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INDEPENDENCE AND SIMPLICITY IN JONSSON THEORIES WITH ABSTRACT GEOMETRY

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Abstract. The concepts of forking and independence are examined in the framework of the study of Jonsson theories and the fixed Jonsson spectrum. The axiomatically given property of nonforking satisfies the classical notion of nonforking in the sense of S. Shelah and the approach to this concept by Laskar-Poizat. On this basis, the simplicity of the Jonsson theory is determined and the Jonsson analog of the Kim-Pillay theorem is given. Abstract pregeometry on definable subsets of the Jonsson theory's semantic model is defined. The properties of Morley rank and degree for definable subsets of the semantic model are considered. A criterion of uncountable categoricity for the hereditary Jonsson theory in the language of central types is proved.

Keywords: Jonsson theory, existentially closed model, Morley rank, cosemanticness, Jonsson spectrum, Jonsson set, a fragment of Jonsson set, Jonsson independence, Jonsson nonforking, Jonsson simplicity, central type, strong minimality, pregeometry, modular geometry.

1. Introduction

This article is related to the study of important concepts of the modern Model Theory, such as independence, simplicity, forking, Morley rank, strong minimality, modular geometry in the framework of the study of Jonsson theories. The relevance of the study of these issues, first of all, is dictated by the fact that Jonsson theories, generally speaking, are not complete, but at the same time, all the above concepts have been defined and have a significant development within the framework of the

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study of complete theories. Nevertheless, such classical objects as groups, Abelian groups, fields of fixed characteristic, modules, important classes of lattices, and polygons are examples of algebraic systems whose theories of classes are Jonsson theories.

In the classical Model Theory, one of the modern trends is the "geometric"approach to the study of the abstract independence's properties of the considered structures. Its essence lies in specifying special conditions in the form of axioms, which must be satisfied by definable subsets of the considered models. For example, in the case when definable closures coincide with algebraic closures of given subsets, one can notice that the structural properties of such a model in the sense of dimension are very similar to vector spaces.

In this article, we adapt the elements independence concept of the considered semantic model by an axiomatic way, setting some binary relation of nonforking and the corresponding geometry of definable subsets of the semantic model of the Jonsson spectrum's central class. Moreover, we transfer the corresponding results for complete theories in the framework of studying a simple class from the Jonsson spectrum, where as a simple class means the Jonsson analog (Definition 29) of a simple complete theory. An important role in the description of the Jonsson spectrum is played the central types of cosemanticness classes of the Jonsson spectrum of the axiomatizable class of an arbitrary signature's models. The central type of Jonsson theory is a certain syntactic invariant, which is uniquely determined with some enrichment. The admissibility of the language enrichment presupposes the preservation of definability of the considered type with the kind of stability [1], [2] corresponding to the given enrichment. It is well known that enrichment with predicate and constant is an admissible enrichment. In this work, enrichment occurs due to the unary predicate and constants. It should be noted that the emergence of new types of stability with various enrichments of the signature, and at the same time generalizing the concept of classical stability, in itself is an interesting scientific fact, and this problem has been actively studied within the framework of the study of complete theories [1], [2]. Moreover, it turned out that the concept of Jonssonnes is not always preserved even with admissible enrichments. At the moment, we do not know how to get around this rather complicated obstacle in the general case, which compels us to restrict ourselves to the framework of the so-called hereditary theories. A theory is said to be hereditary if, for any admissible enrichment, it preserves the property of Jonssonness.

All the main results are proved for some fixed Jonsson spectrum. Earlier, results related to the Jonsson spectrum has already been obtained, which allow us to note that this is not a simple generalization of the concept of Jonsson theory, but a certain syntactic invariant of an arbitrary model of an arbitrary signature in connection with the fact that the concept of cosemanticness generalizes the concept of elementary equivalence. For example, in the papers [3], [4], results were obtained that generalize the classical classification theorems about the elementary equivalence of Abelian groups and modules. It is also clear that the study of the properties of the elements of a fixed Jonsson theory's semantic model using modern methods associated with the study of the nonforking property in the framework of simple theories will allow us to study in more detail the structure of imperfect Jonsson theories. A striking example of such theories is the group theory.

In earlier works on the study of Jonsson theories [3], [4], [5], [6], has been studied well enough the connection between Jonsson theory and its class of existentially closed models and allowed to highlight the topics of positive Jonsson theories [7] and to define new problems in this direction.

The work consists of 6 sections, including an introduction. The second section provides information on Jonsson theories and their models. In the third section, we study the axiomatic definition of the concept of nonforking on Jonsson subsets of some semantic model. In the fourth section, we study the geometry of strongly minimal sets on subsets of some fixed semantic model. In the fifth section, Theorem 16 is proved, which connects the notion of nonforking, given on subsets of a fixed semantic model, and the notion of independence. The essence of this result is the Jonsson analog of the well-known Kim-Pillay result (Theorem 14). In the final sixth section, we present a criterion for uncountable categoricity (Theorem 19) in the language of strong minimality of the central type of the hereditary class of a fixed Robinson spectrum's cosemanticness. It is easy to see that the concept of Robinson spectrum is a special case of Jonsson spectrum, since in the considered spectrum, instead of Jonsson theories we consider their special cases Jonsson universal theories.

We give a brief survey of the main results of this work, namely, Theorems 16 and 19.

