\prec \{s_n(y)\}.$$

We say that a Banach ideal $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is *fully symmetric* if conditions $y \in \mathcal{C}_E$, $x \in \mathcal{K}(\mathcal{H})$, $x \prec\prec y$ entail that $x \in \mathcal{C}_E$ and $\|x\|_{\mathcal{C}_E} \leq \|y\|_{\mathcal{C}_E}$. It is clear that $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$

is a fully symmetric ideal if and only if $(E, \|\cdot\|_E)$ is a fully symmetric sequence space.

Examples of fully symmetric ideals include $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty)$ as well as the Banach ideals $(\mathcal{C}^p, \|\cdot\|_p)$ for all $1 \le p < \infty$. It is clear that $\mathcal{C}^1 \subset \mathcal{C}_E \subset \mathcal{K}(\mathcal{H})$ for every symmetric sequence space $E \subset c_0$ with $\|x\|_{\mathcal{C}_E} \le \|x\|_1$ and $\|y\|_\infty \le \|y\|_{\mathcal{C}_E}$ for all $x \in \mathcal{C}^1$ and $y \in \mathcal{C}_E$.

Remark 2. *If $x, y, y_k \in \mathcal{K}(\mathcal{H})$ are such that $y_k \prec\prec x$ for all $k \in \mathbb{N}$ and $\|y_k - y\|_\infty \to 0$ as $k \to \infty$, then $y \prec\prec x$.*

Indeed, since $y_k \prec\prec x$, it follows that $\sum\limits_{n=1}^{m} s_n(y_k) \le \sum\limits_{n=1}^{m} s_n(x)$ for all $m, k \in \mathbb{N}$. By [12, Ch.II, § 2, Sec. 3, Corollary 2.3]*, $|s_n(y_k) - s_n(y)| \le \|y_k - y\|_\infty \to 0$, hence $\sum\limits_{n=1}^{m} s_n(y_k) \to \sum\limits_{n=1}^{m} s_n(y)$ as $k \to \infty$ for every $m \in \mathbb{N}$. Therefore*

$$\sum_{n=1}^{m} s_n(y) = \lim_{k\to\infty} \sum_{n=1}^{m} s_n(y_k) \le \sum_{n=1}^{m} s_n(x)$$

for all m.

2.4. **Dunford-Schwartz operators and conditional expectation.** A linear operator $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is called a *Dunford-Schwartz operator* if

$$\|T(x)\|_1 \le \|x\|_1 \text{ for all } x \in \mathcal{C}^1 \quad \text{and} \quad \|T(x)\|_\infty \le \|x\|_\infty \text{ for all } x \in \mathcal{B}(\mathcal{H}).$$

In what follows, we will write $T \in DS$ to indicate that T is a Dunford-Schwartz operator.

Any fully symmetric ideal $\mathcal{C}_E$ is an exact interpolation space in the Banach pair $(\mathcal{C}^1, \mathcal{B}(\mathcal{H}))$ (see [7, Theorem 2.4]), in particular, $T(\mathcal{C}_E) \subset \mathcal{C}_E$ and $\|T\|_{\mathcal{C}_E \to \mathcal{C}_E} \le 1$ for all $T \in DS$. Hence $T(\mathcal{K}(\mathcal{H})) \subset \mathcal{K}(\mathcal{H})$, and the restriction of T on $\mathcal{K}(\mathcal{H})$ is a linear contraction (also denoted by T). We note that if $T \in DS$, then $A_n(T) \in DS$; also, $T(x) \prec\prec x$ and $A_n(T)(x) \prec\prec x$ for any $x \in \mathcal{K}(\mathcal{H})$ and $n \in \mathbb{N}$.

We need the following Theorem on the existence of conditional expectation from $\mathcal{B}(\mathcal{H})$ into von Neumann subalgebra $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ (see, for example, [21], [22]).

Theorem 5. *Let $\mathcal{N}$ be a von Neumann subalgebra in $\mathcal{B}(\mathcal{H})$ such that the restriction of the canonical trace τ on $\mathcal{N}$ is a semifinite trace. Then there exists a unique positive linear map $U : \mathcal{B}(\mathcal{H}) \to \mathcal{N}$ (conditional expectation on $\mathcal{N}$), having the following properties:*

(i) $\tau(x) = \tau(U(x))$ for all $x \in \mathcal{C}^1$;
(ii) $U(x) = x$ for all $x \in \mathcal{N}$;
(iii) $\|U\|_{\mathcal{B}(\mathcal{H}) \to \mathcal{N}} = 1$.

