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## LOCALLY FREE SUBGROUPS OF ONE-RELATOR GROUPS

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**ABSTRACT.** Let  $G_1 = \langle x_1, \dots, x_s; [x_1, x_{n+1}][x_2, x_{n+2}] \dots [x_n, x_{2n}]S \rangle$ ,  $G_2 = \langle a, x_1, \dots, x_s; [a, x_1][a, x_2] \dots [a, x_n]S \rangle$  be an one-relator groups. We find conditions on  $S$  and  $n$  under which the normal closure of each  $(n-1)$ -generated subgroup of  $G_1$  and of each 3-generated subgroup of  $G_2$  is locally free.

**Keywords:** one-relator group, locally free group,  $n$ -free group.

### 1. INTRODUCTION

In this paper we find conditions when the normal closure of each  $n$ -generated subgroup of an one-relator group is a locally free group.

A group is said to be  $n$ -free if each of its  $n$ -generated subgroup is free. An example of such a group is the fundamental group

$$A_n = \langle x_1, x_2, \dots, x_{2n}; [x_1, x_2][x_3, x_4] \dots [x_{2n-1}, x_{2n}] \rangle$$

of genus  $n$  orientable surface. There are quite a lot of articles related to  $n$ -free groups, we will note only a few of them.

In [1] it is found the conditions under which every subgroup of infinite index of a fundamental group of a surface is free. In particular, this implies the well-known fact that all subgroups of infinite index of  $A_n$  are free.

The proof of lemma [2] without changes is suitable for establishing the following result: the group

$$\langle x_1, x_2, \dots, x_{2n}, z_1, \dots, z_m; [x_1, x_2][x_3, x_4] \dots [x_{2n-1}, x_{2n}]S \rangle \quad (n \geq 2),$$

where  $S$  is a word in the alphabet  $z_1, \dots, z_m$ , is  $n$ -free. A detailed proof of this fact is given in [3] (proposition 2) and in [4] (lemma 1.2.3).

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It follows from [5] that a free product with amalgamation  $G = F *_C F'$ , where the factors  $F$  and  $F'$  are free groups and the amalgamated subgroup  $C$  is a maximal cyclic subgroup of each factor, is 2-free. Corollary 4.2[6] says that this group  $G$  is in fact 3-free. In [6] it is defined a class  $C_k$  of groups which contains the class of all  $k$ -free groups, and it is proved if  $G = K *_K F'$ , where  $K$  and  $K'$  belongs to  $C_3$  and the amalgamated subgroup  $C$  is a maximal cyclic subgroup of each factor, then  $G \in C_3$ .

In [7] it is found the conditions on the defining relations of a group under which it is 2-free. In [8] so-called fully residually free groups were studied. It is proved in [8], if such group is 2-free then it is 3-free. Interesting results about  $n$ -free groups can also be found in [9 – 13].

In this paper we generalize the notion of a  $n$ -free group. We consider a group

$$G_1 = \langle x_1, \dots, x_s; [x_1, x_{n+1}][x_2, x_{n+2}] \dots [x_n, x_{2n}]R \rangle,$$

where  $x_{n+1}, x_{n+2}, x_{n+3}, \dots, x_{2n}$  do not occur in  $R$ . We prove that if  $t_1, \dots, t_{n-1} \in G_1$  then  $G_1' \langle t_1, \dots, t_{n-1} \rangle^{G_1}$  is a free group.

Also we investigate a group  $G_2$  which have the representation

$$G_2 = \langle a, x_1, \dots, x_s; P \rangle,$$

where  $P = P(a, x_1, x_2, \dots, x_s) = [a, x_1][a, x_2] \dots [a, x_n]S$  ( $n \geq 7$ ) and  $x_1, \dots, x_7$  do not occur in  $S$ . We show that if  $t_1, t_2, t_3 \in G_2$  then  $G_2' \langle t_1, t_2, t_3 \rangle^{G_2}$  is a locally free group. Such group were also studied in [14].

## 2. PRELIMINARY REMARKS

We recall some definitions and notation. As usual,  $\langle S \rangle$  is a group generated by  $S$ ;  $\langle t_1, \dots, t_n \rangle^G$  is a normal closure of a subgroup  $\langle t_1, \dots, t_n \rangle$  in  $G$  (otherwise, it is a normal subgroup of  $G$  generated by  $t_1, \dots, t_n$ );  $G'$  is the commutant of  $G$ ;  $Z(G)$  is the center of  $G$ ;  $[x, y] = x^{-1}y^{-1}xy$ ;  $\mathbf{Z}$  is a set of integers.

An embedding of a group  $A$  into  $B$  is any homomorphism  $\varphi : A \rightarrow B$  which is an isomorphism of  $A$  onto  $A^\varphi$ . A group  $A$  is said to be embeddable in a group  $B$  if there exists an embedding  $\varphi : A \rightarrow B$ .

A group is locally free if each of its finitely generated subgroups is free.

We say that a set  $\{f_1, \dots, f_r\}$  of numbers has the property (E), if it contains exactly one largest non-zero positive number or exactly one smallest negative non-zero number. If this set does not have the property (E), we write not(E).

Let  $X = \{x_1, x_2, \dots\}$  be an alphabet,  $F(X)$  be a free group freely generated by the set  $X$ , and  $\alpha$  be an homomorphism of  $F(X)$  onto some group  $G$ . If  $\{P, Q, R, \dots\}$  is a set of defining words for  $G$  in the alphabet  $X$  under the map  $\alpha$ , then  $G$  can suitably be represented thus:

$$(1) \quad G = \langle X; P, Q, R, \dots \rangle.$$

In what follows we identify symbols  $x_i$  with their corresponding generating elements  $x_i^\alpha$ . The elements of  $X$  are called generating symbols for  $G$ . We shall suppose that  $X$  contains fixed symbols  $x_1, \dots, x_n$ .

If  $T$  is a group word in the alphabet  $X$  then  $\sigma_{x_i}(T)$  ( $i = 1, \dots, n$ ), or simply  $\sigma_i(T)$ , denotes the sum of exponents over  $x_i$  in  $T$ , and  $[T]$  denotes the set elements from  $X$  occurring in  $T$ .

Let  $t \in G$  and  $T = T(x_1, \dots, x_n)$  be a group word in  $X$  for which  $T^\alpha = t$ ; then, by definition, we put  $\sigma_i(t) = \sigma_i(T)$ . Throughout what follows, the definition of  $\sigma_i(t)$  does not depend on the choice of  $T$ .

Often we will write  $a_{ij}$  instead of  $\sigma_j(t_i)$ .

If  $\tilde{t} = (t_1, \dots, t_k)$  is an  $k$ -tuple of elements  $t_1, \dots, t_k$  of  $G$  it is legitimate to consider the following matrix with integer entries:

$$M(G, \tilde{t}) = M(G, t_1, \dots, t_k) = \begin{pmatrix} \sigma_1(t_1) & \dots & \sigma_n(t_1) \\ \dots & \dots & \dots \\ \sigma_1(t_k) & \dots & \sigma_n(t_k) \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{k1} & \dots & a_{kn} \end{pmatrix}.$$

Let  $L$  be a subgroup of  $G$  generated by elements  $t_1, \dots, t_k$ . We consider the following elementary row transformations of the matrix  $M(G, \tilde{t})$ :

- 1) multiplication of elements of the  $j$ -th row by  $-1$ ,
- 2) replacement of the  $j$ -th row by the difference between the  $j$ -th and  $m$ -th rows,  $m \neq j$ .