The concept of nonforking has been systematically studied in [8] in the framework of the study of model-theoretical properties stable theories. To study the abstract property of independence due to the work [9], it turned out to be enough to work in the class of simple theories, which does not necessarily contain stable theories. In connection with this fact, it seems to us interesting to adapt the concept of a simple theory in the framework of the study of Jonsson theories, and then move to a more general situation and consider this concept in a more general context of the perfect Jonsson spectrum for any model of an arbitrary signature. Theorem 16 demonstrates exactly this, i.e. a Jonsson analog of the Kim-Pillay theorem was obtained (Theorem 14), and the concept of nonforking, defined axiomatically on subsets of the semantic model of the cosemanticness class of a fixed spectrum, made it possible to determine the relation of Jonsson system of independence on this cosemanticness class.

The notion of categoricity is central to the study of the structural properties of complete theories' models. A natural desire is to find Jonsson analogs of results related to the concept of categoricity. In this paper, one of the main results is the description of the uncountable categoricity of some hereditary class of the Robinson spectrum. There are well-known results for complete theories related to uncountable categoricity, for example:

Theorem 1 (Erimbetov, Lachlan, Baldwin [10], p. 529)**.** *For a countable complete theory T to be a ω_1-categorical, it is necessary and sufficient that T have a non-two-cardinal strongly minimal formula $\varphi(x, a)$.*

Theorem 2 (Morley [11], p. 152)**.** *A theory T if and only if it is a ω_1-categorical if any of its countable models has a simple proper elementary extension.*

In this paper, we have proved Theorem 19, which naturally relates to both of the above Theorems 1, 2 through the necessary conditions of these theorems and generalizes them using the notion of an algebraically prime model extension, moreover, since the central type corresponds to the center of the enriched Jonsson theory under consideration (and the center is a complete theory), we obtain a refinement of Theorem 1 within the framework of studying uncountably categorical Robinson theories.

It should be also noted that in this article in an implicit form there is a formed open question about the description of hereditary Jonsson theories.

All of the above considerations led us to write this article.

2. Basic concepts and results concerning Jonsson theories

Let us give well-known definitions of concepts and results related to Jonsson theories, which are necessary for study independence and simplicity within the framework of Jonssonness. Statements that are given without proof include the link where these statements were obtained.

Definition 1 ([10], p. 80). *A theory T is called a Jonsson theory if*

(1) *T has an infinite model;*
(2) *T is inductive, i.e. T is equivalent to the set of $\forall\exists$-sentences;*
(3) *T has the joint embedding property (JEP), i.e. any two models A, B of the theory T are isomorphically embedded in some model C of the theory T;*
(4) *T has the amalgamation property (AP), i.e. if for any A, B, $C \models T$ such that $f_1 : A \to B$, $f_2 : A \to C$ are isomorphic embeddings, there are $D \models T$ and isomorphic embeddings $g_1 : B \to D$, $g_2 : C \to D$, such that $g_1 f_1 = g_2 f_2$.*

Examples of Jonsson theories are the theories of well-known classical algebras such as groups, Abelian groups, Boolean algebras, linear orders, fields of fixed characteristic, and polygons.

Definition 2 ([12], p. 529). *Let $\kappa \geq \omega$. A model M of the theory T is called κ-universal for T, if each model of the theory T of cardinality strictly less than κ is isomorphically embeddable into M.*

Definition 3 ([12], p. 529). *Let $\kappa \geq \omega$. A model M of the theory T is called κ-homogeneous for T, if for any two models A and A_1 of the theory T, which are submodels of M, cardinality is strictly less than κ, and isomorphism $f : A \to A_1$, for each extension B of the model A, that is a submodel of M and the model of the theory T of cardinality is strictly less than κ there exists an extension B_1 of the model A_1, which is a submodel of M, and the isomorphism $g : B \to B_1$, continuing f.*

A homogeneous-universal model for T is called a κ-homogeneous-universal model for T of cardinality κ, where $\kappa \geq \omega$.

Theorem 3 ([12], p. 529). *Every Jonsson theory T has a κ^+-homogeneous-universal model of cardinality 2^κ. Conversely, if T is inductive, has an infinite model, and has a ω^+-homogeneous-universal model, then T is a Jonsson theory.*

Theorem 4 ([12], p. 529). *Let T be a Jonsson theory. Two models A and B, κ-homogeneous-universal for T, are elementarily equivalent.*

Definition 4 ([12], p. 529). *The semantic model C_T of the Jonsson theory T is called the ω^+-homogeneous-universal model of the theory T.*

For any Jonsson theory, a semantic model always exists, so it plays an important role as a semantic invariant.

From the definition of the semantic model it follows that:

Proposition 1 ([13], p. 160). *Any two semantic models of the Jonsson theory T are elementarily equivalent to each other.*

Lemma 1 ([13], p. 162). *The semantic model C_T of the Jonsson theory T is T-existentially closed.*

Definition 5 ([13], p. 161). *The semantic completion (center) of the Jonsson theory T is called the elementary theory T^* of the semantic model C_T of the theory T, i.e. $T^* = Th(C_T)$.*

Definition 6 ([13], p. 162). *A Jonsson theory T is called perfect if every semantic model of T is a saturated model of T^*.*

Theorem 5. *Let T be an arbitrary Jonsson theory, then the following conditions are equivalent:*
1) the theory T is perfect;
2) T^* is a model companion of the theory T.

Proof. Follows from Theorem 2.3.6 [13]. □

Let us denote by E_T the class of all existentially closed models of the theory T.