Moreover, the conditional expectation U is projection of norm one from $(L^p(\mathcal{B}(\mathcal{H}), \tau), \|\cdot\|_p) = (\mathcal{C}^p, \|\cdot\|_p)$ onto $(L^p(\mathcal{N}, \tau), \|\cdot\|_p)$, $1 \le p < \infty$.

Thus, the conditional expectation $U : \mathcal{B}(\mathcal{H}) \to \mathcal{N} \subset \mathcal{B}(\mathcal{H})$ is a positive Dunford-Schwartz operator.

3. Individual ergodic theorem in fully symmetric ideals of compact operators

Let $\mathcal{H}$, $\tau : \mathcal{B}_+(\mathcal{H}) \to [0, \infty]$, and $\mathcal{C}^1$ be as above. Below we give a proof of Theorem 2 (i).

Proof. Since $T(\mathcal{C}^2) \subset \mathcal{C}^2$, $\|T\|_{\mathcal{C}^2\to\mathcal{C}^2} \leq 1$ and the Banach space $\mathcal{C}^2$ is reflexive, by the mean ergodic theorem [6, Ch. VIII, § 5, Corollary 4], the sequence $\{A_n(T)(x)\}$ converges strongly in $\mathcal{C}^2$, that is, for every $x \in \mathcal{C}^2$ there exists $\widehat{x} \in \mathcal{C}^2$ such that $\|A_n(T)(x) - \widehat{x}\|_2 \to 0$. As $\|\xi\|_\infty \leq \|\xi\|_2$ for all $\xi \in l^2$, it follows that $\|x\|_\infty \leq \|x\|_2$ for all $x \in \mathcal{C}^2$. Consequently,

$$\|A_n(T)(x) - \widehat{x}\|_\infty \to 0 \quad \text{for every} \quad x \in \mathcal{C}^2.$$

Let now $x \in \mathcal{K}(\mathcal{H})$ and $\varepsilon > 0$. Then there exists $x_\varepsilon \in \mathcal{F}(\mathcal{H}) \subset \mathcal{C}^2$ such that $\|x - x_\varepsilon\|_\infty < \varepsilon/4$. Since the sequence $A_n(T)(x_\varepsilon)$ converges uniformly, there exists $N = N(\varepsilon)$ such that

$$\|A_m(T)(x_\varepsilon) - A_n(T)(x_\varepsilon)\|_\infty < \frac{\varepsilon}{2} \quad \text{whenever} \quad m, n \geq N.$$

Therefore,

$$\|A_m(T)(x) - A_n(T)(x)\|_\infty \leq \|A_m(T)(x - x_\varepsilon) - A_n(T)(x - x_\varepsilon)\|_\infty + \|A_m(T)(x_\varepsilon) - A_n(T)(x_\varepsilon)\|_\infty \leq 2\|x - x_\varepsilon\|_\infty + \frac{\varepsilon}{2} < \varepsilon.$$

for all $m, n \geq N$. Thus, since $\mathcal{C}_E \subset \mathcal{K}(\mathcal{H})$ and the space $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty)$ is complete, it follows that for any $x \in \mathcal{C}_E$ there exists $\widehat{x} \in \mathcal{K}(\mathcal{H})$ such that $\|A_n(T)(x) - \widehat{x}\|_\infty \to 0$. Using Remark 2, we obtain that $\widehat{x} \prec\prec x$, hence $\widehat{x} \in \mathcal{C}_E$. □

Now we give a proof of the part *(ii)* of Theorem 2. We begin with a Dunford-Schwartz operator acting in $(l^\infty, \|\cdot\|_\infty)$, that is, when a linear operator $T : l^\infty \to l^\infty$ is such that $\|T(\xi)\|_1 \leq \|\xi\|_1$ for all $\xi \in l^1$ and $\|T(\xi)\|_\infty \leq \|\xi\|_\infty$ for all $\xi \in l^\infty$ (writing $T \in DS$). The following Theorem is a commutative version of Theorem 2 *(ii)* (proof see in [2, Theorem 3.3]).

Theorem 6. *If $\xi \in l^\infty \setminus c_0$, then there exists $T \in DS$ such that the averages $A_n(T)(\xi)$ do not converge coordinate-wise, hence uniformly.*

Assume first that $(\mathcal{H}, (\cdot,\cdot))$ is a separable infinite-dimensional complex Hilbert space. Fix an orthonormal basis $\{\varphi_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$. Let p_n be the one-dimensional projection on the linear subspace $\mathbb{C} \cdot \varphi_n \subset \mathcal{H}$. It is clear that $p_m p_n = 0$ for all $m, n \in \mathbb{N}$, $n \neq m$.

