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ON BAND PRESERVING ORTHOGONALLY ADDITIVE
OPERATORS

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ABSTRACT. In this paper, we investigate a new class of operators on vector lattices. We say that an orthogonally additive operator $T : E \rightarrow E$ on a vector lattice E is band preserving if $T(D) \subset \{D\}^{\perp\perp}$ for every subset D of E . We show that the set of all band preserving operators on a Dedekind complete vector lattice E is a band in the vector lattice of all regular orthogonally additive operators on E which coincides with the band generated by the identity operator. We present a formula for the order projection onto this band and obtain an analytical representation for order continuous band preserving operators on the space of all measurable functions. Finally, we consider the procedure of extending a band preserving map from a lateral band to the whole space.

Keywords: orthogonally additive operator, band preserving operator, disjointness preserving operator, nonlinear superposition operator, vector lattice, lateral ideal, lateral band.

1. INTRODUCTION

Orthogonally additive operators on vector lattices have been first introduced in work [12]. At present, a general theory of these operators is an area of active research (see [3, 5, 10, 13, 15, 20, 21]). Special classes of orthogonally additive operators were studied in works [1, 4, 9, 11, 14, 16, 18]). In this article, we will consider a new class of so called *band preserving* orthogonally additive operators on vector lattices. We will show that the set of all such operators, defined on an order complete vector lattice E , is a band on a vector lattice of all orthogonally additive operators on E , moreover, this band coincides with a band generated by the identity operator. We have found a formula for projecting on that band. We have also obtained an

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analytical representation of order continuous band preserving operators, acting in the space of all measurable almost everywhere finite functions. In the last section of this article, a procedure of extending a band preserving mapping from a lateral strip to the whole space is studied. Note that the theory of linear band preserving operators is widely represented in mathematical literature (see [7]).

All necessary information on vector lattices and positive linear operators on them can be found in studies [6, 7]. All vector lattices considered in the text below are Archimedes ones.

Let E be a vector lattice. A net $(x_\alpha)_{\alpha \in \Lambda} \subset E$ is called *order convergent* to an element $x \in E$ (denoted as $x_\alpha \xrightarrow{(o)} x$), if there exists a net $(u_\alpha)_{\alpha \in \Lambda}$ in E_+ , such that $u_\alpha \downarrow 0$ and $|x_\alpha - x| \leq u_\alpha$ for all indices $\alpha \in \Lambda$ satisfying the equality $\alpha \geq \alpha_0$ for some $\alpha_0 \in \Lambda$. Two elements x, y of a vector lattice E are called *disjoint* (denoted as $x \perp y$), if $|x| \wedge |y| = 0$. The sum $x + y$ of two disjoint elements x and y is denoted by $x \sqcup y$. A notation $x = \bigsqcup_{i=1}^n x_i$ means that $x = \sum_{i=1}^n x_i$ and $x_i \perp x_j$, when $i \neq j$.

An element $y \in E$ is called a *fragment* of the element $x \in E$, if $y \perp (x - y)$. The notation $y \sqsubseteq x$ expresses the fact that y is a fragment of the element x . The set of all fragments of an element $x \in E$ is denoted by \mathcal{F}_x . A positive linear operator $\pi : E \rightarrow E$ is called an *order projection*, if the following conditions hold:

- $\pi = \pi^2$;
- $0 \leq \pi \leq I$, where I is the identity operator in E .

A set of all order projection in E is denoted by $\mathfrak{B}(E)$. A set $\mathfrak{B}(E)$, ordered by a relation $\pi \leq \rho \Leftrightarrow \pi \circ \rho = \pi$, is a Boolean algebra with respect to Boolean operations:

$$\pi \wedge \rho := \pi \circ \rho; \pi \vee \rho := \pi + \rho - \pi \circ \rho; \bar{\pi} = I - \pi.$$

An element x of a vector lattice E is called a *projection element*, if a strip $\{x\}^{\perp\perp}$, generated by the element x , admits order projecting. It is said that a vector lattice E is a lattice *with principal projection property*, if every element of the lattice E is a projection one. Note that every order σ -complete vector lattice is a lattice with projections on the main strips. We will say that E is a *vector lattice with projections*, if there exists an order projector on every strip B in E . We will denote the order projector on the strip $\{D\}^{\perp\perp}$, generated by the set $D \subset E$, by π_D . The notation π_x will be used for the order projector on the strip $\{x\}^{\perp\perp}$.

Let E be a vector lattice and F an actual vector space. We will call an arbitrary mapping from E to F an *operator*. An operator $T : E \rightarrow F$ is called *orthogonally additive*, if $T(x + y) = Tx + Ty$ for all disjoint elements $x, y \in E$. It is clear that $T(0) = 0$. The set of all orthogonally additive operators is a real vector space with respect to natural linear operations of vector addition and multiplication of a vector by an element of a field.

Let E and F be vector lattices. An orthogonally additive operator $T : E \rightarrow F$ is called:

- *positive*, if $Tx \geq 0$ for every $x \in E$;
- *order bounded*, if T maps order bounded sets in E into order bounded sets in F ;
- *fragmentary-order bounded* or a *Popov operator*, if for every $x \in E$ the set $T(\mathcal{F}_x)$ is order bounded in F ;
- *regular*, if $T = S_1 - S_2$, where S_1, S_2 are positive orthogonally additive operators from E to F .

The vector spaces of all fragmentary-order bounded, positive, regular, and arbitrary additive operators from E to F are designated by $\mathcal{P}(E, F)$, $\mathcal{OA}_+(E, F)$, $\mathcal{OA}_r(E, F)$ and $\mathcal{OA}(E, F)$ respectively.

The vector space $\mathcal{OA}(E, F)$ possesses a relation of partial order. We set that $S \leq T$, if $T - S \geq 0$. Note that in the case of order completeness of the vector lattice F , for the ordered space $\mathcal{OA}_r(E, F)$ the following theorem is true.

Lemma 1 ([19], Theorem 3.6). *Let E and F be vector lattices, moreover, the lattice F is order complete. Then $\mathcal{OA}_r(E, F)$ is an order complete vector lattice and $\mathcal{OA}_r(E, F) = \mathcal{P}(E, F)$. Furthermore, for every $S, T \in \mathcal{OA}_r(E, F)$ and $x \in E$, the following formulas hold:*

- (1) $(T \vee S)x := \sup\{Ty + Sz : x = y \sqcup z\}$.
- (2) $(T \wedge S)x := \inf\{Ty + Sz : x = y \sqcup z\}$.
- (3) $(T)^+x := \sup\{Ty : y \sqsubseteq x\}$.
- (4) $(T)^-x := -\inf\{Ty : y \sqsubseteq x\}$.
- (5) $|Tx| \leq |T|x$.