Theorem 6 ([13], p. 162). *If the Jonsson theory T is perfect, then $E_T = Mod(T^*)$, where $T^* = Th(C_T)$.*

Proposition 2 ([10], p. 368). *If the theory T is inductive, then any model of the theory T is embedded in some existentially closed model of the theory T.*

Definition 7 (T.G. Mustafin [13], p. 175). *We say that the Jonsson theory T_1 is cosemantic to the Jonsson theory T_2 ($T_1 \bowtie T_2$), if $C_{T_1} = C_{T_2}$, where C_{T_i} is the semantic model of the theory T_i, $i = 1, 2$.*

Let T be some Jonsson theory of fixed signature σ and $\mathrm{Mod}(T)$ be the class of all models of the theory T. Consider an arbitrary model A from $\mathrm{Mod}(T)$. Let us call the *Jonsson spectrum* of model A the set:

$$JSp(A) = \{T \mid T \text{ is a Jonsson theory in the language } \sigma \text{ and } A \in \mathrm{Mod}(T)\}.$$

It is easy to see that the cosemantic relation on a set of theories is an equivalence relation. Then we can consider the $JSp(A)/_{\bowtie}$ factor set of the Jonsson spectrum of the model A with respect to $\bowtie$.

Definition 8. *Let A and B be models of the same signature. We say that the model A JSp-cossemantic to the model B ($A \underset{JSp}{\bowtie} B$), if*

$$JSp(A)/_{\bowtie} = JSp(B)/_{\bowtie}.$$

3. Special forking and independence relations for fragments of Jonsson sets

One of the most important concepts in the modern Model Theory is the concept of forking. With the help of this concept, we can estimate the dependence of the properties of elements on each other on the first-order language. It should be noted that this concept was introduced by S. Shelah [8] to solve a very important problem on the spectrum of an arbitrary complete theory. Over time, the specialist from Model Theory, having appreciated the depth and significance of the concept of forking, began to look for new approaches to its simpler explanation. One of the well-known sources in this direction is the famous work of the French mathematicians B. Poizat and D. Lascar [14], in which the concept of forking was redefined within some order. Later, other mathematicians noticed that it is possible to consider the abstract properties of the independence of the model elements from each other and relate this to the first-order properties of the types of these elements for nonforking. In particular, as an example, we can cite the following monograph by D. Baldwin [15], where he considered the axiom system that defines the abstract property of independence.

The central concept of this section is the concept of a fragment of a Jonsson set, which was defined by A.R. Yeshkeyev in the paper [16], and some of its model-theoretical properties were considered in the papers [17], [18], [19].

Definition 9 ([16], p. 38). *A set X is called a Jonsson set in the theory T, if it satisfies the following properties:*

1. *X is a definable subset of C_T, where C_T is a semantic model of the theory T;*
2. *$dcl(X)$ is a carrier of existentially closed submodel C_T, where $dcl(X)$ is definable closure of X.*

As can be seen from the definition, the concept of a Jonsson set is in very good agreement with the concept of a basis for a linear space. Note that linear spaces are a special case of modules, and the theory of modules is Jonsson theory.

Consider a countable language L, a complete for existential sentences perfect Jonsson theory T in the language L and its semantic model C_T. Let$X$ be a Jonsson set in T and M be an existentially closed submodel of the semantic model C_T, where $\mathrm{dcl}(X) = M$. Then let $\mathrm{Th}_{\forall\exists}(M) = Fr(X)$, where $Fr(X)$ is the *Jonsson fragment* of the Jonsson set X.

Since the concept of forking is a central concept to the stability theory, there is a natural desire to study it from different points of view (especially if we want to use it). For this purpose, we describe the forking axiomatically. The classical concept of forking belongs to S. Shelah [8], we recall it.

Definition 10 ([8], p. 43). *The set of formulas $\{\varphi(\bar{x}, \bar{a}_i) : i < k\} = p$ is called k-inconsistent for some natural number k if every finite subset p of cardinality k is inconsistent, that is*

$$\models \neg(\exists \bar{x})(\varphi(\bar{x}, \bar{a}_{i_1}) \wedge ... \wedge \varphi(\bar{x}, \bar{a}_{i_k}))$$

for each $i_1 < ... < i_k < k$.

Definition 11 ([8], p. 85). *The partial type p is divisible over the set relative to $k \in \omega$ if there exist a formula $\varphi(\bar{x}, \bar{a})$ and the sequence $\langle \bar{a}_i : i \in \omega \rangle$, such that:*

1. *$p \vdash \varphi(\bar{x}, \bar{a})$;*
2. *$\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{a}_i/A)$ for all i;*
3. *$\{\varphi(\bar{x}, \bar{a}_i) : i \in \omega\}$ is k-inconsistent.*

p is divisible over A, if p is divisible over A relative to some k.