For any $\xi = \{\xi_n\}_{n=1}^\infty \in l^\infty$ and $h = \sum\limits_{n=1}^{\infty} (h, \varphi_n)\varphi_n \in \mathcal{H}$ we set

$$x_\xi(h) = \sum_{n=1}^{\infty} \xi_n (h, \varphi_n)\varphi_n = \sum_{n=1}^{\infty} \xi_n p_n(h).$$

It is clear that $x_\xi \in \mathcal{B}(\mathcal{H})$ and $x_\xi = (wo) - \sum\limits_{n=1}^{\infty} \xi_n p_n$, where (wo) stands for the weak operator topology. In addition,

$$\mathcal{N} = \{x_\xi \in \mathcal{B}(\mathcal{H}) : \ \xi = \{\xi_n\}_{n=1}^\infty \in l^\infty\}$$

is the smallest commutative von Neumann subalgebra in $\mathcal{B}(\mathcal{H})$ containing all projections p_n. Besides, the restriction of the trace τ on $\mathcal{N}$ is a semifinite trace.

Define the linear map $\Phi : (\mathcal{N}, \|\cdot\|_\infty) \to (l^\infty, \|\cdot\|_\infty)$ by setting $\Phi(x_\xi) = \xi$. By definition of Φ, we have $\Phi(\mathcal{N}) = l^\infty$. Using [17, Ch. 1, § 1.1, E. 1.1.11], we see that $\|x_\xi\|_\infty = \|\xi\|_\infty = \|\Phi(x_\xi)\|_\infty$, that is, Φ is a linear surjective isometry. Since $\xi = \{\xi_n\}_{n=1}^\infty \geq 0$ whenever $x_\xi \in \mathcal{N}_+$, the map Φ is positive. Therefore, Φ is a positive linear surjective isometry.

If $(E, \|\cdot\|_E) \subset c_0$ is a symmetric sequence space and $\mathcal{N}_E = \mathcal{N} \cap \mathcal{C}_E$, then for any $x_\xi = \sum\limits_{n=1}^{\infty} \xi_n p_n \in \mathcal{N}_E$ we have that $\{s_n(x_\xi)\}_{n=1}^{\infty} = \{\xi_n^*\} \in E$, hence $\{\xi_n\} \in E$. In addition, $\|x_\xi\|_{\mathcal{C}_E} = \|\{\xi_n^*\}\|_E = \|\{\xi_n\}\|_E$. Consequently, the restriction $\Phi|_{\mathcal{N}_E} : (\mathcal{N}_E, \|\cdot\|_{\mathcal{C}_E}) \to (E, \|\cdot\|_E)$ is a positive linear surjective isometry (we denote this restriction also by Φ).

Below we give a proof of Theorem 2 (ii).

Proof. Let $0 \le x \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H})$. Assume first that $\mathcal{H}$ is separable. Since $x \notin \mathcal{K}(\mathcal{H})$, it follows that there exists a spectral projection $e_\lambda(|x|),\ \lambda > 0$, such that $\tau(e_\lambda(|x|)^\perp) = \infty$. Choose an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ such that $e_\lambda(|x|)^\perp \ge p_{n_i}$ for some sequence $\{n_i\}_{i=1}^{\infty}$, where p_n is the one-dimensional projection on the subspace $\mathbb{C} \cdot \varphi_n \subset \mathcal{H}$.

Let $\mathcal{N} = \{x_\xi \in \mathcal{B}(\mathcal{H}) :\ \xi = \{\xi_n\}_{n=1}^{\infty} \in l_\infty\}$ be the smallest commutative von Neumann subalgebra in $\mathcal{B}(\mathcal{H})$ containing all projections p_n. Since the restriction of the trace τ on $\mathcal{N}$ is a semifinite trace, it follows by Theorem 5 that there exists a conditional expectation $U : \mathcal{B}(\mathcal{H}) \to \mathcal{N}$ such that

$$0 \le y = U(x) \ge U(\lambda\, e_\lambda(|x|)^\perp) \ge \lambda\, U(p_{n_i}) = \lambda p_{n_i} \quad \text{for all} \quad i \in \mathbb{N}.$$

Consequently, $y \notin \mathcal{K}(\mathcal{H})$ and $y = x_\xi \in \mathcal{N}$, where $0 \le \xi = \{\xi_n\}_{n=1}^{\infty} \in l^\infty \setminus c_0$. Besides, by definition of Φ, we have $\Phi(y) = \xi$.