Each of these transformations effects a transformation of the set  $t_1, \dots, t_k$  to a new set of generators of the subgroup  $L$ ; this new set may be derived from the old by corresponding transformations:

- 1') replacement of  $t_j$  by  $t_j^{-1}$ ,
- 2') replacement of  $t_j$  by  $t_j t_m^{-1}$  for  $m \neq j$ .

It is easy to see that the matrix  $M(G, \tilde{t})$  with integer entries may be brought by transformations of the above two types to a trapezoidal form. We may thus assume that the generators  $t_1, \dots, t_k$  of  $L$  have been so chosen that  $M(G, \tilde{t})$  is a trapezoidal matrix with only zeros below the principal diagonal (i.e.  $a_{ij} = 0$  for all  $i > j$ ) and positive elements on the principal diagonal (i.e.  $a_{ii} \geq 0$  for all  $i$ ). We will always assume that the matrix  $M(G, \tilde{t})$  has this form.

We will also consider

- 3) permutation of the columns of the matrix  $M(G, \tilde{t})$ .

This transformation corresponds to

- 3') renaming of generators of the group  $G$ .

Note that transformation 3' changes the defining relations of the group  $G$ .

We recall the Reidemeister-Schreier rewriting process. Let  $G$  have representation

$$(2) \quad G = \langle x_1, x_2, \dots; P, Q, R, \dots \rangle,$$

$H \leq G$ , and  $N$  be the preimage of a subgroup  $H$  under the homomorphism  $\alpha$ . Choose the Schreier system  $S$  of representatives of right cosets of the classes  $F(X)$  modulo  $N$ . A representative of the coset  $Nu$  in  $S$  is denoted by  $\bar{u}$ . Consider  $\tau : F(X) \rightarrow F(Y)$  which maps an element  $u = x_{i_1}^{\epsilon_1} \dots x_{i_r}^{\epsilon_r}$  ( $\epsilon_l = \pm 1$ ) of  $F(X)$  to an element  $\tau(u) = y_{s_1, x_{i_1}}^{\epsilon_1} \dots y_{s_r, x_{i_r}}^{\epsilon_r}$  of  $F(Y)$ , where  $s_j = \overline{x_{i_1}^{\epsilon_1} \dots x_{i_{j-1}}^{\epsilon_{j-1}}}$  if  $\epsilon_j = 1$ , and  $s_j = \overline{x_{i_1}^{\epsilon_1} \dots x_{i_j}^{\epsilon_j}}$  if  $\epsilon_j = -1$ . Note that the restriction of  $\tau$  to  $K$  is an homomorphism [15].

A representation of the group  $H$  relative to the mapping  $y_{s,x} \rightarrow (sx\bar{s}x^{-1})^\alpha$  is obtained by appealing to the representation theorem for subgroups [15, theorem 2.4]. Namely,  $Y$  is taken to be the set of generators for  $H$ , and as the set of defining words we take

$$\{\tau(sPs^{-1}), \tau(sQs^{-1}), \dots \mid s \in S\} \cup \{y_{s,x} \mid sx = \bar{s}x \text{ B } F(X)\}.$$

In particular, if  $H = \langle x_2, x_3, \dots \rangle^G$ , and  $G/H$  is an infinite cyclic group, then we take the Schreier system of representatives of the right cosets of  $F(X)$  modulo  $N$  to be the set  $S = \{x_1^k \mid k \in \mathbf{Z}\}$ . Then, by the subgroup representation theorem,

$$H = \langle \{y_{x_1^k, x_1}, y_{x_1^k, x_2}, \dots\}_{k \in \mathbf{Z}}; \{y_{x_1^k, x_1}, \tau(x_1^k P x_1^{-k}), \dots\}_{k \in \mathbf{Z}} \rangle.$$

In order to pass from the above-given representation to a new one, repeated use will be made of so-called Tietze transformations, detailed information about which is contained in [15].

Later we shall also use the following corollary of the Magnus freedom theorem [15, corollary 4.10.2]: if

$$H = \langle x, c, \dots, t; R(x^p, c, \dots, t) \rangle, \quad p \neq 0,$$

then the subgroup  $G$  of  $H$  generated by the elements  $x^p, c, \dots, t$ , has the representation

$$G = \langle b, c, \dots; R(b, c, \dots, t) \rangle,$$

where  $b$  corresponds to the element  $x^p$ ,  $c$  to the element  $c, \dots, t$  to the element  $t$  of  $G$ .

### 3. LOCALLY FREE SUBGROUPS

Let's fix the matrices:

$$A = \begin{pmatrix} 1 & a_{12} & p_2 a_{13} & p_2 p_3 a_{14} & \dots & p_2 p_3 p_4 \dots p_{k-2} a_{1, k-1} \\ 0 & 1 & a_{23} & p_3 a_{24} & \dots & p_3 p_4 \dots p_{k-2} a_{2, k-1} \\ 0 & 0 & 1 & a_{34} & \dots & p_4 \dots p_{k-2} a_{3, k-1} \\ & & & \dots & & \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} p_2 p_3 p_4 \dots p_{k-2} p_{k-1} a_{1, k} \\ p_3 p_4 \dots p_{k-2} p_{k-1} a_{2, k} \\ p_4 \dots p_{k-2} p_{k-1} a_{3, k} \\ \dots \\ a_{k-1, k} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_{k-1} \end{pmatrix},$$

where  $p_2 \neq 0, \dots, p_{k-1} \neq 0$ .

**Lemma 1.** *It is stated that the column  $(q_1, \dots, q_{k-1})'$  is a solution of the system of equations  $AX = B$ , where*

$$q_1 = (-1)^{k-2} \begin{vmatrix} a_{12} & a_{13} & a_{14} & \dots & a_{1, k-1} & a_{1, k} \\ p_2 & a_{23} & a_{24} & \dots & a_{2, k-1} & a_{2, k} \\ 0 & p_3 & a_{34} & \dots & a_{3, k-1} & a_{3, k} \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & p_{k-1} & a_{k-1, k} \end{vmatrix},$$

$$q_2 = (-1)^{k-3} \begin{vmatrix} a_{23} & a_{24} & a_{25} & \dots & a_{2, k-1} & a_{2, k} \\ p_3 & a_{34} & a_{35} & \dots & a_{3, k-1} & a_{3, k} \\ 0 & p_4 & a_{45} & \dots & a_{4, k-1} & a_{4, k} \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & p_{k-1} & a_{k-1, k} \end{vmatrix},$$

$$\dots$$

$$q_{k-2} = - \begin{vmatrix} a_{k-2, k-1} & a_{k-2, k} \\ p_{k-1} & a_{k-1, k} \end{vmatrix},$$