Consider some examples of orthogonally additive operators.

Example 1. One of the most important examples is a nonlinear integral Urysohn operator. Let (A, Σ, μ) and (B, Ξ, ν) be spaces with full σ -finite measures, and $(A \times B, \mu \times \nu)$ their completed product. Let $K : A \times B \times \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying the following conditions

- (C₀) $K(s, t, 0) = 0$ for $\mu \times \nu$ -almost all $(s, t) \in A \times B$;
- (C₁) $K(\cdot, \cdot, r)$ is $\mu \times \nu$ -measurable for all $r \in \mathbb{R}$;
- (C₂) $K(s, t, \cdot)$ is continuous on \mathbb{R} for $\mu \times \nu$ -almost all $(s, t) \in A \times B$.

We denote by $L_0(B, \Xi, \nu)$ or $L_0(\nu)$ the ordered space of classes of equivalence of ν -measurable almost everywhere finite real-valued functions defined on B , where a partial order $f \leq g$ is given as $f(t) \leq g(t)$ ν -almost everywhere on B . It is well known that $L_0(\nu)$ is an order complete vector lattice.

For a given function $f \in L_0(\nu)$, the function $|K(s, \cdot, f(\cdot))|$ is ν -measurable for μ -almost all $s \in A$ and $h_f(s) := \int_B |K(s, t, f(t))| d\nu(t)$ is also a μ -measurable function. We introduce the following notation:

$$\text{Dom}_B(K) := \{f \in L_0(\nu) : h_f \in L_0(\mu)\}.$$

Then the operator $T : \text{Dom}_B(K) \rightarrow L_0(\mu)$ is defined,

$$T(f)(s) := \int_B K(s, t, f(t)) d\nu(t) \quad \mu - \text{a.e.}$$

Let E and F be order ideals in $L_0(\nu)$ and $L_0(\mu)$, therefore, K also satisfies the conditions (C₀)-(C₂). Then, if $E \subseteq \text{Dom}_B(K)$ and $T(E) \subseteq F$, the orthogonally additive operator acting from $T : E \rightarrow F$, called an *integral Urysohn operator*, is defined.

A special case of an integral Urysohn operator is a Hammerstein operator, given by the formula

$$T(f)(s) := \int_B K(s, t)u(t, f(t)) d\nu(t) \quad \mu - \text{a.e.},$$

where $K(\cdot, \cdot)$ is a $\mu \times \nu$ -measurable function on $A \times B$ and $u : B \times \mathbb{R} \rightarrow \mathbb{R}$ is a function, such that $u(t, \cdot)$ is continuous on \mathbb{R} for ν -almost all $t \in B$, $u(\cdot, r)$ is

ν -measurable for every $r \in \mathbb{R}$ and $u(t, 0) = 0$ for ν -almost all $t \in B$ (to satisfy the condition C_0).

Example 2. Every positive linear operator $T \in L_+(E, F)$ defines a positive orthogonally additive operator $G : E \rightarrow F$, where

$$G(f) := T(f_+), \quad f \in E.$$

Example 3. Recall that the vector space \mathbb{R}^m , $m \in \mathbb{N}$ is a vector lattice with coordinate-wise order: for all $x, y \in \mathbb{R}^m$, we have that $x \leq y$ if and only if $e_i^*(x) \leq e_i^*(y)$ for every $i = 1, \dots, m$, where $(e_i^*)_{i=1}^m$ are coordinate functionals on \mathbb{R}^m . Suppose that $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Then $T \in \mathcal{P}(\mathbb{R}^n, \mathbb{R}^m)$ if and only if there exist real-valued functions $T_{i,j} : \mathbb{R} \rightarrow \mathbb{R}$, $1 \leq i \leq m$, $1 \leq j \leq n$, such that $T_{i,j}(0) = 0$ and

$$e_i^*(T(x_1, \dots, x_n)) = \sum_{j=1}^n T_{i,j}(x_j),$$

In this case, we write that $T = (T_{i,j})$.

2. ELEMENTARY PROPERTIES OF BAND PRESERVING OPERATORS

In this section, we introduce the notion of a band preserving operator and establish some properties of these operators.

Definition 1. Let E be a vector lattice and $T : E \rightarrow E$. The operator T is called band preserving, if for every $D \subset E$, the inclusion $T(D) \subset \{D\}^{\perp\perp}$ is valid. The operator $T : E \rightarrow E$ preserves disjointness, if $Tx \perp Ty$ for every $x, y \in E$, such that $x \perp y$.

Definition 2. Let E be a vector lattice with projections. We will say that the operator $T : E \rightarrow E$ commutes with projectors, if $T\pi = \pi T$ for all $\pi \in \mathfrak{B}(E)$.

Proposition 1. Let E be a vector lattice with projections on the main strips. Then every operator $T : E \rightarrow E$, commuting with projectors, is orthogonally additive and maintains disjointness.

Proof. Suppose that $T\pi = \pi T$ for every $\pi \in \mathfrak{B}(E)$ and $x, y \in E$ is a pair of arbitrary disjoint elements. Then we can write that

$$\begin{aligned} T(x + y) &= T(\pi_x + \pi_y)(x + y) \\ &= T(\pi_x \vee \pi_y)(x + y) \\ &= (\pi_x \vee \pi_y)T(x + y) \\ &= (\pi_x + \pi_y)T(x + y) \\ &= \pi_x T(x + y) + \pi_y T(x + y) \\ &= T\pi_x(x + y) + T\pi_y(x + y) \\ &= Tx + Ty. \end{aligned}$$

Therefore, the operator T is orthogonally additive. We will show that T preserves disjointness. We take a pair of disjoint elements $x, y \in E$. Then the order projections π_x and π_y are disjoint in the Boolean algebra $\mathfrak{B}(E)$ and $\pi_x u \perp \pi_y v$ for all $u, v \in E$. From here we obtain

$$Tx = T\pi_x(x) = \pi_x Tx \perp \pi_y Ty = T\pi_y(y) = Ty.$$

□

Proposition 2. *Let E be a vector lattice with projections. Then the following statements hold:*

- (1) *every operator $T: E \rightarrow E$, commuting with projections, is a band preserving orthogonally additive operator;*
- (2) *every band preserving orthogonally additive operator $T: E \rightarrow E$ commutes with projections.*