Next, by Theorem 6, there exists an operator $S : l^\infty \to l^\infty$, $S \in DS$, such that the sequence $\{A_n(S)(\xi)\}$ does not converge uniformly. Consider the operator

$$T = \Phi^{-1} S \Phi U : \mathcal{B}(\mathcal{H}) \to \mathcal{N} \subset \mathcal{B}(\mathcal{H}).$$

It is clear that $T \in DS$. Since $U : \mathcal{B}(\mathcal{H}) \to \mathcal{N}$ is a conditional expectation and $y = U(x)$, it follows that $U(y) = y$, $U\Phi^{-1} = \Phi^{-1}$, and $T^k(y) = \Phi^{-1} S^k \Phi(y)$ for each $k \in \mathbb{N}$.

Since Φ^{-1} is an isometry and

$$A_n(T)(y) = \frac{1}{n+1} \sum_{k=0}^{n} T^k(y) = \Phi^{-1}\left(\frac{1}{n+1}\sum_{k=0}^{n} S^k \Phi(y)\right) = \Phi^{-1}(A_n(S)(\xi)),$$

for all $n \in \mathbb{N}$, it follows that the sequence $\{A_n(T)(y)\}_{n=1}^{\infty}$ does not converge uniformly.

Now, as above, $y = U(x) \in \mathcal{N}$ entails $T^k(x) = \Phi^{-1} S^k \Phi(y) = T^k(y)$ for all $k \in \mathbb{N}$. Therefore, we have

$$A_n(T)(x) - A_n(T)(y) = \frac{1}{n+1}(x - y),$$

and it follows that the sequence $\{A_n(T)(x)\}_{n=1}^{\infty}$ also does not converge uniformly.

Let now $\mathcal{H}$ be non-separable, and let $0 \le x \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H})$. Since $x \notin \mathcal{K}(\mathcal{H})$, it follows that there exists a spectral projection $e_\lambda(|x|),\ \lambda > 0$, such that $\tau(e_\lambda(|x|)^\perp) = \infty$. Choose an orthonormal basis $\{\varphi_j\}_{j\in J}$ in $\mathcal{H}$ such that $e_\lambda(|x|)^\perp \ge p_{j_n}$ for some sequence $\{j_n\}_{n=1}^{\infty}$, where p_j is the one-dimensional projection on the subspace $\mathbb{C} \cdot \varphi_j \subset \mathcal{H}$. If $p = \sup\limits_{n\in\mathbb{N}} p_{j_n}$, then $\mathcal{H}_0 = p(\mathcal{H})$ is a separable infinite-dimensional Hilbert subspace in $\mathcal{H}$ such that $\mathcal{K}(\mathcal{H}_0) = p\mathcal{K}(\mathcal{H})p$.

Since $z = pxp \in \mathcal{B}_+(\mathcal{H}_0)$ and $z \ge \lambda p e_\lambda(|x|)^\perp p \ge \lambda p$, it follows that $z \in \mathcal{B}_+(\mathcal{H}_0) \setminus \mathcal{K}(\mathcal{H}_0)$. In view of the above, there exists a Dunford-Schwartz operator

$D_0 : \mathcal{B}(\mathcal{H}_0) \to \mathcal{B}(\mathcal{H}_0)$ such that the sequence $\{A_n(D_0)(z)\}_{n=1}^{\infty}$ does not converge uniformly.

It is clear that $D(y) = D_0(pyp)$, $y \in \mathcal{B}(\mathcal{H})$, is a Dunford-Schwartz operator in $\mathcal{B}(\mathcal{H})$ such that $D^k(x) = D_0^k(z)$ for each $k \in \mathbb{N}$. Then

$$A_n(D)(x) - A_n(D_0)(z) = \frac{1}{n+1}(x - z),$$

and we conclude that the sequence $\{A_n(D)(x)\}_{n=1}^{\infty}$ does not converge uniformly. $\square$

Note that the commutative version of Theorem 2 (ii) for symmetric spaces of measurable functions was obtained in [5].

4. Mean ergodic theorem in fully symmetric ideals of compact operators

In this section, our goal is to prove Theorem 4. So, let $(E, \|\cdot\|_E) \subset c_0$ be a fully symmetric sequence space, and let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a fully symmetric ideal generated by $(E, \|\cdot\|_E)$. Let us show that the mean ergodic theorem, generally speaking, is not true in $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$, in the cases when $E = l^1$ as sets, or when $(E, \|\cdot\|_E)$ is non-separable space.