$$q_{k-1} = a_{k-1, k}.$$

PROOF. First we find  $x_i$  by Cramer' rule. Then we put the  $i$ -th column in the place of the last one. Using the formula for the decomposition of the determinant by column it is easy to see that

$$q_i = (-1)^{k-i-1} \times \begin{vmatrix} a_{i,i+1} & p_{i+1}a_{i,i+2} & p_{i+1}p_{i+2}a_{i,i+3} & \cdots & p_{i+1}\cdots p_{k-2}a_{i,k-1} & p_{i+1}\cdots p_{k-1}a_{i,k} \\ 1 & a_{i+1,i+2} & p_{i+2}a_{i+1,i+3} & \cdots & p_{i+2}\cdots p_{k-2}a_{i+1,k-1} & p_{i+2}\cdots p_{k-1}a_{i+1,k} \\ 0 & 1 & a_{i+2,i+3} & \cdots & p_{i+3}\cdots p_{k-2}a_{i+2,k-1} & p_{i+3}\cdots p_{k-1}a_{i+2,k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & a_{k-1,k} \end{vmatrix}$$

Now multiply the 2nd row by  $p_{i+1}$ , the 3rd row by  $p_{i+1}p_{i+2}$ ,  $\dots$ , the last row by  $p_{i+1}p_{i+2}\cdots p_{k-1}$ . Then we divide the 2nd column by  $p_{i+1}$ , the 3rd column by  $p_{i+1}p_{i+2}$ ,  $\dots$ , column by  $p_{i+1}p_{i+2}\cdots p_{k-1}$ . We obtain that

$$q_i = (-1)^{k-i-1} \begin{vmatrix} a_{i,i+1} & a_{i,i+2} & a_{i,i+3} & \cdots & a_{i,k-1} & a_{i,k} \\ p_{i+1} & a_{i+1,i+2} & a_{i+1,i+3} & \cdots & a_{i+1,k-1} & a_{i+1,k} \\ 0 & p_{i+2} & a_{i+2,i+3} & \cdots & a_{i+2,k-1} & a_{i+2,k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & p_{k-1} & a_{k-1,k} \end{vmatrix}.$$

□

Now we present the method for the embedding a group  $G$  in a suitable group  $C$  with the specially selected representation in which the matrix  $M(C, t_1^\psi, \dots, t_k^\psi)$  has zero  $k$ -th column.

#### Construction of the embedding.

Let

$$G = \langle a, x_1, \dots, x_s; R(a, x_1, \dots, x_s) \rangle,$$

be a group and  $t_1, \dots, t_k$  be elements of  $G$ . Suppose that the matrix  $M(G, \tilde{t})$  has the form:

$$(3) \quad M(G, \tilde{t}) = \begin{pmatrix} p_1 & a_{12} & a_{13} & \cdots & a_{1,k-1} & a_{1k} & \cdots \\ 0 & p_2 & a_{23} & \cdots & a_{2,k-1} & a_{2k} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & p_{k-1} & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \end{pmatrix},$$

where  $a_{ij} = \sigma_j(t_i)$  (note that  $a_{k,k+1}$  is any element or is missing). Assume that the group  $C$  has the representation

$$(4) \quad C = \langle a, x_1, \dots, x_s; R_1 \rangle,$$

where  $R_1 = R(a, x_1, \dots, x_{k-1}, x_k^m, x_{k+1}, \dots)$  and  $m = p_1 \dots p_{k-1}$ . By corollary of the Magnus freedom theorem formulated above, there exists the embedding  $\psi$  of  $G$  into  $C$  such that  $a^\psi = a$ ,  $x_i^\psi = x_i$  ( $i = 1, 2, \dots, k-1, k+1, \dots$ ),  $x_k^\psi = x_k^m$ .

Let  $q_1, \dots, q_{k-1}$  be the same as in lemma 1. Clearly, in generators

$$a, y_1 = x_1 x_k^{q_1}, y_2 = x_2 x_k^{q_2 p_1}, \dots, y_{k-1} = x_{k-1} x_k^{q_{k-1} p_1 p_2 \cdots p_{k-2}}, x_k, x_{k+1}, \dots, x_s$$

$C$  has the following representation:  $C = \langle a, y_1, \dots, y_{k-1}, x_k, \dots, x_s; R_2 \rangle$ , where

$$R_2(a, y_1, \dots, y_{k-1}, x_k, \dots, x_s) = R(a, y_1 x_k^{-q_1}, y_2 x_k^{-q_2 p_1}, y_3 x_k^{-q_3 p_1 p_2}, \dots, y_{k-1} x_k^{-q_{k-1} p_1 p_2 \cdots p_{k-2}}, x_k^m, x_{k+1}, \dots, x_s).$$

Rename generators  $y_1, \dots, y_{k-1}$  by  $x_1, \dots, x_{k-1}$  respectively, we see that  $C$  in new generators has the representation

$$(5) \quad C = \langle a, x_1, \dots, x_{k-1}, x_k, \dots, x_s; R_2(a, x_1, \dots, x_s) \rangle.$$

Express elements  $t_1^\psi, \dots, t_k^\psi$  through these new generators. Suppose that  $t_i$  is a value of  $T_i(a, x_1, \dots, x_s)$  on generators  $a, x_1, \dots, x_s$  of  $G$ . We see

$$t_i^\psi = T_i(a, x_1 x_k^{-q_1}, x_2 x_k^{-q_2 p_1}, x_3 x_k^{-q_3 p_1 p_2}, x_{k-1} x_k^{-q_{k-1} p_1 p_2 \dots p_{k-2}}, x_k^m, x_{k+1}, \dots, x_s).$$

Therefore

$$\sigma_k(t_i^\psi) = -a_{i1}q_1 - a_{i2}p_1q_2 - a_{i3}p_1p_2q_3 - \dots - a_{i,k-1}p_1p_2 \dots p_{k-2}q_{k-1} + a_{i,k}m,$$

where  $i = 1, \dots, k-1$ . As  $M(G, \tilde{t})$  is a trapezoidal matrix we note that  $a_{ij} = 0$  for all  $j, j < i$ . Hence

$$\begin{aligned} \sigma_k(t_i^\psi) &= -p_1 \dots a_{ii}q_i - a_{i,i+1}p_1 \dots p_i q_{i+1} - \dots - a_{i,k-1}p_1 \dots p_{k-2}q_{k-1} + a_{i,k}m = \\ &= -p_1 \dots p_i (q_i + a_{i,i+1}q_{i+1} + a_{i,i+2}p_{i+1}q_{i+2} + \dots + a_{i,k-1}p_{i+1} \dots p_{k-1} - a_{i,k}p_{i+1} \dots p_{k-1}). \end{aligned}$$

The expression in parentheses coincides with the  $i$ -th row of the matrix  $AX - B$  (if  $x_j$  replace by  $q_j$ ) therefore it is equal to 0. Thus

$$\sigma_k(t_i^\psi) = 0 \text{ for } i = 1, \dots, k-1.$$

Since  $a_{k1} = a_{k2} = \dots = a_{kk} = 0$  and  $\sigma_k(t_k^\psi) = a_{k1}q_1 - a_{k2}p_1q_2 - a_{k3}p_1p_2q_3 - \dots - a_{k,k-1}p_1p_2 \dots p_{k-2}q_{k-1} + a_{k,k}m$ , we obtain that  $\sigma_k(t_k^\psi) = 0$ .