Proof. We will show that (1) is true. Suppose that $T\pi = \pi T$ for every $\pi \in \mathfrak{B}(E)$. Since in E there exist order projections on every main band, then according to Proposition 1, the operator T is orthogonally additive. Suppose that D is an arbitrary subset in E . Then

$$T(D) \subset T\pi_D(E) = \pi_D T(E) \subset \pi_D(E) = \{D\}^{\perp\perp}.$$

We will show that (2) is true. Suppose that B is a strip in E , $T: E \rightarrow E$ is a band preserving orthogonally additive operator and x is an arbitrary element of E . Then there exists a unique representation $x = x_1 + x_2$, where $x_1 \in B$ and $x_2 \in B^\perp$. It is clear that $Tx_1 \in B$, $Tx_2 \in \{B\}^\perp$. Now we can write that

$$\begin{aligned} T\pi_B(x) &= T\pi_B(x_1 + x_2) = T\pi_B(x_1); \\ \pi_B Tx &= \pi_B T(x_1 + x_2) = \\ &= \pi_B Tx_1 + \pi_B Tx_2 = \\ \pi_B Tx_1 &\Rightarrow T\pi_B = \pi_B T. \end{aligned}$$

□

A set of all band preserving additive operators on a vector lattice E is denoted by $\mathcal{N}(E)$. Consider a typical example of a band preserving orthogonally additive operator.

Example 4. Let (A, Σ, μ) be a space with a σ -finite measure. We will say that a function $N: A \times \mathbb{R} \rightarrow \mathbb{R}$ is *superpositionnally measurable* or *super-measurable*, for brevity, if

- (C₁') the mapping $N(\cdot, g(\cdot))$, considered as a function of one variable, is measurable for every function $g \in L_0(\mu)$.

We will say that N is a \mathfrak{S} -function, if it is super-measurable and

- (C₀) $N(s, 0) = 0$ for μ -almost all $s \in A$.

We can pair every \mathfrak{S} -function $N: A \times \mathbb{R} \rightarrow \mathbb{R}$ with a *superposition operator* $T_N: L_0(\mu) \rightarrow L_0(\mu)$ defined by the rule

$$(T_N f)(s) = N(s, f(s)) \pm f \in L_0(\mu).$$

Note that nonlinear superposition operator also known as Nemytskii's operators are widely represented in the literature (see the book [8]).

The next statement is well known to specialists. However, for completeness, we will provide the proof.

Proposition 3. *Suppose that (A, Σ, μ) is a space with a σ -finite measure and $N: A \times \mathbb{R} \rightarrow \mathbb{R}$ is a \mathfrak{S} -function. Then $T_N \in \mathcal{N}(L_0(\mu))$.*

Proof. Suppose that $\Sigma_0 = \{D \in \Sigma: \mu(D) = 0\}$. It is clear that Σ_0 is an ideal of the Boolean algebra Σ . Then $\Sigma' := \Sigma/\Sigma_0$ is a factor algebra referred to as an *algebra of measurable sets modulo of sets of zero measure*. It is well known that the Boolean algebra $\mathfrak{B}(L_0(\mu))$ of all order projectors of the vector lattice $L_0(\mu)$ is isomorphic to

the Boolean algebra Σ' . In fact, there is an order projector π_D , which is an operator of multiplying by a characteristic function of a measurable set 1_D , corresponding to every class of equivalency of measurable sets $D \in \Sigma'$ (see [6, 1.6]). We will show that

$$N(s, x1_D(s)) = N(s, x)1_D(s) \quad \text{for every } D \in \Xi'.$$

Suppose that $s \in D$. Then

$$N(s, x1_D(s)) = N(s, x) = N(s, x)1_D(s).$$

Now suppose that $s \notin D$. Then under condition (C_0) we obtain

$$N(s, x1_D(s)) = N(s, 0) = 0 = N(s, x)1_D(s).$$

We take arbitrary $f \in L_0(\mu)$ and $\pi = \pi_D \in \mathfrak{B}(L_0(\mu))$. Then we can write that

$$T\pi(f) = T(f1_D) = N(\cdot, f1_D(\cdot)) = N(\cdot, f(\cdot))1_D(\cdot) = \pi T(f)$$

and the inclusion $T_N \in \mathcal{N}(L_0(\nu))$ is established. \square

Every band preserving operator is automatically fragmentary-order bounded.

Proposition 4. *Let E be a vector lattice with projections on the main strips and $T \in \mathcal{N}(E)$. Then $T \in \mathcal{P}(E)$.*

Proof. We take an arbitrary element $x \in E$. Suppose that $y \in \mathcal{F}_x$. Since $y \perp (x-y)$, by Proposition 1 we obtain that $Ty \perp T(x-y)$. Due to the fact that the operator T is orthogonally additive, we get that $T(\mathcal{F}_x) \subseteq \mathcal{F}_{Tx}$. Hence, $|Ty| \leq |Tx|$ and T is fragmentary-order restricted. \square

3. THE PROJECTION BAND OF BAND PRESERVING OPERATORS

In this section, we will show that for an order complete vector lattice E , the set of all band preserving orthogonally additive operators $\mathcal{N}(E)$ is a strip in $\mathcal{O}\mathcal{A}_r(E)$.

Theorem 1. *Let E be an order complete vector lattice. Then $\mathcal{N}(E)$ is a strip in $\mathcal{O}\mathcal{A}_r(E)$, coinciding with $\{I\}^{\perp\perp}$ and for all $T, S \in \mathcal{N}(E)$, $x \in E$, the following equalities hold:*

- (1) $(T \vee S)x = Tx \vee Sx$;
- (2) $(T \wedge S)x = Tx \wedge Sx$;
- (3) $(T)^+x = (Tx)^+$;
- (4) $(T)^-x = (Tx)^-$;
- (5) $|T|x = |Tx|$.