Definition 12 ([8], p. 85). *The type p (not necessarily complete) fork over A in T if there are formulas $\{\varphi(\bar{x}, \bar{a}_i) : i \in \omega\}$, such that:*

1. *$p \vdash \bigvee_{0 \leq i \leq n} \varphi_i(\bar{x}, \bar{a}_i)$;*
2. *$\varphi_i(\bar{x}, \bar{a}_i)$ is divisible over A for each i.*

Let $B \subset A$, $p \in S_n(A)$. Then the type p is called *definable* over B if for each formula $\varphi(\bar{x}, \bar{v})$ of the language $L(\bar{x})$ there is a tuple $\bar{b} \in B$ and a formula $\psi(\bar{v}, \bar{u}) \in L$, such that $\forall \bar{a} \in A\ \varphi(\bar{x}, \bar{a}) \in p$ if and only if $\models \psi(\bar{a}, \bar{b})$. The formula $\psi(\bar{v}, \bar{b})$ from $L(B)$ in this case is called *φ-defining* type p by the formula. A type p is called *completely definable over B* if there exist $C \models T$, $q \in S_n(C)$, such that $A \subseteq C$, $p = q \restriction A$, q is defined over B. If $A \models T$, $p \in S_n(A)$ and p define over B, where $B \subseteq A$, then the mapping $\alpha : L(\bar{x}) \to L(B)$, putting each the formula $\varphi(\bar{x}, \bar{v})$ from $L(\bar{x})$ accordance with φ-defining type p the formula is called *schema* over B, defining p. Obviously, if d and d' are schemas over B, defining the same type, then $\models d(\varphi) \leftrightarrow d'(\varphi)$ for any formula $\varphi \in L(\bar{x})$. If $A \models T$; d is a schema that defines $p \in S_n(A)$, $A \subset A_1$, then $\{\varphi(x, \bar{a}_1) | \bar{a}_1 \in A_1, \varphi \in L(\bar{x})\}$, $\models d(\varphi)(\bar{a}_1)$ is called an *heir* p over A_1.

Let T be a Jonsson theory, $S^J(X)$ be the set of all existential complete n-types over X consistent with T for every finite n.

Definition 13 ([13], p. 201)**.** *We say that a Jonsson theory T is a J-λ-stable if for any T-existentially closed model A, for any subset X of the set A such that $|X| \leq \lambda$ it follows that $|S^J(X)| \leq \lambda$. The Jonsson theory T is called a J-stable if it is a J-λ-stable for some λ.*

In connection with the above definition, we have the following result:

Theorem 7 ([3], p. 868)**.** *Let T be a perfect Jonsson theory complete for $\exists$-sentences, $\lambda \geq \omega$. Then the following conditions are equivalent:*

(1) *T is a J-λ-stable;*
(2) *T^* is a λ-stable, where T^* is the center of Jonsson theory T.*

For complete theories, an axiomatic approach was already known for defining the notion of nonforking, for example, in [20]. In this paper, we will also approach the adaptation of the concept of Jonsson nonforking by the axiomatic way.

Let $\mathcal{A}$ be the class of all Jonsson subsets of the $\exists$-saturated semantic model C_T (i.e., any 1-$\exists$-type over any subset of the given model is realized in it) of some Jonsson theory T, $\mathcal{P}$ is the class of all existential types (not necessarily complete). Let $JNF \subseteq \mathcal{P} \times \mathcal{A}$ be some binary relation. Let us write in the form of axioms some conditions imposed on *JNF* (Jonsson nonforking).

Axiom 1. If $(p, A) \in JNF$ and $f : A \to B$ are isomorphic embeddings, then $(f(p), f(A)) \in JNF$.

Axiom 2. If $(p, A) \in JNF$ and $q \subseteq p$, then $(q, A) \in JNF$.

Axiom 3. If $A \subseteq B \subseteq C$ and $p \in S^J(C)$, then $(p, A) \in JNF$ if and only if $(p, B) \in JNF$ and $(p \restriction B, A) \in JNF$.

Axiom 4. If $A \subseteq B$, $dom(p) \subseteq B$ and $(p, A) \in JNF$, then there exist $q \in S^J(B)$ such that $p \subseteq q$ and $(q, A) \in JNF$.

Axiom 5. There is a cardinal μ, such that if $A \subseteq B \subseteq C$, $p \in S^J(B)$ and $(p, A) \in JNF$, then $\left|\{q \in S^J(C) : p \subseteq q, (q, A) \in JNF\}\right| < \mu$.

Axiom 6. There is a cardinal $\varkappa$, such that for any $p \in \mathcal{P}$ and every $A \in \mathcal{A}$, if $(p, A) \in JNF$, then there is $A_1 \subseteq A$, such that $|A_1| < \varkappa$ and $(p, A_1) \in JNF$.

Axiom 7. If $p \in S^J(A)$, then $(p, A) \in JNF$.

Let $Fr(X)$ be a fragment of some Jonsson set X, where X is a subset of the semantic model C_T of some Jonsson theory T. Then we have the following result.

Theorem 8. *The following conditions are equivalent:*

(1) *the relation JNF satisfies axioms 1-7 with respect to the fragment* $Fr(X)$;
(2) $Fr(X)^*$ *is a stable and for all* $p \in \mathcal{P}$, $A \in \mathcal{A}$ $(p, A) \in JNF \Leftrightarrow p$ *does not fork over* A *(in the classical sense of S. Shelah* [8]*), where* $Fr(X)^*$ *is the center of the fragment* $Fr(X)$.

The following well-known facts will be used in the proof of the theorem:

Fact 1 (Ramsey's theorem [21], p. 173)**.** Let I be an infinite set, $n < \omega$ and let $[I]^n$ be the family of all subsets of the set I, which consist exactly of n elements. If $[I]^n = A_0 \cup ... \cup A_{k-1}$, $k < \omega$ and $A_i \cap A_j = \emptyset$ for $i < j < k$, then there is an infinite set $J \subset I$ such that $[J]^n \subset A_i$ for some $i < k$.