So we found the embedding  $\psi : G \rightarrow C$  of  $G$  in a group  $C$  with the representation (5), such that the matrix consisting of the first  $k$  columns of the matrix  $M(C, t_1^\psi, \dots, t_k^\psi)$  is triangular with a zero angle under the principal diagonal and with a zero  $k$ -th column.

A similar construction also appeared in [16].  $\square$

**Lemma 2.** *Let a group  $G$  have the representation*

$$G = \langle a, x_1, \dots, x_s; R(a, x_1, \dots, x_s) \rangle$$

and  $t_1, \dots, t_k \in G$ . Suppose that the matrix  $M(G, \tilde{t})$  has the form (3). Let a group  $C$  have the representation (5), where  $m = p_1 p_2 \dots p_{k-1}$ ,  $(q_1, \dots, q_{k-1})'$  is a solution of a system of equations  $AX = B$  from lemma 1. Then if  $\langle a, x_1, \dots, x_{k-1}, x_{k+1}, x_{k+2}, \dots, x_s \rangle^C$  is a locally free (or free) group then  $G' \langle t_1, \dots, t_k \rangle^G$  is also a locally free group (respectively, free group).

PROOF. According to the above construction of the embedding, the matrix  $M(C, t_1^\psi, \dots, t_k^\psi)$  has zero  $k$ -th column. This means that  $C' \langle t_1^\psi, \dots, t_k^\psi \rangle^C$  is contained in a subgroup  $\langle a, x_1, \dots, x_{k-1}, x_{k+1}, x_{k+2}, \dots, x_s \rangle^C$ , which by condition is locally free. Hence  $C' \langle t_1^\psi, \dots, t_k^\psi \rangle^C$  is a locally free group. As  $\psi : G \rightarrow C$  is an embedding and  $(G')^\psi \langle t_1, \dots, t_k \rangle^G \subseteq C' \langle t_1^\psi, \dots, t_k^\psi \rangle^C$ , then  $G' \langle t_1, \dots, t_k \rangle^G$  is a locally free group.  $\square$

**Lemma 3.** *Let a group  $G$  have the representation*

$$G = \langle a, b, c, d, \dots, u, h, v, \dots, w, \dots, z; PR \rangle,$$

where  $P = [a, bh^{-f_1}][a, ch^{-f_2}][a, dh^{-f_3}] \dots [a, uh^{-f_{k-1}}][a, h^m][a, v] \dots [a, w]$ . Suppose that the set  $\{f_1, f_2, \dots, f_{k-1}, -m\}$  of numbers satisfies the property (E). If

(i)  $h \notin [R]$  or

(ii)  $a \notin [R]$  and  $\sigma_h(R) = 0$ ,  
then the normal closure  $N = \langle a, b, c, d, \dots, u, v, \dots, w, \dots, z \rangle^G$  generated by all generators except  $h$  is a locally free group.

PROOF. We proceed as B. Magnus [15] did when he proved the freedom theorem for one-relator groups.

Since  $G/N$  is an infinite cyclic group then  $N$  is generated by all elements

$$(6) \quad a_s = h^s a h^{-s}, b_s = h^s b h^{-s}, \dots, w_s = h^s w h^{-s}, \dots, z_s = h^s z h^{-s}.$$

By section 2 the group  $N$  in generators (6) is defined by the set of  $P_k$  ( $k \in \mathbf{Z}$ ) defining relations, where  $P_k = \tau(h^k P R h^{-k})$ . We have

$$\begin{aligned} P_0 &= P(a_0, \dots, u_{f_{k-1}}, \dots, z_0) R_0 = \\ &= a_0^{-1} b_{f_1}^{-1} a_{f_1} b_{f_1} \dots a_0^{-1} u_{f_{k-1}}^{-1} a_{f_{k-1}} u_{f_{k-1}} a_0^{-1} a_{-m} [a_0, v_0] \dots [a_0, w_0] R_0, \end{aligned}$$

where the word  $R_0$  is obtained by rewriting the word  $R$  through generators (6). We note that from of  $a_l$ -symbols, only  $a_0$  can be occurred in  $R_0$ .

Let the word  $P_i$  be defined as ( $R_i$  obtained from  $R_0$  by adding to each index of the variable included in  $R_0$  the number  $i$ )

$$\begin{aligned} P_i &= P(a_i \dots u_{f_{k-1}+i}, \dots, z_i) R_i = \\ &= a_i^{-1} b_{f_1+i}^{-1} a_{f_1+i} b_{f_1+i} \dots a_0^{-1} u_{f_{k-1}+i}^{-1} a_{f_{k-1}+i} u_{f_{k-1}+i} a_i^{-1} a_{-m+i} [a_i, v_i] \dots [a_i, w_i] R_i, \end{aligned}$$

i.e. the number  $i$  added to each index of the variable included in  $P_0$ .

Let  $\mu$  and  $M$  respectively, be smallest and largest indexes of  $a_l$ -symbols included in  $P_0$ .

For every  $i = 0, 1, 2, \dots$ , we represent  $N_{-i,i}$  by generators  $S_i$  and defining relations  $\Sigma_i$  as follows. As a generating set  $S_i$ , take the set of generators from (6), except for those  $a_k$  for which  $k < \mu - i$  or  $k > M + i$ , and take the set  $\{P_{-i}, \dots, P_i\}$  of defining words to be  $\Sigma_i$ . Thus

$$N_{-i,i} = \langle S_i; P_{-i}, \dots, P_i \rangle.$$

In [15, theorem 4.10, case 3] these groups  $N_{-i,i}$  are introduced for any one-relator groups in which the sum of the exponents in our notation for  $h$  is 0. In [15] it is proved that we can assume that  $N_{-i,i}$  is a subgroup of  $N_{-i-1,i+1}$  generated by elements from (6) included in the list of generators of  $N_{-i,i}$ . In addition, in [15] it is noted that the union an ascending sequence of groups  $N_{-i,i}$  coincides with  $N$ . We need to check that  $N_{-i,i}$  is a free group. To do this, use the Tietze transformations.

First, we assume that the set  $\{f_1, f_2, \dots, f_{k-1}, -m\}$  contains exactly one largest non-zero positive number. Deleting, via the Tietze transformations,  $a_{M+i}$  (i.e.  $a_s$  with largest index) from the set of generators of  $N_{-i,i}$  and the defining relation  $P_i$ . Since  $a_{M+i}$  is not occurred in  $P_{-i}, \dots, P_{i-1}$ , these defining relations will not change. We continue this process of sequentially removing generators. Delete  $a_{M+i-1}, a_{M+i-2}, \dots$  from the set of generators and, respectively,  $P_{i-1}, P_{i-2}, \dots$  from the set of defining relations. As a result, we get that  $N_{-i,i}$  can be defined by an empty set of defining relations. It means that  $N_{-i,i}$  is a free group.