Proof. According to Propositions 2 and 4, the set $\mathcal{N}(E)$ is a linear subspace of the order complete vector lattice $\mathcal{O}\mathcal{A}_r(E)$. Suppose that $T, S \in \mathcal{N}(E)$ and $x \in E$. Then we can write that

$$(T \vee S)x = \sup\{Ty + Sz : x = y \sqcup z\} \geq Tx \vee Sx.$$

Note that if $x = y \sqcup z$, then $y = \pi_y x = \pi_y y$, $z = \pi_z x = \pi_z z$, $\pi_y \perp \pi_z$, $\pi_x = \pi_y + \pi_z$, $\pi_x(Tx \vee Sx) = Tx \vee Sx$. From here we imply that

$$\begin{aligned} Ty + Sz &= T\pi_y y + S\pi_z z \\ &= T\pi_y x + S\pi_z x \\ &= \pi_y Tx + \pi_z Sx \\ &\leq \pi_y(Tx \vee Sx) + \pi_z(Tx \vee Sx) \\ &\leq \pi_x(Tx \vee Sx) \\ &= Tx \vee Sx. \end{aligned}$$

Taking the supremum in the left-hand side of the inequality above by all fragments $y, z \in \mathcal{F}_x$ such that $x = y \sqcup z$, we obtain the required estimate

$$(T \vee S)x \leq Tx \vee Sx.$$

Therefore, $(T \vee S)x = Tx \vee Sx$ for all $x \in E$. Now we will establish the order formulas for the infimum of the two operators, modulus, positive and negative parts.

$$\begin{aligned} (T \wedge S)x &= -\left((-T) \vee (-S)x\right) = -\left((-Tx) \vee (-Sx)\right) = Tx \wedge Sx; \\ T^+x &= (T \vee 0)x = Tx \vee 0 = (Tx)^+; \\ T^-x &= (-T \vee 0)x = -Tx \vee 0 = (Tx)^-; \\ |T|x &= (T \vee (-T))x = Tx \vee (-Tx) = |Tx|. \end{aligned}$$

Now we will show that $\mathcal{N}(E)$ is a strip in $\mathcal{O}\mathcal{A}_r(E)$. To do that, it is enough to establish the equality $\mathcal{N}(E) = \{I\}^{\perp\perp}$. First we will affirm the inclusion $\mathcal{N}(E) \subset \{I\}^{\perp\perp}$. Suppose that $T \in \mathcal{N}(E) \cap \mathcal{O}\mathcal{A}_+(E)$. Then $Tx \in \{x\}^{\perp\perp} = \{|x|\}^{\perp\perp}$ for every $x \in E$ and $Tx = \sup_n \{n|x| \wedge Tx\}$. It is required to establish that $T = \sup_n \{n|I| \wedge T\}$.

Since $n|I|, T \in \mathcal{N}(E) \cap \mathcal{O}\mathcal{A}_+(E)$, $n \in \mathbb{N}$, we have that

$$\begin{aligned} (n|I| \wedge T)x &= n|I|(x) \wedge Tx = n|x| \wedge Tx \Rightarrow \\ \sup_n \{n|I| \wedge T\}x &= \sup_n \{n|x| \wedge Tx\} = Tx \end{aligned}$$

and the inclusion $\mathcal{N}(E) \subset \{I\}^{\perp\perp}$ is verified. Now we will show that $\{I\}^{\perp\perp} \subset \mathcal{N}(E)$. We take an arbitrary operator $T \in \{I\}^{\perp\perp} \cap \mathcal{O}\mathcal{A}_+(E)$. We have to prove that for every $x \in E$ the condition $Tx \in \{x\}^{\perp\perp}$ holds. Assume the opposite. That means that there exists $x \in E$ such that there is a decomposition $Tx = x_1 + x_2$, where $x_1 \in \{x\}^{\perp\perp}$, $x_2 \in \{x\}^\perp$, and $x_2 \neq 0$. At the same time, $T = \sup_n \{n|I| \wedge T\}$. Note that for every $n \in \mathbb{N}$ the following formulas hold:

$$\begin{aligned} (n|I| \wedge T)x &= \inf\{n|I|y + Tz : x = y \sqcup z\} = \\ &= \inf\{n|y| + Tz : x = y \sqcup z\} \leq (n|x| \wedge Tx). \end{aligned}$$

From here we imply that

$$Tx = \sup_n \{n|I| \wedge T\}x \leq \sup_n \{n|x| \wedge Tx\} \in \{x\}^{\perp\perp}.$$

We have obtained a contradiction. The equality $\mathcal{N}(E) = \{I\}^{\perp\perp}$ is established. \square

Note that without a quite strict requirement of order completeness of the vector lattice E , the ordered space $\mathcal{O}\mathcal{A}_r(E)$ is not a vector lattice. However, under certain

conditions, imposed on the lattice E , every band preserving orthogonally additive operator on E possesses a modulus.

Proposition 5. *Let E be a vector lattice with projections and $T \in \mathcal{N}(E)$. Then there exists an operator $G \in \mathcal{N}(E) \cap \mathcal{OA}_+(E)$, such that $G = T \vee (-T)$.*

Proof. Note, first of all, that due to Propositions 1 and 2, every operator $T \in \mathcal{N}(E)$ preserves disjointness. Now consider the mapping $G : E \rightarrow E$, defined by the formula

$$Gx := |Tx|, \quad x \in E.$$

G is an orthogonally additive operator as a composition of the operator T , preserving disjointness, and the operator $u \rightarrow |u|$ on the left.

For every $x \in E$ we have that

$$Tx \leq |Tx| = Gx; \quad (-Tx) \leq |Tx| = Gx.$$

Therefore, $T \leq G$ and $(-T) \leq G$. We will show that $G \in \mathcal{N}(E)$. To do that, according to Proposition 2, it is enough to show that G commutes with projections. We take an arbitrary order projection $\pi \in \mathfrak{B}(E)$ and $x \in E$. Then

$$G\pi(x) = |T\pi(x)| = |\pi Tx| = \pi|Tx| = \pi Gx.$$

Assume that $S : E \rightarrow E$ is a positive orthogonally additive operator, such that $T \leq S$ and $(-T) \leq S$. Then for every $x \in E$ we can write

$$\begin{aligned} Tx \leq Sx; \quad (-Tx) \leq Sx &\Rightarrow \\ Tx \vee (-Tx) = |Tx| = Gx &\leq Sx \end{aligned}$$

and hence, we obtain that $G = T \vee (-T)$. □

Let E be an order complete vector lattice. According to Lemma 1, the vector lattice $\mathcal{OA}_r(E)$ is order complete, which implies that there exists an order projection $\mathfrak{p} : \mathcal{OA}_r(E) \rightarrow \mathcal{N}(E)$ on the strip $\mathcal{N}(E)$. Below we will provide an explicit formula which allows to calculate \mathfrak{p} .

We will denote by $\mathfrak{D}_0(E)$, or \mathfrak{D}_0 for brevity, the set of all finite decompositions of the unit operator I :

$$\mathfrak{D}_0 = \left\{ (\pi_i) : \pi_k \wedge \pi_j = 0, k \neq j; \sum_{i=1}^n \pi_i = I; n \in \mathbb{N} \right\}.$$

Note that in the definition of the set \mathfrak{D}_0 mentioned above, we mean linear decompositions of the operator I , that is, $(\pi_i)_{i=1}^n$ is a family of positive, pairwise disjoint linear operators, such that $\sum_{i=1}^n \pi_i = I$.