Fact 2 (Lemma 14.9 [22], p. 73)**.** Let T be a stable theory, M be a saturated model of cardinality μ^+, and the types $p_1, p_2 \in S(M)$ does not fork over A. Then if $p_1 \restriction A = p_2 \restriction A$, then there exists an elementary monomorphism f, that is identical on A such that $f(d_1) \sim d_2$, where d_1, d_2 are schemas that define p_1 and p_2 respectively.

Proof of Theorem 8.

(1) $\Rightarrow$ (2). Let $\lambda = 2^{\rho \cdot |T| \cdot \mu}$, where λ, ρ and μ are cardinals corresponding to Axioms 1-7. Obviously, that $\lambda^\rho = \lambda$. Let $|A| = \lambda$. If $p \in S^J(A)$, then, by Axiom 7, $(p, A) \in JNF$, and by Axiom 6, there exists $A_p \subseteq A$ such that $|A_p| < \rho$ and $(p, A_p) \in JNF$. Then by Axiom 3 $(p \restriction A_p, A_p) \in JNF$. Denote $p \restriction A_p$ by $g(p)$. Then, according to Axiom 5, $|\{q \in S^J(A) : g(q) = g(p)\}| < \mu$. Hence,

$$|S^J(A)| \leq |\{g(p) : p \in S^J(A)\}| \cdot \mu \leq |A^\rho| \cdot 2^{\rho \cdot |T|} \cdot \mu \leq \lambda^\rho \cdot \lambda \cdot \lambda = \lambda^\rho = \lambda.$$

Thus, the theory $Fr(X)$ is a J-λ-stable. Note that $Fr(X)$ is a Jonsson theory, and since $Fr(X)$ is a J-λ-stable, this theory has a saturated model in any regular cardinality, therefore $Fr(X)$ is a perfect Jonsson theory. Then, according to Theorem 7, we conclude that $Fr(X)^*$ is a λ-stable.

Let $(p, A) \in JNF$. Let us show that p does not fork over A. Let $B = dom(p)$. Then, by Axiom 4, there exists $q \in S^J(B)$ such that $p \subseteq q$ and $(q, A) \in JNF$. Let us prove that q does not fork over A (then p does not fork over A by Axiom 2). Let's assume the opposite. Then, by the definition of forking and perfectness of the theory $Fr(X)$, there is a finite set of existential formulas Σ such that $q \vdash \vee\{\varphi : \varphi \in \Sigma\}$ and each formula $\varphi \in \Sigma$ is divisible over A. Let $C = B \cup D$, D be the set of constants appearing in at least one of the formulas from Σ. According to Axiom 4, there exists $q_0 \in S^J(C)$ such that $q \subseteq q_0$ and $(q_0, A) \in JNF$. Obviously, $q_0 \vdash \vee\{\varphi : \varphi \in \Sigma\}$, therefore there is $\varphi(\bar{x}, \bar{a}) \in q_0 \cap \Sigma$. Using Fact 1, the compactness theorem and the divisibility of $\varphi(\bar{x}, \bar{a})$ over A, we can show the existence of the sequence $\langle \bar{a}_\alpha : \alpha < \mu^+ \rangle$ and elementary monomorphisms f_α, $\alpha < \mu^+$, identical on A so that $\bar{a}_0 = \bar{a}$, $\bar{a}_\alpha = f_\alpha(\bar{a})$, where $\alpha < \mu^+$ and $\{\varphi(\bar{x}, \bar{a}_\alpha) : \alpha < \mu^+\}$ k is inconsistent for some $k < \omega$.

Let $E = C \cup \{\bar{a}_\alpha : \alpha < \mu^+\}$ and $q_\alpha = f_\alpha(q_0)$, where $0 < \alpha < \mu^+$. According to Axiom 1, $(q_\alpha, A) \in JNF$, where $\alpha < \mu^+$. According to Axiom 4, there exist $q'_\alpha \in S^J(E)$ such that $q_\alpha \subseteq q'_\alpha$ and $(q'_\alpha, A) \in JNF$. It is clear that $\varphi(\bar{x}, \bar{a}_\alpha) \in q'_\alpha$ and $q \subseteq q'_\alpha$, where $\alpha < \mu^+$. We have $|\{q'_\alpha : \alpha < \mu^+\}| = \mu^+$, since $\{\varphi(\bar{x}, \bar{a}_\alpha) : \alpha < \mu^+\}$ k is inconsistent. We have obtained a contradiction with Axiom 5. Therefore, q does not fork over A. Thus, we have that if $(p, A) \in JNF$, then p does not fork over A.

Let's prove it in the opposite direction. Let p does not fork over A. Since the theory $Fr(X)$ is perfect, then $Fr(X)^*$ is model complete (Theorem 5), and we only need to work with existential types and consider $\exists$-saturated existential closed models of the theory $Fr(X)$. We need to prove that $(p, A) \in JNF$. Let $M \supseteq A$, $M \supseteq dom(p)$, $|M| > 2^{\rho \cdot |T| \cdot \mu}$ and M be $\exists$-saturated model of the theory $Fr(X)^*$, $t \in S^J(M)$, $p \subseteq t$, t does not fork over A. By Axiom 7 $(t \upharpoonright A, A) \in JNF$ and by Axiom 5 there exists $q \in S^J(M)$ such that $q \supseteq t \upharpoonright A$ and $(q, A) \in JNF$. As shown above, $(q, A) \in JNF$ implies that q does not fork over A. According to Fact 2, there exists an automorphism f of the model M identity on A such that $t = f(q)$. Then by Axiom 1 $(t, A) \in JNF$, and by Axiom 2 $(p, A) \in JNF$. Therefore, the implication (1) $\Rightarrow$ (2) is proved.