In the case when the set  $\{f_1, f_2, \dots, f_{k-1}, -m\}$  contains exactly one smallest non-zero negative number, we will do the same, starting with the deletion of  $a_{\mu-i}$ .

Since the union of an increasing sequence of free groups is a locally free group, we see that  $N$  is a locally free group. □

If  $P = [a, h^m]$  ( $m \neq 0$ ) we have a special case of lemma 3. In fact, as shown in [2, Lemma 1], the following more general statement is true.

**Lemma 4.** *Let a group  $G$  have the representation*

$$G = \langle a, b, c, d, \dots, u, h, v, \dots, w, \dots, z; [a, h^m]R \rangle \quad (m \neq 0).$$

*If  $a \notin [R]$  and  $\sigma_h(R) = 0$ , then the normal closure  $N = \langle a, b, c, d, \dots, u, v, \dots, w, \dots, z \rangle^G$  generated by all generators except  $h$  is a free group.*

**Theorem 1.** *Let a group  $G$  have the representation*

$$G = \langle x_1, \dots, x_s; [x_1, x_{n+1}][x_2, x_{n+2}] \dots [x_n, x_{2n}]R \rangle$$

*where  $x_{n+1}, x_{n+2}, x_{n+3}, \dots, x_{2n} \notin [R]$ . If  $t_1, \dots, t_{n-1} \in G$  then  $G' \langle t_1, \dots, t_{n-1} \rangle^G$  is a free group.*

PROOF. If there exists  $i$  ( $i \leq n$ ) such that  $i$ -th column of  $M(G, \tilde{t})$  is zero, we apply lemma 4 ( $h = x_i$ ,  $a = x_{i+n} \notin [R]$ ,  $m = 1$ ). By lemma 4  $\langle x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_s \rangle^G$  is a free group. But  $G' \langle t_1, \dots, t_{n-1} \rangle^G \subseteq \langle x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_s \rangle^G$  therefore  $G' \langle t_1, \dots, t_{n-1} \rangle^G$  is a free group. We will assume that the first  $n$  columns of the matrix  $M(G, \tilde{t})$  are non-zero.

Let's apply the transformations 1), 2) defined before. By renaming (if necessary) generators  $x_1, \dots, x_n$  of  $G$ , we can assume that the matrix  $M(G, \tilde{t})$  has the form

$$(7) \quad M(G, \tilde{t}) = \begin{pmatrix} p_1 & a_{12} & a_{13} & \dots & a_{1,k-1} & a_{1k} & \dots \\ 0 & p_2 & a_{23} & \dots & a_{2,k-1} & a_{2k} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & p_{k-1} & a_{k-1,k} & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

for some  $k$  ( $k \leq n$ ), where  $p_1 > 0, \dots, p_{k-1} > 0$ . In particular, at the intersection of the row with the number  $> k$  and the column with the number  $< n + 1$ , there is a zero.

By the construction of the embedding described earlier, there is an embedding  $\psi : G \rightarrow C$  of  $G$  into  $C$  in which the  $k$ -th ( $k \leq n$ ) column of the matrix  $M(G, \tilde{t}^\psi)$  is zero. We can assume that the defining relation of the group  $C$  has the form  $[x_j, x_k^{p_1 \dots p_{k-1}}]T$  ( $j > k$ ,  $x_j \notin [T]$ ). Let's apply 4 ( $h = x_k$ ,  $a = x_j \notin [T]$ ,  $\sigma_k(T) = 0$ ,  $m = p_1 \dots p_{k-1}$ ). By lemma 4,  $N = \langle x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_s \rangle^C$  is a free group. Since  $(G' \langle t_1, \dots, t_{n-1} \rangle^G)^\psi \subseteq N$  then by lemma 2,  $G' \langle t_1, \dots, t_{n-1} \rangle^G$  is a free group.  $\square$

#### 4. NORMAL CLOSURES OF 3-GENERATED SUBGROUPS

In this section, we will consider a group  $G = \langle a, x_1, \dots, x_s; P \rangle$ , where

$$P = P(a, x_1, x_2, \dots, x_s) = [a, x_1][a, x_2] \dots [a, x_n]S \quad (n \geq 7)$$

and  $x_1, \dots, x_7 \notin [S]$ . We will prove that a normal closure of every 3-generated subgroup of this group is a locally free group.

Let  $t_1, t_2, t_3$  be any elements of  $G$ ,  $R = \langle t_1, t_2, t_3 \rangle^G$  be a normal closure of the subgroup  $\langle t_1, t_2, t_3 \rangle$  in  $G$ .

First, we will assume (lemmas 5, 6) that the matrix  $M(G, \tilde{t})$  does not contain zero columns.

**Lemma 5.** *Let's suppose that there are three columns in  $M(G, \tilde{t})$  with numbers  $i, j, u$  such that*

$$\sigma_i(t_2) = \sigma_j(t_2) = \sigma_u(t_2) = 0, \quad \sigma_i(t_3) = \sigma_j(t_3) = \sigma_u(t_3) = 0.$$

*Then  $RG'$  is a locally free group.*

PROOF. Renaming generators  $\{x_1, \dots, x_n\}$  of  $G$ , we can assume that  $i = 1, j = 2, u = 3$ , i.e. the matrix  $M(G, \tilde{t})$  has the form  $\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \dots \\ 0 & 0 & 0 & a_{24} & \dots \\ 0 & 0 & 0 & a_{34} & \dots \end{pmatrix}$ , where  $a_{11} \neq 0, a_{12} \neq 0, a_{13} \neq 0$ . We can suppose that  $a_{11} \neq -a_{12}$ .

Note that when we rename generators, the defining relation is transformed into the relation denoted by  $P_1(a, x_1, x_2, \dots, x_s)$ .

We apply the construction of the embedding described earlier. According to it, there is the embedding ( $k = 2, q_1 = a_{12}, m = a_{11}$ )  $\psi : G \rightarrow C$ , where  $C$  has the representation

$$\langle a, x_1, x_2, \dots, x_{2n}; P_1(a, x_1 x_2^{-a_{12}}, x_2^{a_{11}}, x_3, \dots, x_s) \rangle.$$

Since the 2nd column of the matrix  $M(C, t_1^\psi, t_2^\psi, t_3^\psi)$  is zero, then  $\langle t_1^\psi, t_2^\psi, t_3^\psi \rangle$  is contained in the normal closure  $N = \langle a, x_1, x_3, x_4, \dots, x_s \rangle^C$ . By lemma 3 (here  $q_1 = a_{12}, m = a_{11}, a_{12} \neq -a_{11}$ ) the subgroup  $N$  is locally free. As  $(RG')^\psi \subseteq NC' = N$  then  $(RG')^\psi$ , and hence  $RG'$ , are locally free groups.  $\square$