Proposition 1. *There exists $T \in DS$ such that the averages $A_n(T)$ do not converge strongly in $(\mathcal{C}^1, \|\cdot\|_1)$.*

Proof. Let $S : l^\infty \to l^\infty$ be the Dunford-Schwartz operator defined by

$$S(\{\xi_n\}_{n=1}^{\infty}) = \{0, \xi_1, \xi_2, \dots\}, \quad \{\xi_n\}_{n=1}^{\infty} \in l^\infty.$$

If $\xi = \{1, 0, 0, \dots\} \in l^1$, then

$$\|A_{2n-1}(S)(\xi) - A_{n-1}(S)(\xi)\|_1$$
$$= \left\| \frac{1}{2n}\{\underbrace{1, 1, \dots, 1}_{2n}, 0, 0, \dots\} - \frac{1}{n}\{\underbrace{1, 1, \dots, 1}_{n}, 0, 0, \dots\} \right\|_1 = 1.$$

Consequently, the sequence $\{A_n(S)(\xi)\}$ does not converge in the norm $\|\cdot\|_1$.

Let p_n, $p = \sup\limits_{n \in \mathbb{N}} p_n$, $\mathcal{H}_0 = p(\mathcal{H})$,

$$\mathcal{N}(\mathcal{H}_0) = \left\{ x_\xi = (wo) - \sum_{n=1}^{\infty} \xi_n p_n \in \mathcal{B}(\mathcal{H}_0) : \xi = \{\xi_n\}_{n=1}^{\infty} \in l^\infty \right\},$$

$\Phi : \mathcal{N}(\mathcal{H}_0) \to l^\infty$ and $U : \mathcal{B}(\mathcal{H}_0) \to \mathcal{N}(\mathcal{H}_0)$ be the same as in the proof of Theorem 2 (ii). Then

$$T = \Phi^{-1} S \Phi U : \mathcal{B}(\mathcal{H}_0) \to \mathcal{N}(\mathcal{H}_0) \subset \mathcal{B}(\mathcal{H}_0)$$

is a positive Dunford-Schwartz operator. In addition, for $\xi = \{1, 0, 0, \dots\} \in l^1$ and $x_\xi = \Phi^{-1}(\xi)$ we have that $x_\xi \in \mathcal{N}(\mathcal{H}_0) \cap \mathcal{C}^1$ and $U(x_\xi) = x_\xi$ (see proof of Theorem 2 (ii)). Consequently,

$$T(x_\xi) = \Phi^{-1} S \Phi U(x_\xi) = \Phi^{-1} S \Phi(x_\xi).$$

Now, repeating the proof of Theorem 2 (ii), we conclude that the averages

$$\{A_n(T)(x_\xi)\}$$

do not converge in the norm $\|\cdot\|_1$. $\square$

Proposition 2. *If $(E, \|\cdot\|_E) \subset c_0$ is non-separable fully symmetric sequence space, then there exists $T \in DS$ such that the averages $A_n(T)$ do not converge strongly in $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$.*

Proof. If $(E, \|\cdot\|_E) \subset c_0$ is a non-separable fully symmetric sequence space, then there exists $\xi = \{\xi_n\}_{n=1}^{\infty} = \{\xi_n^*\}_{n=1}^{\infty} \in E$ such that $\xi_n \downarrow 0$ and

$$\|\{\underbrace{0, 0, \dots, 0}_{n+1}, \xi_{n+2}, \dots\}\|_E \downarrow \alpha > 0. \tag{1}$$

Let the operator $S \in DS$ be defined as in the proof of Proposition 1. Then $S^k(\xi) = \{\underbrace{0, 0, \dots, 0}_{k}, \xi_1, \xi_2, \dots\}$ and

$$\sum_{k=0}^{n} S^k(\xi) = \{\eta_m^{(n)}\}_{m=1}^{\infty},$$

where

$$\eta_m^{(n)} = \xi_1 + \xi_2 + \ldots + \xi_m \quad \text{for} \quad 1 \le m \le n+1$$

and

$$\eta_m^{(n)} = \xi_{m-n} + \xi_{m-n+1} + \ldots + \xi_m \quad \text{for} \quad m > n+1.$$

Since $\xi_n \downarrow 0$, given $1 \le m \le n+1$, we have

$$0 \le \frac{1}{n+1}\,\eta_m^{(n)} \le \frac{1}{n+1}\sum_{k=1}^{n+1} \xi_k \to 0 \quad \text{as} \quad n \to \infty,$$

implying that $A_n(S)(\xi) \to 0$ coordinate-wise.