Theorem 2. *Let E be an order complete vector lattice and $T \in \mathcal{OA}_+(E)$. Then the projection $\mathfrak{p}(T)$ of the operator T on the band $\mathcal{N}(E)$ can be calculated by the formula*

$$\mathfrak{p}(T) = \inf \left\{ \sum_{i=1}^n \pi_i T \pi_i : (\pi_i) \in \mathfrak{D}_0 \right\}.$$

Proof. For the operator $T \in \mathcal{OA}_+(E)$, we put

$$\mathfrak{A}(T) := \left\{ \sum_{i=1}^n \pi_i T \pi_i : (\pi_i) \in \mathfrak{D}_0 \right\}.$$

Note that $\mathfrak{A}(T)$ is a downward directed set of positive orthogonally additive operators. We will illustrate a general construction for the simplest case of two decompositions: $(\pi_1, \pi_2), (\pi_{11}, \pi_{12}, \pi_2) \in \mathfrak{D}_0$, where $\pi_1 = \pi_{11} + \pi_{12}$. Then

$$\begin{aligned} \pi_1 T \pi_1 + \pi_2 T \pi_2 &= \pi_1 T (\pi_{11} + \pi_{12}) + \pi_2 T \pi_2 \\ &\geq \pi_{11} T \pi_{11} + \pi_{12} T \pi_{12} + \pi_2 T \pi_2. \end{aligned}$$

Due to the order completeness of the lattice $\mathcal{OA}_r(E)$, there exists an operator $R(T) := \inf \mathfrak{A}(T)$. We will show that the following conditions are satisfied:

- (1) $0 \leq R(T) \leq T$;
- (2) the mapping $R: \mathcal{OA}_+(E) \rightarrow \mathcal{OA}_+(E)$ is additive;
- (3) $R(T) = T \Leftrightarrow T \in \mathcal{N}(E)$;
- (4) $R(R(T)) = R(T)$.

The condition (1) is trivial. We will prove (2). Suppose that $T_1, T_2 \in \mathcal{OA}_+(E)$. For all $(\pi_i), (\pi_j), (\pi_k) \in \mathfrak{D}_0$ such that $(\pi_k) \geq (\pi_i)$ and $(\pi_k) \geq (\pi_j)$, we have

$$\begin{aligned} \sum_i \pi_i T_1 \pi_i + \sum_j \pi_j T_2 \pi_j &\geq \sum_k \pi_k (T_1 + T_2) \pi_k \\ &\geq R(T_1 + T_2). \end{aligned}$$

Taking the infimum in the left-hand side of the formula above by all decompositions from \mathfrak{D}_0 , we obtain that $R(T_1) + R(T_2) \geq R(T_1 + T_2)$. On the other hand, for every decomposition $(\pi_k) \in \mathfrak{D}_0$, the chain of formulas hold:

$$\begin{aligned} \sum_k \pi_k (T_1 + T_2) \pi_k &= \sum_k \pi_k T_1 \pi_k + \sum_k \pi_k T_2 \pi_k \\ &\geq R(T_1) + R(T_2), \end{aligned}$$

and taking the infimum in the left-hand side of the inequality by all the finite decompositions, we obtain that $R(T_1 + T_2) \geq R(T_1) + R(T_2)$.

We will prove the condition (3). Suppose that $0 \leq T \in \mathcal{N}(E)$. Then we can write

$$\sum_i \pi_i T \pi_i = \sum_i \pi_i^2 T = \sum_i \pi_i T = T.$$

Taking the infimum in the left-hand side of the equality by all $(\pi_i) \in \mathfrak{D}_0$, we obtain that $R(T) = T$.

Now suppose that $R(T) = T$. We will show that $T \in \mathcal{N}(E)$. Indeed, we take $\pi \in \mathfrak{B}(E)$. Then

$$T = R(T) \leq \pi T \pi + (I - \pi) T (I - \pi) \leq T.$$

From here it follows that $\pi T = \pi T \pi$. Therefore, $\pi T \leq T \pi$. Moreover, since

$$T = (\pi + I - \pi) T (\pi + (I - \pi)) = T + (I - \pi) T \pi + \pi T (I - \pi),$$

we have that $(I - \pi) T \pi = 0$. Then

$$T \pi = (\pi + (I - \pi)) T \pi = \pi T \pi \leq \pi T.$$

From here we imply that $\pi T = T \pi$.

It remains to check the condition (4). Suppose that $T \in \mathcal{OA}_+(E)$ and $W := R(T)$. For every $\rho \in \mathfrak{B}(E)$, we have that

$$\begin{aligned} W(\rho) &= \inf \left\{ \sum_i \pi_i T \pi_i \rho : (\pi_i) \in \mathfrak{D}_0 \right\} \\ &= \inf \left\{ \sum_i \pi'_i T \pi'_i \rho : (\pi'_i) \in \mathfrak{D}_0(\rho) \right\} \\ &= \inf \left\{ \sum_i \rho(\pi'_i) T \pi'_i : (\pi'_i) \in \mathfrak{D}_0(\rho) \right\} \\ &= \rho \inf \left\{ \sum_i \pi'_i T \pi'_i : (\pi'_i) \in \mathfrak{D}_0(\rho) \right\} \\ &= \rho \inf \left\{ \sum_i \pi_i T \pi_i : (\pi_i) \in \mathfrak{D}_0 \right\} = \rho W. \end{aligned}$$

Thus, we have shown that $W \in \mathcal{N}(E)$. Due to (3), we get $R(W) = W$. Using Kantorovich theorem ([7], Theorem 1.10), and using the proven property (2), we can continue the additive mapping R up to a linear positive operator from $\mathcal{OA}_r(E)$ in $\mathcal{OA}_r(E)$. For that operator, we keep the same notation R . According to conditions (1) and (4), R is a projector. Condition (3) shows that R is a projector on the band $\mathcal{N}(E)$. \square

Note that a similar theorem for linear order bounded band preserving operators has been established in paper [22].

4. BAND PRESERVING OPERATORS IN SPACES OF MEASURABLE FUNCTIONS

In this section, we will study the band preserving operators in the space of measurable, almost everywhere finite functions.