**Lemma 6.** *Let the matrix  $M(G, \tilde{t})$  have the form*

$$M(G, \tilde{t}) = \begin{pmatrix} p_1 & a_{12} & a_{13} & a_{14} & \dots \\ 0 & p_2 & a_{23} & a_{24} & \dots \\ 0 & 0 & 0 & a_{34} & \dots \end{pmatrix},$$

*where  $p_1 > 0, p_2 > 0, a_{23} > 0$ . Then  $RG'$  is a locally free group.*

PROOF. We apply the construction of embedding described earlier. According to it, there is the embedding ( $k = 3, m = p_1 p_2$ )  $\psi : G \rightarrow C$ , where  $C$  has the representation

$$C = \langle a, x_1, x_2, \dots, x_s; P(a, x_1 x_2^{-q_1}, x_2^{-p_1 q_2}, x_3^{p_1 p_2}, \dots, x_s) \rangle,$$

$(q_1, q_2)$  is a solution of a system of equations

$$x_1 p_1 + x_2 p_1 a_{12} = p_1 p_2 a_{13}, \quad x_2 p_1 p_2 = p_1 p_2 a_{23},$$

$$\text{i.e. } q_1 = - \begin{vmatrix} a_{12} & a_{13} \\ p_2 & a_{23} \end{vmatrix}, \quad q_2 = a_{23}.$$

Since the 3rd column of the matrix  $M(C, t_1^\psi, t_2^\psi, t_3^\psi)$  is zero then  $\langle t_1^\psi, t_2^\psi, t_3^\psi \rangle$  is contained in the normal closure  $N = \langle a, x_1, x_2, x_4, \dots, x_s \rangle^C$ . Since numbers  $q_2 p_1, -p_1 p_2$  have different signs then numbers  $q_1, q_2 p_1, -p_1 p_2$  contain exactly one the non-zero largest or smallest number. By lemma 3 the subgroup  $N$  is locally free. Since  $(RG')^\psi \subseteq NC' = N$  then  $(RG')^\psi$ , and hence  $RG'$ , are locally free groups.  $\square$

Let the matrix  $M(G, \tilde{t})$  have the form

$$M = M(G, \tilde{t}) = \begin{pmatrix} p_1 & a_{12} & a_{13} & a_{14} & a_{15} & \dots \\ 0 & p_2 & a_{23} & a_{24} & a_{25} & \dots \\ 0 & 0 & p_3 & a_{34} & a_{35} & \dots \end{pmatrix},$$

where  $p_1 > 0, p_2 > 0, p_3 > 0, a_{34} > 0, a_{35} > 0$ .

We apply the construction of embedding described earlier. According to it ( $k = 4, m = p_1 p_2 p_3$ ), there is the embedding  $\psi : G \rightarrow C_0$  of  $G$  into the group  $C_0$  from construction, where  $C_0$  has the representation

$$C_0 = \langle a, x_1, x_2, \dots, x_n; P(a, x_1 x_2^{-q_1}, x_2^{-p_1 q_2}, x_3^{-p_1 p_2 q_3}, x_4^{p_1 p_2 p_3}, x_5, \dots, x_s) \rangle.$$

Here  $(q_1, q_2, q_3)$  is a solution of a system of equations

$$x_1 p_1 + x_2 p_1 a_{12} + x_3 p_1 p_2 a_{13} = p_1 p_2 p_3 a_{14},$$

$$x_2 p_1 p_2 + x_3 p_1 p_2 a_{23} = p_1 p_2 p_3 a_{24},$$

$$x_3 p_1 p_2 p_3 = p_1 p_2 p_3 a_{34},$$

in particular,  $q_3 = a_{34} > 0$ .

Rename generators of  $G$  twice. First, we denote  $x_3$  by  $x_4$ ,  $x_4$  by  $x_5$ ,  $x_5$  by  $x_3$ , then we denote  $x_4$  by  $x_5$ ,  $x_5$  by  $x_4$ . This corresponds to a permutation of columns the matrix  $M(G, \tilde{t})$ . The following matrix are obtained

$$\overline{M} = \begin{pmatrix} p_1 & a_{12} & a_{14} & a_{15} & \dots \\ 0 & p_2 & a_{24} & a_{25} & \dots \\ 0 & 0 & a_{34} & a_{35} & \dots \end{pmatrix}, \quad \widetilde{M} = \begin{pmatrix} p_1 & a_{12} & a_{13} & a_{15} & a_{14} & \dots \\ 0 & p_2 & a_{23} & a_{25} & a_{24} & \dots \\ 0 & 0 & p_3 & a_{35} & a_{34} & \dots \end{pmatrix}.$$

Again, in each of these cases, we use our construction. We have groups  $C_1, C_2$  of the form

$$C_1 = \langle a, x_1, x_2, \dots, x_n; P_1(a, x_1 x_2^{-\overline{q}_1}, x_2^{-p_1 \overline{q}_2}, x_3^{-p_1 p_2 \overline{q}_3}, x_4^{p_1 p_2 a_{34}}, x_5, \dots, x_s) \rangle,$$

$$C_2 = \langle a, x_1, x_2, \dots, x_n; P_2(a, x_1 x_2^{-\widetilde{q}_1}, x_2^{-p_1 \widetilde{q}_2}, x_3^{-p_1 p_2 \widetilde{q}_3}, x_4^{p_1 p_2 p_3}, x_5, \dots, x_s) \rangle,$$

and embeddings  $\overline{\psi} : G \rightarrow C_1, \widetilde{\psi} : G \rightarrow C_2$  ( $P_1, P_2$  is obtained from  $P$  by renaming suitable generators).

For each matrix a corresponding system of equations arises. Its solutions for the matrix  $\overline{M}$  are denoted by  $\overline{q}_1, \overline{q}_2, \overline{q}_3$ , for the matrix  $\widetilde{M}$  by  $\widetilde{q}_1, \widetilde{q}_2, \widetilde{q}_3$ . By lemma 1

$$(8) \quad q_1 = \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ p_2 & a_{23} & a_{24} \\ 0 & p_3 & a_{34} \end{vmatrix}, \quad q_2 = - \begin{vmatrix} a_{23} & a_{24} \\ p_3 & a_{34} \end{vmatrix}, \quad q_3 = a_{34},$$

$$(9) \quad \overline{q}_1 = \begin{vmatrix} a_{12} & a_{14} & a_{15} \\ p_2 & a_{24} & a_{25} \\ 0 & a_{34} & a_{35} \end{vmatrix}, \quad \overline{q}_2 = - \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix}, \quad \overline{q}_3 = a_{35},$$

$$(10) \quad \widetilde{q}_1 = \begin{vmatrix} a_{12} & a_{13} & a_{15} \\ p_2 & a_{23} & a_{25} \\ 0 & p_3 & a_{35} \end{vmatrix}, \quad \widetilde{q}_2 = - \begin{vmatrix} a_{23} & a_{25} \\ p_3 & a_{35} \end{vmatrix}, \quad \widetilde{q}_3 = a_{35}.$$

We will assume that each of sets

$$(11) \quad q_1, q_2 p_1, q_3 p_1 p_2, -p_1 p_2 p_3,$$

$$(12) \quad \overline{q}_1, \overline{q}_2 p_1, \overline{q}_3 p_1 p_2, -p_1 p_2 a_{34},$$

$$(13) \quad \widetilde{q}_1, \widetilde{q}_2 p_1, \widetilde{q}_3 p_1 p_2, -p_1 p_2 p_3,$$

does not have the property (E) (i.e. it has the property not(E)).