Assume that there exists $\widehat{\xi} \in E$ such that $\|A_n(S)(\xi) - \widehat{\xi}\|_E \to 0$. Then we have $\|A_n(S)(\xi) - \widehat{\xi}\|_\infty \to 0$; in particular, $A_n(S)(\xi) \to 0$ coordinate-wise, hence $\widehat{\xi} = 0$.

On the other hand, as $\xi_n \downarrow 0$, we obtain

$$\begin{aligned} A_n(S)(\xi) = \Big\{ &\frac{\xi_1}{n+1}, \frac{\xi_1+\xi_2}{n+1}, \dots, \frac{\xi_1+\xi_2+\ldots+\xi_{n+1}}{n+1}, \frac{\xi_2+\xi_3+\ldots+\xi_{n+2}}{n+1}, \\ &\frac{\xi_3+\xi_4+\ldots+\xi_{n+3}}{n+1}, \dots, \frac{\xi_{m-n}+\xi_{m-n+1}+\ldots+\xi_m}{n+1}, \dots \Big\} \\ &\ge \{\underbrace{0, 0, \dots, 0}_{n+1}, \xi_{n+2}, \dots\}. \end{aligned}$$

Therefore, in view of (1), $\|A_n(S)(\xi)\|_E \ge \alpha$, implying that the sequence $\{A_n(S)(\xi)\}$ does not converge in the norm $\|\cdot\|_E$.

Now, if we define the Dunford-Schwartz operator $T \in DS$ as in the proof of Proposition 1, then repeating its proof for $x = \Phi^{-1}(\xi)$, we conclude that the sequence $\{A_n(T)(x)\}$ does not converge in $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$. □

Fix $T \in DS$. By Theorem 2 (i), for every $x \in \mathcal{K}(\mathcal{H})$ there exists $\widehat{x} \in \mathcal{K}(\mathcal{H})$ such that $\|A_n(T)(x) - \widehat{x}\|_\infty \to 0$ as $n \to \infty$. Therefore, one can define a linear operator $P_T : \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ by setting $P_T(x) = \widehat{x}$. Then we have

$$\|P_T(x)\|_\infty = \lim_{n\to\infty} \|A_n(T)(x)\|_\infty \le \|x\|_\infty,$$

Besides, since the unit ball in $(\mathcal{C}^1, \|\cdot\|_1)$ is closed in measure topology [8, Proposition 3.3] and $\|A_n(T)(x)\|_1 \le \|x\|_1$ for all $x \in \mathcal{C}^1$, it follows that $\|P_T(x)\|_1 \le \|x\|_1$, $x \in \mathcal{C}^1$. Consequently, $\|P_T\|_{\mathcal{C}^1 \to \mathcal{C}^1} \le 1$, and, according to [3, Proposition 1.1], there exists a unique operator $\widehat{P} \in DS$ such that $\widehat{P}(x) = P_T(x)$ whenever $x \in \mathcal{K}(\mathcal{H})$. In what follows, we denote $\widehat{P}$ by P_T.

Lemma 1. *If $T \in DS$ and $x \in \mathcal{K}(\mathcal{H})$, then*

$$P_T T(x) = P_T(x) = T P_T(x).$$

Proof. We have

$$\|(I-T)A_n(T)(x)\|_\infty = \left\| \frac{(I-T^{n+1})(x)}{n+1} \right\|_\infty \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

On the other hand,

$$TA_n(T)(x) = \frac{1}{n+1} \sum_{k=0}^{n} T^k(Tx) \overset{\|\cdot\|_\infty}{\longrightarrow} P_T(T(x)),$$

implying that

$$(I-T)A_n(T)(x) = A_n(T)(x) - TA_n(T)(x) \overset{\|\cdot\|_\infty}{\longrightarrow} P_T(x) - P_T T(x),$$

hence $P_T T(x) = P_T(x)$.

Now, as $\|A_n(T)(x) - P_T(x)\|_\infty \to 0$, we have $\|T(A_n(T)(x)) - T(P_T(x))\|_\infty \to 0$ as $n \to \infty$, and the result follows. □

Corollary 1. *If $T \in DS$ and $x \in \mathcal{K}(\mathcal{H})$, then*

$$T^k(P_T(x)) = P_T(x) \quad \textit{for all} \quad k \in \mathbb{N}, \quad \textit{and} \quad P_T^2(x) = P_T(x).$$

We need the following property of separable symmetric sequence spaces [9, Proposition 2.2].