(2) $\Rightarrow$ (1). It is easy to see that this follows from the proof of Theorem 19.1 from [23] with a generalization of the corresponding concepts to Jonsson analogs.

Consider gain of Lemma 19.7 from [23].

Let T be some Jonsson theory, M, N be existentially closed submodels of the semantic model C_T of T. If $A \subseteq M \cap N$, $p \in S^J(M)$, $q \in S^J(N)$, then $p \geq_A q$ means that for any existential formula $\varphi(\bar{x}, \bar{y}) \in L(A)$ from the fact that there is a tuple $\bar{m} \in M$ and $\varphi(\bar{x}, \bar{m}) \in p$ it follows that there is a tuple $\bar{n} \in N$ such that $\varphi(\bar{x}, \bar{n}) \in q$; $p \sim_A q$ means that $p \geq_A q$ and $q \geq_A p$.

$$[p]^A \rightleftharpoons \{q : \exists N \models T, q \in S^J(N), p \sim_A q\}.$$

It is easy to see that the relation $\geq_A$ induces a similar relation between classes, which is a partial ordering relation. If $A = \emptyset$, then the index A for $\geq_A$ and $\sim_A$ will be omitted. Each equivalence class in $\sim_A$ is determined uniquely by the set of existential formulas from $L(A)$, representable in each type of this class.

Definition 14. *Let $A \subseteq M$, $\exists$-formula $\varphi(\bar{x}, \bar{y}) \in L(A)$ be called representable in $p \in S^J(M)$ if there is a tuple $\bar{m} \in M$ such that $p \vdash \varphi(\bar{x}, \bar{m})$.*

Thus, it is obvious that the number of equivalence classes in $\sim_A$ is at most, than $2^{|L(A)|} = 2^{|L| \cdot |A|}$.

Equivalence classes with respect to $\sim_A$ will be denoted by ξ^A. If $p \in S^J(A)$, then Ω_p denotes a partially ordered set

$$\langle \{\xi^A : \exists M \supseteq A, p_1 \in S^J(M), p \subseteq p_1, p_1 \in \xi^A\}; \geq \rangle.$$

Definition 15. *Let $p \in S^J(M)$, $M \in E_T$, A be a Jonsson set, and $M \subseteq A$. The extension q ($q \in S^J(A)$) of type p to A is called an heir of p if for all $\varphi(x, \bar{v}_n) \in \Sigma_{n+1}(M \cup x)$ and $\bar{a} \in A^n$ such that $q \vdash \varphi(x, \bar{a})$, there is a tuple $\bar{m} \in M^n$ such that $p \vdash \varphi(x, \bar{m})$.*

Thus, if $M \prec_\exists N$ and $p \in S^J(M)$, then the heirs of the type p on N are exactly these types $q \in S^J(N)$ such that $p \sim_M q$.

We have the following technical lemmas.

Lemma 2. *There exists a maximum element in Ω_p.*

Proof. It suffices to prove that Ω_p satisfies the condition of Zorn's lemma. Let $\{\xi_i : i \in I\}$ be a linear chain in Ω_p, $J_i = \{j \in I : \xi_j \geq \xi_i\}$, $i \in I$. It is clear that the set $\{J_i : i \in I\}$ is centered, that is, the intersection of any finite number of terms is non-empty. Let D be an ultrafilter over I, containing $\{J_i : i \in I\}$; M, $p_i \in S^J(M)$ such that $M \supseteq A$, $p_i \in \xi_i$, $i \in I$. Let $p' \in S^J(M)$ be such that for every tuple $\bar{m}$ of elements from M, for every existential formula $\varphi(\bar{x}, \bar{y})$ of the language $L(A)$ $\varphi(\bar{x}, \bar{m}) \in p'$ if and only if $\{i \in I : \varphi(\bar{x}, \bar{m}) \in p_i\} \in D$. Obviously, that $p \subseteq p'$.

Let us show that $p' \geq p_i$ for any $i \in I$. Indeed, let $\varphi(\bar{x}, \bar{y})$ be some existential formula of the language $L(A)$, $\bar{m} \in M$, $\varphi(\bar{x}, \bar{m}) \in p'$. Then $K = \{j \in I : \varphi(\bar{x}, \bar{m}) \in$

$p_j\} \in D$, but $J_i \in D$, so $K \cap J_i \in D$. Let $j \in K \cap J_i$. It is clear that $\varphi(\bar{x}, \bar{m}) \in p_j$. But $j \in J_i$, whence $p_j \geq_A p_i$, therefore, $\varphi(\bar{x}, \bar{y})$ is representable in p_i. Thus $p' \geq_A p_i$, $i \in I$. It follows that $[p']^A \geq_A \xi_i$, $i \in I$. □

Lemma 3. *If T is a J-stable, existentially complete Jonsson theory, $p \in S^J(M)$, $M \subseteq B$, $p' \in S^J(B)$ is the heir of type p, then $p \sim_M p'$.*