**Lemma 7.**

$$(14) \quad \tilde{q}_1 - \frac{p_3}{a_{34}}\bar{q}_1 = \frac{a_{35}}{a_{34}}q_1,$$

$$(15) \quad \tilde{q}_2 - \frac{p_3}{a_{34}}\bar{q}_2 = \frac{a_{35}}{a_{34}}q_2.$$

PROOF. We calculate

$$\begin{aligned} \tilde{q}_1 - \frac{p_3}{a_{34}}\bar{q}_1 &= \begin{vmatrix} a_{12} & a_{13} & a_{15} \\ p_2 & a_{23} & a_{25} \\ 0 & p_3 & a_{35} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{14} \frac{p_3}{a_{34}} & a_{15} \\ p_2 & a_{24} \frac{p_3}{a_{34}} & a_{25} \\ 0 & a_{34} \frac{p_3}{a_{34}} & a_{35} \end{vmatrix} = \\ &= \begin{vmatrix} a_{12} & a_{13} - a_{14} \frac{p_3}{a_{34}} & a_{15} \\ p_2 & a_{23} - a_{24} \frac{p_3}{a_{34}} & a_{25} \\ 0 & 0 & a_{35} \end{vmatrix} = a_{35} \begin{vmatrix} a_{12} & a_{13} - a_{14} \frac{p_3}{a_{34}} \\ p_2 & a_{23} - a_{24} \frac{p_3}{a_{34}} \end{vmatrix} = \\ &= a_{35} \left( \begin{vmatrix} a_{12} & a_{13} \\ p_2 & a_{23} \end{vmatrix} - \frac{p_3}{a_{34}} \begin{vmatrix} a_{12} & a_{14} \\ p_2 & a_{24} \end{vmatrix} \right). \end{aligned}$$

Since

$$q_1 = a_{34} \begin{vmatrix} a_{12} & a_{13} \\ p_2 & a_{23} \end{vmatrix} - p_3 \begin{vmatrix} a_{12} & a_{14} \\ p_2 & a_{24} \end{vmatrix},$$

then the equality (14) is true. Also we have

$$\begin{aligned} \tilde{q}_2 - \frac{p_3}{a_{34}}\bar{q}_2 &= - \begin{vmatrix} a_{23} & a_{25} \\ p_3 & a_{35} \end{vmatrix} + \begin{vmatrix} a_{24} \frac{p_3}{a_{34}} & a_{25} \\ a_{34} \frac{p_3}{a_{34}} & a_{35} \end{vmatrix} = \\ &= \begin{vmatrix} -a_{23} + a_{24} \frac{p_3}{a_{34}} & a_{25} \\ 0 & a_{35} \end{vmatrix} = \frac{a_{35}}{a_{34}}(-a_{23}a_{34} + a_{24}p_3) = \frac{a_{35}}{a_{34}}q_2. \end{aligned}$$

□

**Lemma 8.** *From the equalities*

$$q_1 = -p_1p_2p_3, \quad \bar{q}_1 = -p_1p_2a_{34}, \quad \tilde{q}_1 = -p_1p_2p_3$$

*no more than one is true.*

PROOF. Suppose that  $\bar{q}_1 = -p_1p_2a_{34}$ ,  $\tilde{q}_1 = -p_1p_2p_3$ . By (14)

$$-p_1p_2p_3 + \frac{p_3}{a_{34}}p_1p_2a_{34} = \frac{a_{35}}{a_{34}}q_1.$$

This means that  $q_1 = 0$ . Now we see that the set  $\{0, q_2p_1, a_{34}p_1p_2, -p_1p_2p_3\}$  has the property (E). We got a contradiction.

Assume that  $q_1 = -p_1p_2p_3$ ,  $\bar{q}_1 = -p_1p_2a_{34}$ . By (14)

$$\tilde{q}_1 + \frac{p_3}{a_{34}}p_1p_2a_{34} = -\frac{a_{35}}{a_{34}}p_1p_2p_3.$$

Hence  $\tilde{q}_1 = -p_1p_2p_3(1 + \frac{a_{35}}{a_{34}}) \neq -p_1p_2p_3$ . We see that

$$\tilde{q}_1 < 0, \quad -p_1p_2p_3 < 0, \quad \tilde{q}_1 \neq -p_1p_2p_3, \quad \tilde{q}_3p_1p_2 = a_{35}p_1p_2 > 0,$$

then the set  $\{\tilde{q}_1, \tilde{q}_2p_1, \tilde{q}_3p_1p_2, -p_1p_2p_3\}$  has the property (E), what's wrong.

Let  $q_1 = -p_1p_2p_3$ ,  $\tilde{q}_1 = -p_1p_2p_3$ . Since  $q_3p_1p_2 > 0$ ,  $\tilde{q}_3p_1p_2 > 0$ , not(E) for sets  $\{q_1, q_2p_1, q_3p_1p_2, -p_1p_2p_3\}$  and  $\{\tilde{q}_1, \tilde{q}_2p_1, \tilde{q}_3p_1p_2, -p_1p_2p_3\}$  bring us to equalities  $q_2p_1 = q_3p_1p_2$ ,  $\tilde{q}_2p_1 = \tilde{q}_3p_1p_2$ . It means  $q_2 = a_{34}p_2$  and  $\tilde{q}_2 = a_{35}p_2$ . By (15)

$$a_{35}p_2 - \frac{p_3}{a_{34}}\tilde{q}_2 = \frac{a_{35}}{a_{34}}a_{34}p_2.$$

Hence  $\bar{q}_2 = 0$ . We see that the set  $\{\bar{q}_1, 0, a_{35}p_1p_2, -p_1p_2p_3\}$  has (E). We got a contradiction.  $\square$

**Lemma 9.**  $\tilde{q}_1 \neq -p_1p_2p_3$ .

PROOF. Suppose  $\tilde{q}_1 = -p_1p_2p_3$ . As  $\tilde{q}_3p_1p_2 > 0$ , from not(E) for the set  $\{\tilde{q}_1, \tilde{q}_2p_1, \tilde{q}_3p_1p_2, -p_1p_2p_3\}$  it follows that  $\tilde{q}_2p_1 = \tilde{q}_3p_1p_2$ . Hence  $\tilde{q}_2 = a_{35}p_2$ .