Proposition 3. *Let $(E, \|\cdot\|_E)$ be a separable symmetric sequence space and $E \neq l^1$ as sets. If $\mathcal{C}_E \ni y_n \prec\prec x \in \mathcal{C}_E$ for every $n \in \mathbb{N}$ and $\|y_n\|_\infty \to 0$ as $n \to \infty$, then $\|y_n\|_{\mathcal{C}_E} \to 0$ as $n \to \infty$.*

Now we can finalize the proof of Theorem 4:

Proof. (i) ⇒ (ii): Proposition 2 implies that E is separable. If $E = l^1$ as sets, then the norms $\|\cdot\|_E$ and $\|\cdot\|_1$ are equivalent [18, Part II, Ch. 6, § 6.1]. Therefore, in view of Proposition 1, we would have that item (i) in Theorem 4 is not true.

(ii) ⇒ (i): Let $(E, \|\cdot\|_E)$ be separable, $E \neq l^1$ as sets, and let $T \in DS$. If $x \in \mathcal{C}_E$ and $y = x - P_T(x)$, then $P_T(y) = 0$, which, by Theorem 2 (i), implies $\|A_n(T)(y)\|_\infty \to 0$. Since E is a separable symmetric sequence space, $E \neq l^1$ as sets, and $A_n(T)(y) \prec\prec y \in \mathcal{C}_E$, it follows from Proposition 3 that

$$\|A_n(T)(y)\|_{\mathcal{C}_E} \to 0. \tag{2}$$

Since $P_T(z) \prec\prec z$ for all $z \in \mathcal{K}(\mathcal{H})$, it follows that $A_n(T)(P_T(x)) \prec\prec P_T(x) \prec\prec x$, hence $A_n(T)(P_T(x)) - P_T(x) \prec\prec 2x$. Next, as $A_n(T)(P_T(x)) \overset{\|\cdot\|_\infty}{\longrightarrow} P_T(x)$, Proposition 3 entails

$$\|A_n(T)(P_T(x)) - P_T(x)\|_{\mathcal{C}_E} \to 0. \tag{3}$$

Now, utilizing (2) and (3), we obtain

$$\begin{aligned} \|A_n(T)(x) - P_T(x)\|_{\mathcal{C}_E} &= \|A_n(T)(x) - A_n(T)(P_T(x)) + A_n(T)(P_T(x)) - P_T(x)\|_{\mathcal{C}_E} \\ &\leq \|A_n(T)(y)\|_{\mathcal{C}_E} + \|A_n(T)(P_T(x)) - P_T(x)\|_{\mathcal{C}_E} \to 0 \end{aligned}$$

as $n \to \infty$. □

Now we give applications of Theorems 2 and 4 to Orlicz and Lorentz ideals of compact operators.

1. Let Φ be an *Orlicz function*, that is, $\Phi : [0,\infty) \to [0,\infty)$ is convex, continuous at 0, $\Phi(0) = 0$ and $\Phi(u) > 0$ if $u > 0$ (see, for example, [10, Ch. 2, § 2.1], [15, Ch. 4]). Let

$$l^\Phi(\mathbb{N}) = \left\{ \xi = \{\xi_n\}_{n=1}^\infty \in l^\infty : \ \sum_{n=1}^\infty \Phi\left(\frac{|\xi_n|}{a}\right) < \infty \ \text{ for some } \ a > 0 \right\}$$

be the *Orlicz sequence space*, and let

$$\|\xi\|_\Phi = \inf\left\{ a > 0 : \ \sum_{n=1}^\infty \Phi\left(\frac{|\xi_n|}{a}\right) \le 1 \right\}$$

be the *Luxemburg norm* in $l^\Phi(\mathbb{N})$. It is well-known that $(l^\Phi(\mathbb{N}), \|\cdot\|_\Phi)$ is a fully symmetric sequence space.