Proof. Obviously, $p \geq_M p'$. Let $\varphi(\bar{x}, \bar{y})$ be an existential formula of the language $L(M)$, $\bar{b} \in B$, $\varphi(\bar{x}, \bar{b}) \in p'$, $\varphi(\bar{x}, \bar{b}) = \psi(\bar{x}, \bar{b}, \bar{m}_1)$, $\bar{m}_1 \in M$. And let d be the schema defining p (and p'). Then $\varphi(\bar{x}, \bar{b}) \in p' \Rightarrow \models (d\psi)(\bar{b}^\wedge \bar{m}_1) \Rightarrow \models \exists \bar{z}(d\psi)(\bar{z}^\wedge \bar{m}_1) \Rightarrow M \models \exists \bar{z}(d\psi)(\bar{z}^\wedge \bar{m}_1)$, therefore there exists $\bar{b}' \in M$ such that $M \models (d\psi)(\bar{b}'^\wedge \bar{m}_1)$, therefore $\psi(\bar{x}, \bar{b}', \bar{m}_1) \in p \Rightarrow \varphi(\bar{x}, \bar{b}') \in p$. □

Lemma 4. *If T is a J-stable, existentially complete Jonsson theory, $p \in S^J(M)$, $M \subseteq B$, $q \in S^J(B)$, $p \subseteq q$ and $p \sim_M q$, then q is the heir of p.*

Proof. Let d be the schema that defines the type p. Obviously, the formula $\theta(\bar{x}, \bar{y}) = \neg(\varphi(\bar{x}, \bar{y}) \leftrightarrow (d\varphi)(\bar{y}))$ is not representable in p. Since $q \geq_A p$, $\theta(\bar{x}, \bar{y})$ is not representable in q. Then for any $\bar{b} \in B$ $\neg\theta(\bar{x}, \bar{b}) \in q$. It follows that for any $\bar{b} \in B$ $\varphi(\bar{x}, \bar{b}) \in q$ if and only if $\models (d\varphi)(\bar{b})$. □

Lemma 5. *If $A \subseteq M \cap N$, $p \in S^J(M)$, $q \in S^J(N)$, $p \upharpoonright A = q \upharpoonright A$ and p, q does not fork over A, then $p \sim_A q$.*

Proof. Let $M \cup N \subseteq \tilde{M}$, $\tilde{M}$ be $\exists$-saturated model T of cardinality $> 2^{|L(A)|}$. Then there exists a f automorphism $\tilde{M}$, identical on A and $f(p') = q'$, where p', q' are the heirs of p, q over $\tilde{M}$ respectively. Then, applying Lemma 2, we have $p \sim_M p' \sim_A q' \sim_N q$. But $\sim_{M,N}$ implies $\sim_A$. Therefore, $p \sim_A q$. □

Lemma 6. *If T is a J-stable, existentially complete Jonsson theory, $p \in S^J(A)$, then Ω_p has a unique maximal (i.e. largest) element.*

Proof. Let $M \supseteq A$ be an arbitrary model of T, p' be arbitrary not forking over A extension of p over M. Let us show that $p' \geq_A q$ for any $q \supseteq p$, where q is a complete type over some model N. Let $\bar{c} \models q$, and M_1 be a model containing A such that $t(M_1, N \cup \bar{c})$ does not fork over A. It is obvious that $t(M_1, A \cup \bar{c})$ does not fork over A. According to the symmetry property of forking, $t(\bar{c}, M_1 \cup A)$ $(= t(\bar{c}, M_1))$ does not fork over A (and contains p). By Lemma 4, $p' \sim_A t(\bar{c}, M_1)$.

Now we will show that $t(\bar{c}, M_1 \cup N)$ does not fork over $A \cup N$ $(= N)$. Assume that this is not the case. Then there is a tuple $\bar{m} \in M_1$ such that $t(\bar{c}, \bar{m} \cup N)$ forks over N, whence $t(\bar{m}, N \cup \bar{c})$ forks over N. But $t(\bar{m}, N \cup \bar{c})$ does not fork over A and even more so N. We got a contradiction.

So, $t(\bar{c}, M_1 \cup N)$ does not fork over N. By Lemma 2, we have $t(\bar{c}, M_1 \cup N) \geq_A t(\bar{c}, N) = q$. Then $p' \sim_A t(\bar{c}, M_1) \geq_A t(\bar{c}, M_1 \cup N) \geq_A q$. Therefore, $[p']^A$ is the largest element of Ω_p. □

Let denote by $\beta^J(p)$ the largest element of Ω_p.

Let us introduce the following relation *JNFLP* (Jonsson nonforking according to Lascar-Poizat) on $\mathcal{P} \times \mathcal{A}$.

Definition 16. *Let T be a J-stable, existentially complete Jonsson theory.*

(1) *If $p \in S^J(B)$, $A \subseteq B$, then $(p, A) \in JNFLP \Leftrightarrow \beta^J(p) = \beta^J(p \upharpoonright A)$.*

(2) *If p is an arbitrary existential type, then $(p, A) \in JNFLP \Leftrightarrow$ there is a type $p' \in S^J(A \cup dom(p))$, that $p \subseteq p'$ and $(p', A) \in JNFLP$.*

Theorem 9. *In J-stable existentially complete Jonsson theory, the relation JNFLP satisfies Axioms 1-7.*

Proof. Axioms 1, 2, 3, 4, 7 are trivial to check.