Let (A, Σ, μ) be a space with a σ -finite measure. As in the example 4, we denote by $L_0(A, \Sigma, \mu)$ (or $L_0(\mu)$ for brevity) the set of all equivalence classes of real-valued measurable functions on A . The measurable functions f and g are equivalent, if they coincide almost everywhere on A . The vector space $L_0(\mu)$ is endowed with a metric ρ_{L_0} , given by the formula

$$\rho_{L_0}(f, g) := \int_A \frac{|f(s) - g(s)|}{1 + |f(s) - g(s)|} d\mu \quad (f, g \in L_0(\mu)).$$

The space $L_0(\mu)$ is a complete metric space, moreover, the convergence by the metric ρ_{L_0} coincides with the convergence by the measure. Below we will provide a characterisation of the mentioned convergence in terms of almost everywhere converging sequences.

Lemma 2 ([6], Theorem 1.82). *Let (A, Σ, μ) be a space with a finite measure, then for the sequence $(g_n) \subseteq L_0(\mu)$ and the element $g \in L_0(\mu)$, the following conditions are equivalent:*

- (1) (g_n) converges by measure to g ;
- (2) for every subsequence (g_{n_k}) of the sequence (g_n) , we can extract a subsequence $(g_{n_{k_l}})$, converging μ -almost everywhere to g .

Let the function $N : A \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the following conditions:

- (C₀) $N(s, 0) = 0$ for ν -almost all $s \in A$;
- (C₁) $N(\cdot, r)$ is measurable for every $r \in \mathbb{R}$;
- (C₂) $N(s, \cdot)$ is continuous on \mathbb{R} for ν -almost all $s \in A$.

In this case, we will say that N is a \mathfrak{R} -function. In the case when only the conditions (C₁) and (C₂) hold, N is called a *Caratheodory function*. Note that every super-measurable function N satisfies the condition (C₁). It is also well known that every Caratheodory function N is super-measurable ([8], 1.4). For two super-measurable functions $N, K : A \times \mathbb{R} \rightarrow \mathbb{R}$, we will write that $N \preceq K$, if for every function $g \in L_0(\mu)$, the inequality $N(\cdot, g(\cdot)) \leq K(\cdot, g(\cdot))$ holds μ -almost everywhere on A . We will say that N and K are *super-equivalent* and write that $N \simeq K$, if $N \preceq K$ and $K \preceq N$.

Lemma 3 ([8], Theorem 1.4). *Let (A, Σ, μ) be a space with a σ -finite measure and $N : A \times \mathbb{R} \rightarrow \mathbb{R}$ be a super-measurable function. Then the superposition operator $T_N : L_0(\nu) \rightarrow L_0(\nu)$, defined by the formula*

$$T_N f := N(\cdot, f(\cdot)) \quad (f \in L_0(\mu))$$

is continuous in the space $L_0(\mu)$ with respect to topology, generated by the metric ρ_{L_0} , if and only if N is super-equivalent to some Caratheodory function K .

Recall that the orthogonally additive operator $T : L_0(\mu) \rightarrow L_0(\mu)$ is called *order continuous*, if for every sequence $(x_n)_{n \in \mathbb{N}}$ of elements $L_0(\mu)$, converging to $x \in L_0(\mu)$ almost everywhere, the sequence $(Tx_n)_{n \in \mathbb{N}}$ converges to $Tx \in L_0(\mu)$ almost everywhere.

The next theorem is the main result of this section.

Theorem 3. *Let (A, Σ, μ) be a space with finite measure and $T : L_0(\mu) \rightarrow L_0(\mu)$ be an orthogonally additive operator. Then the following conditions are equivalent:*

- (1) T is an order continuous band preserving operator;
- (2) there exist: a \mathfrak{S} -function $N : A \times \mathbb{R} \rightarrow \mathbb{R}$ and a \mathfrak{R} -function $K : A \times \mathbb{R} \rightarrow \mathbb{R}$ such that $N \simeq K$ and $T = T_N$, that is, the following representation holds:

$$Tf = N(\cdot, f(\cdot)) \quad (f \in L_0(\mu)).$$

Proof. (1) \Rightarrow (2). Let $T : L_0(\mu) \rightarrow L_0(\mu)$ be an order continuous band preserving operator. Then we can define a function $N : A \times \mathbb{R} \rightarrow \mathbb{R}$, where

$$N(\cdot, r) := T(r1_A)(\cdot) \quad (r \in \mathbb{R}).$$

Since $N(\cdot, 0) = T(0) = 0$, we obtain that $N(\cdot, 0) = 0$ is μ -almost everywhere (condition (C₀)). Moreover, $N(\cdot, r)$ is μ -measurable for every $r \in \mathbb{R}$ (condition (C₁)). Now we take a simple function $f = \sum_{i=1}^n r_i 1_{A_i}$, where A_i are pairwise disjoint

measurable subsets of A and $r_i \in \mathbb{R}$, $1 \leq i \leq n$. Then we can write that

$$\begin{aligned}
 Tf &= T\left(\sum_{i=1}^n r_i 1_{A_i}\right) \\
 &= \sum_{i=1}^n T(r_i 1_{A_i}) \\
 &= \sum_{i=1}^n T\pi_{A_i}(r_i 1_A) \\
 &= \sum_{i=1}^n \pi_{A_i} T(r_i 1_A) \\
 &= \sum_{i=1}^n N(\cdot, r_i) 1_{A_i} \\
 &= \sum_{i=1}^n N(\cdot, r_i 1_{A_i}) \\
 &= N\left(\cdot, \sum_{i=1}^n r_i 1_{A_i}\right) \\
 &= N(\cdot, f).
 \end{aligned}$$

Now let f be an arbitrary element of $L_0(\mu)$. Then there exists a subsequence of simple functions (f_n) , order converging to f . From here we conclude that

$$Tf = (o) - \lim_{n \rightarrow \infty} Tf_n = \lim_{n \rightarrow \infty} N(\cdot, f_n(\cdot)) = N(\cdot, f(\cdot)).$$

It is clear that N is a \mathfrak{S} -function and $T = T_N$. We will show that the operator T_N satisfies the conditions of Lemma 3, that is, it is continuous with respect to topology, generated by the metric ρ_{L_0} . We will prove it by contradiction. Assume that there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of measurable functions, converging in measure to $f \in L_0(\mu)$ such that the sequence $(Tf_n)_{n \in \mathbb{N}}$ does not converge in measure to Tf . For $n \in \mathbb{N}$, $\delta > 0$, we put

$$D_{n,\delta} := \{t \in A : |Tf(t) - Tf_n(t)| \geq \delta\}.$$

Then, transitioning if required to the subsequence $(f_{n_k})_{k \in \mathbb{N}}$, we can imply that there exist $\delta, \varepsilon > 0$, such that $\mu(D_{n_k,\delta}) \geq \varepsilon$, for every $k \in \mathbb{N}$. Using Lemma 2 and the order continuity of the operator, we will find a subsequence $(f_{n_{k_l}})_{l \in \mathbb{N}}$ of the sequence $(f_{n_k})_{k \in \mathbb{N}}$, almost everywhere converging to f . Then due to order continuity of the operator T , we obtain that $\lim_{l \rightarrow \infty} \mu(D_{n_{k_l},\delta}) = 0$. We have arrived at a contradiction.