Since $\Phi(u) > 0,\ u > 0$, it follows that $\sum\limits_{n=1}^\infty \Phi(a^{-1}) = \infty$ for each $a > 0$, hence $\mathbf{1} = \{1,1,...\} \notin l^\Phi(\mathbb{N})$ and $l^\Phi(\mathbb{N}) \subset c_0$. Therefore, we can define Orlicz ideal of compact operators

$$\mathcal{C}^\Phi = \mathcal{C}_{l^\Phi(\mathbb{N})}, \quad \|x\|_\Phi = \|x\|_{\mathcal{C}_{l^\Phi(\mathbb{N})}},\ x \in \mathcal{C}^\Phi.$$

By Theorem 2 (i) we obtain that given Dunford-Schwartz operator T and $x \in \mathcal{C}^\Phi$, there exists $\widehat{x} \in \mathcal{C}^\Phi$ such that $\|A_n(T)(x) - \widehat{x}\|_\infty \to 0$ as $n \to \infty$ (cf. Theorem 3.2 [4]).

It is said that an Orlicz function Φ satisfies (Δ_2)-*condition at* 0 if there exist $u_0 \in (0,\infty)$ and $k > 0$ such that $\Phi(2u) < k\,\Phi(u)$ for all $0 < u < u_0$. It is well known that an Orlicz function Φ satisfies (Δ_2)-condition at 0 if and only if $(l^\Phi(\mathbb{N}), \|\cdot\|_\Phi)$ is separable (see [10, Ch. 2, § 2.1, Theorem 2.1.17], [15, Ch. 4, Proposition 4.a.4]). In addition, $l^\Phi(\mathbb{N}) = l^1$ as sets, if and only if $\limsup\limits_{u\to 0} \frac{\Phi(u)}{u} > 0$ (see [15, Ch. 4, Proposition 4.a.5], [18, Ch. 16, § 16.2]).

Thus, using Theorem 4, we obtain that the averages $A_n(T)$ converge strongly in $\mathcal{C}^\Phi$ for any Dunford-Schwartz operator T if and only if Φ satisfies (Δ_2)-condition at 0 and $\lim\limits_{u\to 0} \frac{\Phi(u)}{u} = 0$.

2. Let ψ be a concave function on $[0,\infty)$ with $\psi(0) = 0$ and $\psi(t) > 0$ for all $t > 0$, and let

$$\Lambda_\psi(\mathbb{N}) = \left\{ \xi = \{\xi_n\}_{n=1}^\infty \in l^\infty : \ \|\xi\|_\psi = \sum_{n=1}^\infty \xi_n^*(\psi(n) - \psi(n-1)) < \infty \right\},$$

the *Lorentz sequence space*. The pair $(\Lambda_\psi(\mathbb{N}), \|\cdot\|_\psi)$ is a fully symmetric sequence space (see, for example, [14, Ch. II, § 5], [18, Part III, Ch. 9, § 9.1]). Besides, if $\psi(\infty) = \infty$, then $\mathbf{1} \notin \Lambda_\psi(\mathbb{N})$ and $\Lambda_\psi(\mathbb{N}) \subset c_0$. In this case we can define Lorentz ideal of compact operators

$$\mathcal{C}_\psi = \mathcal{C}_{\Lambda_\psi(\mathbb{N})}, \quad \|x\|_\psi = \|x\|_{\mathcal{C}_{\Lambda_\psi(\mathbb{N})}},\ x \in \mathcal{C}_\psi,$$

for which is true Theorem 2 (i).

It is well known that $(\Lambda_\psi(\mathbb{N}), \|\cdot\|_\psi)$ is separable if and only if $\psi(+0) = 0$ and $\psi(\infty) = \infty$ (see, for example, [14, Ch. II, § 5, Lemma 5.1], [18, Ch. 9, § 9.3, Theorem 9.3.1]). In addition, $\lim\limits_{t\to\infty} \frac{\psi(t)}{t} > 0$ if and only if the norms $\|\cdot\|_\psi$ and $\|\cdot\|_1$

are equivalent on $\Lambda_\psi(\mathbb{N})$, that is, $\Lambda_\psi(\mathbb{N}) = l^1$ as sets. Therefore, by Theorem 4, we obtain that the averages $A_n(T)$ converge strongly in $\mathcal{C}_\psi$ for any Dunford-Schwartz operator T if and only if $\psi(+0) = 0$, $\psi(\infty) = \infty$ and $\lim\limits_{t\to\infty} \frac{\psi(t)}{t} = 0$.

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Azizkhon Nodirovich Azizov
National University of Uzbekistan,
4, Universitet str.,
Tashkent, 100174, Uzbekistan
Email address: azizov.07@mail.ru

Vladimir Ivanovich Chilin
National University of Uzbekistan,
4, Universitet str.,
Tashkent, 100174, Uzbekistan
Email address: vladimirchil@gmail.com, VladimirChilin@micros.uz