By lemma 8,  $q_1 \neq -p_1p_2p_3$ . Since  $\{q_1, q_2p_1, q_3p_1p_2, -p_1p_2p_3\}$  has not(E) and  $q_3p_1p_2 > 0$ ,  $-p_1p_2p_3 < 0$ , we have that numbers  $q_1, q_2p_1$  are different signs. Hence, as not(E) and  $q_2p_1 = -p_1p_2p_3$ , then  $q_2 = -p_2p_3$ . Similarly, it is proved that  $\bar{q}_2 = -p_2a_{34}$ . By (15)

$$a_{35}p_2 + \frac{p_3}{a_{34}}p_2a_{34} = -\frac{a_{35}}{a_{34}}p_2p_3.$$

Thus  $a_{35}p_2 < 0$ , what's wrong.  $\square$

**Lemma 10.**  $\bar{q}_1 \neq -p_1p_2a_{34}$ .

PROOF. Assume  $\bar{q}_1 = -p_1p_2a_{34}$ . Since  $\bar{q}_3p_1p_2 > 0$ , from not(E) for the set  $\{\bar{q}_1, \bar{q}_2p_1, \bar{q}_3p_1p_2, -p_1p_2a_{34}\}$  we have  $\bar{q}_2p_1 = \bar{q}_3p_1p_2$ , hence  $\bar{q}_2 = a_{35}p_2$ .

By lemma 8  $q_1 \neq -p_1p_2p_3$ . As  $\{q_1, q_2p_1, q_3p_1p_2, -p_1p_2p_3\}$  has not(E) and  $q_3p_1p_2 > 0$ ,  $-p_1p_2p_3 < 0$  then numbers  $q_1, q_2p_1$  are different signs. If  $q_1 < 0$  then this set has (E), hence  $q_1 > 0$ . Therefore, since not(E),  $q_2p_1 = -p_1p_2p_3$ , i.e.  $q_2 = -p_2p_3$ .

By lemma 9  $\tilde{q}_1 \neq -p_1p_2p_3$ . Similarly to the previous one, it is proved that  $\tilde{q}_2 = -p_2p_3$ . By (15)

$$-p_2p_3 - \frac{p_3}{a_{34}}a_{35}p_2 = -\frac{a_{35}}{a_{34}}p_2p_3.$$

It means  $-p_2p_3 = 0$ , what's wrong.  $\square$

**Lemma 11.**  $q_1 \neq -p_1p_2p_3$ .

PROOF. Suppose  $q_1 = -p_1p_2p_3$ . As  $q_3p_1p_2 > 0$ , from not(E) for the set  $\{q_1, q_2p_1, q_3p_1p_2, -p_1p_2p_3\}$  it follows that  $q_2p_1 = q_3p_1p_2$ . Hence  $q_2 = a_{34}p_2$ .

By lemmas 9, 10  $\bar{q}_1 \neq -p_1p_2a_{34}$ ,  $\tilde{q}_1 \neq -p_1p_2p_3$ . As in lemma 10 we show that  $\bar{q}_2 = -a_{34}p_2$ ,  $\tilde{q}_2 = -p_2p_3$ . By (15)

$$-p_2p_3 + \frac{p_3}{a_{34}}a_{34}p_2 = -\frac{a_{35}}{a_{34}}a_{34}p_2.$$

It implies  $p_2 = 0$ . This is not true.  $\square$

**Lemma 12.** *At least one of the sets (11), (12), (13) has the property (E).*

PROOF. Let's assume that none of these sets has property (E). So far, we have been under this assumption. By lemmas 9, 10, 11,  $\tilde{q}_1 \neq -p_1p_2p_3$ ,  $\bar{q}_1 \neq -p_1p_2a_{34}$ ,  $q_1 \neq -p_1p_2p_3$ . As in the proof of lemma 10, we find that  $\tilde{q}_2 = -p_2p_3$ ,  $\bar{q}_2 = -a_{34}p_2$ ,  $q_2 = -p_2p_3$ . By (15)

$$-p_2p_3 + \frac{p_3}{a_{34}}a_{34}p_2 = -\frac{a_{35}}{a_{34}}p_2p_3.$$

Hence  $\frac{a_{35}}{a_{34}}p_2p_3 = 0$ . This is not true.  $\square$

**Theorem 2.** *Let a group  $G$  have the representation*

$$G = \langle a, x_1, \dots, x_s; P \rangle,$$

where  $P = P(a, x_1, x_2, \dots, x_s) = [a, x_1][a, x_2] \dots [a, x_n]S$  ( $n \geq 7$ ) and  $x_1, \dots, x_7 \notin [S]$ . If  $t_1, t_2, t_3 \in G$  then  $G' \langle t_1, t_2, t_3 \rangle^G$  is a locally free group.

PROOF. Let  $R = \langle t_1, t_2, t_3 \rangle^G$  be a normal closure of the subgroup  $\langle t_1, t_2, t_3 \rangle$  in  $G$ . If  $\sigma_i(t_1) = \sigma_i(t_2) = \sigma_i(t_3) = 0$  for some  $i$ , then we get into the situation of lemma 3, assuming that  $h = x_i$ ,  $m = 1$ ,  $f_1 = \dots = f_{k-1} = 0$ . By lemma 3  $N$  is a locally free group, where  $N = \langle a, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_s \rangle^G$ . In this case  $t_1, t_2, t_3 \in N$ , hence  $RG' \subseteq N$ , therefore  $RG'$  is a locally free group.

Note that if there are three columns in the matrix  $M(G, \tilde{t})$  with numbers  $i, j, u$  such that

$$\sigma_i(t_2) = \sigma_j(t_2) = \sigma_u(t_2) = 0, \sigma_i(t_3) = \sigma_j(t_3) = \sigma_u(t_3) = 0,$$

then by lemma 5  $RG'$  is a locally free group.

We can suppose that the matrix  $M(G, \tilde{t})$  has the form

$$M(G, \tilde{t}) = \begin{pmatrix} p_1 & a_{12} & a_{13} & a_{14} & a_{15} & \dots \\ 0 & p_2 & a_{23} & a_{24} & a_{25} & \dots \\ 0 & 0 & p_3 & a_{34} & a_{35} & \dots \end{pmatrix},$$

where  $p_1 > 0, p_2 > 0, p_3 \neq 0, a_{34} \neq 0, a_{35} \neq 0, a_{36} \neq 0, a_{37} \neq 0$ . If it is necessary to replace  $t_3$  with  $t_3^{-1}$  and rename generators of  $G$ , we can assume (since  $n \geq 7$ ) that  $p_3 > 0, a_{34} > 0, a_{35} > 0$ . By lemma 12, in this case at least one of the sets (11), (12), (13) has the property (E), and one can apply lemma 3. This set with the property (E) corresponds to one of matrices  $M_0 = M(C_0, \tilde{t})$ ,  $M_1 = M(C_1, \tilde{t}_1^{\psi}, \tilde{t}_2^{\psi}, \tilde{t}_3^{\psi})$ ,  $M_2 = M(C_2, \tilde{t}_1^{\psi}, \tilde{t}_2^{\psi}, \tilde{t}_3^{\psi})$ . Let's say it's  $M_j$ . Let  $N$  be a normal closure in  $C_j$  of a subgroup generated by all generators of  $C_j$  except  $x_4$ . By lemma 3  $N$  is a locally free group. Hence by lemma 2  $RG'$  is a locally free group. □

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