Now applying Lemma 3, we establish existence of a \mathfrak{K} -function $K : A \times \mathbb{R} \rightarrow \mathbb{R}$, super-equivalent to N .

(2) \Rightarrow (1). Assume that there exist \mathfrak{S} -function $N : A \times \mathbb{R} \rightarrow \mathbb{R}$ and \mathfrak{K} -function $K : A \times \mathbb{R} \rightarrow \mathbb{R}$, such that $N \simeq K$ and for every function $f \in L_0(\mu)$ the following equality holds:

$$(Tf)(t) = N_T(t, f(t)).$$

We will show that the operator $T = T_N$ is order continuous. We take an arbitrary sequence $(f_n)_{n \in \mathbb{N}} \subset L_0(\mu)$, order converging to f and suppose that $D := \{t \in A : f_n(t) \not\rightarrow f(t)\}$. It is clear that $\mu(D) = 0$. We put $G_n := \{t \in A : N(t, f_n(t)) \neq$

$K(t, f_n(t))\}$, $H := \{t \in A : \text{function } K(t, \cdot) \text{ is not continuous}\}$, $G = \bigcup_{n=1}^{\infty} G_n$ and $D' = D \cup G \cup H$. Then $\mu(D') = 0$. It is clear that $A' = A \setminus D'$ is a set of a full measure and for every $t \in A'$ the following equalities hold:

$$\begin{aligned} \lim_{n \rightarrow \infty} |Tf_n(t) - Tf(t)| &= \lim_{n \rightarrow \infty} |N(t, f_n(t)) - N(t, f(t))| = \\ &= \lim_{n \rightarrow \infty} |K(t, f_n(t)) - K(t, f(t))| = 0. \end{aligned}$$

Thus, the order continuity of the operator T_N is proven. The inclusion $T \in \mathcal{N}(L_0(\mu))$ has been established in Proposition 3. \square

5. EXPANDING BAND PRESERVING MAPPINGS

In this section, we will show that a band preserving orthogonally additive mapping, given on a lateral strip, expands up to a band preserving orthogonally additive operator given on the whole space.

Definition 3. Let E be a vector lattice. We will say that a net $(x_\alpha)_{\alpha \in \Lambda} \subset E$ laterally converges to the element $x \in E$, if $x_\alpha \xrightarrow{(o)} x$ and $(x_\beta - x_\gamma) \perp x_\gamma$ for all $\beta, \gamma \in \Lambda$, $\beta \geq \gamma$. In this case, we will write that $x_\alpha \xrightarrow{\text{lat}} x$. An orthogonally additive operator $T : E \rightarrow F$ is called laterally continuous, if for every net $(x_\alpha) \subset E$ laterally converging to the element x , the net (Tx_α) order converges to Tx .

Definition 4. A subset D of a vector lattice E is called a lateral ideal, if the following conditions are satisfied:

- (1) if $x \in D$, then $y \in D$ for every $y \in \mathcal{F}_x$;
- (2) every pair of disjoint elements $x, y \in D$ satisfies the condition $x + y \in D$.

We will now provide some examples.

Example 6. Let E be a vector lattice. Every order ideal E is a lateral ideal.

Example 6. Let E be a vector lattice and $x \in E$. Then \mathcal{F}_x is a lateral ideal in E .

Example 7. Let E, F be vector lattices and $T \in \mathcal{OA}_+(E, F)$. Then $\mathfrak{K}_T := \{x \in E : Tx = 0\}$ is a lateral ideal in E .

Definition 5. Let E, F be vector lattices, D be a lateral ideal in E . A mapping $T : D \rightarrow F$ is called:

- orthogonally additive, if $T(x + y) = Tx + Ty$ for all disjoint elements such that $x, y \in D$;
- positive, if $Tx \geq 0$ for every $x \in D$;
- band preserving, if $T\pi = \pi T$ for every order projector $\pi \in \mathfrak{B}(E)$.

Lemma 4 ([19], Theorem 4.4). Suppose that E and F are vector lattices, moreover, the lattice F is order complete, $D \subset E$ is a lateral ideal in $D \subset E$ and $T : D \rightarrow F$ is a positive order bounded orthogonally additive mapping. Then there exists $\tilde{T}_D \in \mathcal{OA}_+(E, F)$, such that $\tilde{T}_D y = Ty$ for every $y \in D$. Moreover, for every $x \in E$, the following equality holds:

$$\tilde{T}_D x = \sup\{Ty : y \in \mathcal{F}_x \cap D\}.$$

The operator $\tilde{T}_D \in \mathcal{OA}_+(E, F)$ is called a minimal expansion of a positive, orthogonally additive mapping $T : D \rightarrow F$.

For further analysis we will need some preliminary facts. Let E be a vector lattice and $x, y \in E$. Then the following statements hold: (see [14], Propositions 3.8 and 3.9.)

- the binary relation \sqsubseteq is a relation of partial order on E ;
- $x \sqsubseteq y$ if and only if $x^+ \sqsubseteq y^+$ and $x^- \sqsubseteq y^-$.

Lemma 5 ([17], Lemma 2). *Let E be a vector lattice, $x \in E$. Then the set \mathcal{F}_x , partially ordered by the relation \sqsubseteq , is a Boolean algebra with the smallest element 0, the largest element x , and Boolean operations:*

$$\begin{aligned} z \cup y &:= (z^+ \vee y^+) - (z^- \vee y^-), \\ z \cap y &:= (z^+ \wedge y^+) - (z^- \wedge y^-), \\ \bar{z} &:= x - z \quad (y, z \in \mathcal{F}_x). \end{aligned}$$

Moreover, if the vector lattice is order complete, then \mathcal{F}_x is a complete Boolean algebra.

A subset D of the vector lattice E is called *laterally closed*, if it contains the limits of all laterally converging nets $(x_\alpha)_{\alpha \in \Lambda} \subset D$. Laterally closed lateral ideal D is called a *lateral strip* in E .

Example 8. Let E be a vector lattice. Every band E is a lateral band.

Example 9. Let E, F be vector lattices, $T \in \mathcal{O}\mathcal{A}_+(E, F)$ be a laterally continuous operator and $z \in F$. Then the sets \mathfrak{K}_T and \mathcal{F}_z are lateral bands in E and F respectively.

Note that the intersection $D_1 \cap D_2$ of lateral strips D_1 and D_2 is also a lateral band.

The following lemma allows to simplify the analysis of the properties of the expanded operator.

Lemma 6 ([17], Lemma 3). *Let E be an order complete vector lattice and D be a lateral band in E . Then for every $x \in E$, the set $D(x) := \mathcal{F}_x \cap D$ contains the maximal element, of relatively partial order, given by the relation \sqsubseteq .*

Later in the text for the maximal element of the set $D(x)$ in the Boolean algebra \mathcal{F}_x we will keep the notation x^D . It is also worth mentioning that under the conditions of Lemma 4, the following equality holds:

$$\tilde{T}_D x = \sup\{Ty : y \in \mathcal{F}_x \cap D\} = Tx^D.$$

The following theorem is the main result of this section.

Theorem 4. *Let E be an order complete vector lattice, D be a lateral band in E , and $T : D \rightarrow E$ be a positive band preserving mapping. Then the minimal expansion $\tilde{T}_D : E \rightarrow E$ is a positive, band preserving orthogonally additive operator.*

Proof. According to Lemma 4, the operator \tilde{T}_D is correctly defined and $\tilde{T}_D \in \mathcal{O}\mathcal{A}_+(E, F)$. We will show that $\tilde{T}_D \in \mathcal{N}$. Repeating the wording of the line of reasoning from Propositions 1 and 2, we conclude that to prove the theorem it suffices to establish that the equality $\tilde{T}_D \pi = \pi \tilde{T}_D$ holds for every $\pi \in \mathfrak{B}(E)$. Let π be an order projector on the band B in E and $x \in E$. It is clear that B is a lateral band in E and $x^B = \pi(x)$. Since $B \cap D$ is a lateral band in E , we have that

$$(\pi(x))^D = x^{D \cap B} = \pi(x^D).$$

Now we can write that

$$\pi\tilde{T}_D(x) = \pi T(x^D) = T\pi(x^D) = T(\pi(x))^D = \tilde{T}_D\pi(x).$$

From the fact that the element $x \in E$ is arbitrary it follows that the operator \tilde{T}_D is band preserving. \square

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REFERENCES

- [1] N. Abasov, *Completely additive and C -compact operators in lattice-normed spaces*, Ann. Funct. Anal., **11**:4 (2020), 914–928. Zbl 07238083
- [2] N. Abasov, *On the sum of narrow orthogonally additive operators*, Russ. Math., **64**:7 (2020), 1–6. Zbl 07309102
- [3] N. Abasov, M. Pliev, *On extensions of some nonlinear maps in vector lattices*, J. Math. Anal. Appl., **455**:1 (2017), 516–527. Zbl 06756249
- [4] N. Abasov, M. Pliev, *Disjointness-preserving orthogonally additive operators in vector lattices*, Banach J. Math. Anal., **12**:3 (2018), 730–750. Zbl 06946079
- [5] N. Abasov, M. Pliev, *Dominated orthogonally additive operators in lattice-normed spaces*, Adv. Oper. Theory, **4**:1 (2019), 251–264. Zbl 06946453
- [6] Y.A. Abramovich, C.D. Aliprantis, *An invitation to operator theory*, AMS, Providence, 2002. Zbl 1022.47001
- [7] C.D. Aliprantis, O. Burkinshaw, *Positive Operators*, Springer, Berlin, 2006. Zbl 1098.47001
- [8] J.A. Appel, P.P. Zabrejko, *Nonlinear superposition operators*, Cambridge University Press, Cambridge, 2008. Zbl 1156.47052
- [9] O. Fotiy, A. Gumenchuk, I. Krasikova, M. Popov, *On sums of narrow and compact operators*, Positivity, **24**:1 (2020), 69–80. Zbl 07172083
- [10] W.A. Feldman, *A factorization for orthogonally additive operators on Banach lattices*, J. Math. Anal. Appl., **472**:1 (2019), 238–245. Zbl 1419.46016
- [11] O. Maslyuchenko, M. Popov, *On sums of strictly narrow operators acting from a Riesz space to a Banach space*, J. Funct. Spaces (2019), Article 8569409. Zbl 07076555
- [12] J.M. Mazón, S. Segura de León, *Order bounded orthogonally additive operators*, Rev. Roum. Math. Pures Appl., **35**:4 (1990), 329–353. Zbl 0723.47047
- [13] V. Mykhaylyuk, M. Pliev, M. Popov, *The lateral order on Riesz spaces and orthogonally additive operators*, Positivity, **25**:2 (2021), 291–327.
- [14] V. Orlov, M. Pliev, D. Rode, *Domination problem for AM-compact abstract Uryson operators*, Arch. Math., **107**:5 (2016), 543–552. Zbl 06668331
- [15] M. Pliev, *Domination problem for narrow orthogonally additive operators*, Positivity, **21**:1 (2017), 23–33. Zbl 1420.47017
- [16] M. Pliev, X. Fang, *Narrow orthogonally additive operators in lattice-normed spaces*, Sib. Math. J., **58**:1 (2017), 134–141. Zbl 1377.47021
- [17] M. Pliev, M. Popov, *On extension of abstract Urysohn operators*, Sib. Math. J., **57**:3 (2016), 552–557. Zbl 06630917
- [18] M.A. Pliev, F. Polat, M.R. Weber, *Narrow and C -compact orthogonally additive operators in lattice-normed spaces*, Result. Math., **74**:4 (2019), Paper No. 157. Zbl 07112092
- [19] M. Pliev, K. Ramdane, *Order unbounded orthogonally additive operators in vector lattices*, Mediterr. J. Math., **15**:2 (2018), Paper No. 55. Zbl 06879895
- [20] M. Pliev, M. Weber, *Finite elements in some vector lattices of nonlinear operators*, Positivity, **22**:1 (2018), 245–260. Zbl 06861662
- [21] M. Pliev, M. Weber, *Disjointness and order projections in the vector lattices of abstract Uryson operators*, Positivity, **20**:3 (2016), 695–707. Zbl 1394.47052
- [22] A. Schep, *Positive diagonal and triangular operators*, J. Oper. Theory, **3**:2 (2016), 165–178. Zbl 0452.47040

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