Axiom 6 holds for $\varkappa = |L|^+$. Suppose the opposite. Let $p \in S^J(A)$ and for any $A_1 \subseteq A$, if $|A_1| < \varkappa$, then $(p, A_1) \notin JNFLP$. Obviously, $|A| \geq \varkappa = |L|^+$. There is a sequence $\langle A_\alpha : \alpha < |L|^+ \rangle$ such that $|A_\alpha| \leq |L|$, $A_\alpha \subseteq A_\beta$ for $\alpha < \beta < |L|^+$ and $(p \upharpoonright A_{\alpha+1}, A_\alpha) \notin JNFLP$. Let $M \supseteq \bigcup\limits_\alpha A_\alpha$ be an arbitrary existentially closed submodel of the semantic model of T of cardinality $|T|$, $p_\alpha \supseteq p \upharpoonright A_\alpha$ such that $p_\alpha \in S^J(M)$ and $[p_\alpha]^{A_\alpha}$ is the largest element in $\Omega_{(p \upharpoonright A_\alpha)}$. Then $\langle \{p_\alpha : \alpha < |L|^+\}; \geq \rangle$ is a strictly decreasing sequence. Hence, there are formulas $\varphi_\alpha(\bar{x}, \bar{y}_\alpha) \in L$, $\alpha < |L|^+$ such that $\varphi_\alpha(\bar{x}, \bar{y}_\alpha)$ representable in p_α, but not representable in $p_{\alpha+1}$. It is clear that for $\alpha \neq \gamma$ $\varphi_\alpha(\bar{x}, \bar{y}_\alpha) \neq \varphi_\gamma(\bar{x}, \bar{y}_\gamma)$, since there is no set of cardinality $> |L|$ formulas in the language L. Contradiction.

Axiom 5 holds for $\mu = \left(2^{|T|}\right)^+$. Indeed, let $p \in S^J(B)$, $(p, A) \in JNFLP$, $A \subseteq B \subseteq C$. According to Axiom 6, there is $A_0 \subseteq A$ such that $|A_0| \leq |L|$, $(p, A_0) \in JNFLP$.

Case 1: Let C be an existentially closed submodel of the semantic model M of T, $C \models T$. Let $A_0 \subseteq M_0 \preccurlyeq_\exists C$. If $p' \in S^J(C)$, $p \subseteq p'$, $(p', B) \in JNFLP$, then $(p', A_0) \in JNFLP$. Hence, $(p', M_0) \in JNFLP$. Hence p' is the heir of $p' \upharpoonright M_0$. There are no more such types than $|S^J(M_0)| \leq 2^{|T|}$.

Case 2: $C \not\models T$. Then we take a model $N \in E_T$ such that $N \supseteq C$.

$$|\{q \in S^J(C) : p \subseteq q \& (q, A) \in JNFLP\}| \leq$$
$$\leq |\{q \in S^J(N) : p \subseteq q \& (q, A) \in JNFLP\}| \leq 2^{|T|}.$$

□

The following theorem is the gain of Theorem 19.8 [23] and is the main result of this section.

Theorem 10. *If the theory $Fr(X)$ is a J-stable, then the concepts JNF and JNFLP coincide.*

Proof. Follows from Theorem 8 and Theorem 9. □

Thus, for a fixed fragment of some Jonsson subset of the semantic model of some fixed J-stable existentially complete Jonsson theory, the equivalence of binary relations *JNF* and *JNFLP* is proved. Moreover, for *JNF* in this class of theories, a more detailed version of Theorem 10 from [24] was obtained.

The obtained results with these binary relations provide an additional opportunity to characterize the behavior of existential types in the study of the considered fragment of the Jonsson subset of the given Jonsson theory's semantic model.

4. Strongly minimal sets on the pregeometry of the semantic model of a fixed Jonsson theory

The results of this section are natural generalizations of classical results on the properties of the algebraic closure within the framework of the study of Jonsson strongly minimal sets [26].

One of the interesting advances in the modern Model Theory research is the realization of the local properties of the geometry of strongly minimal sets. In [25], E. Hrushovski obtained remarkable results using the geometric theory of stability, which significantly influenced the development of methods and ideas for studying the global properties of structures. These model-theoretical features play an important role in Hrushovski's proof of the Mordell-Lang hypothesis for field functions.

The Model Theory apparatus associated with the notion of strongly minimality is fairly well developed for complete theories. Therefore, transferring from the above apparatus the basic concepts associated with the concept of strongly minimality for fixed formula subsets of the semantic model of Jonsson theory, we set the condition that the semantic model is saturated in its power, i.e. the considered theory must be perfect.

This section is devoted to the study of the basic concepts of local properties of the geometry of strongly minimal sets on formula subsets of some existentially closed model. Within the framework of studying the combinatorial properties of pregeometry defined on Jonsson sets, results are obtained related to strongly minimal Jonsson sets. The minimum structures and, accordingly, the pregeometries and geometries of the minimum structures are determined. The concepts of dimension, independence, and basis in Jonsson strongly minimal structures for Jonsson theories are considered.

Strongly minimal sets and their properties are reflected in the works [25], [26], [27]. A natural generalization would be to consider Jonsson analogs of strongly minimal arbitrary subsets of the semantic model of some fixed Jonsson theory.

The notion of strong minimality, both for sets and theories, played a decisive role in obtaining a result on the description of uncountably categorical theories.

It is clear from the definition of Jonsson sets that they are arranged very simply in the sense of the Morley rank. It turns out that elements from the set-theoretic difference (holes) of the closure and the set have rank 0, i.e. they are all algebraic.

The second point of the usefulness of the definition of the Jonsson set is that by closing a given set, we get some existentially closed models. This, in turn, enables us to study the Morley's rank for an arbitrary fragment of the considersd set. Morley's rank is some ordinal-valued estimate for the independence of the elements of the set generating the fragment. For complete theories, one of the conditions for the correctness of the definition of the Morley's rank is saturation. In the case of Jonsson theories, when they are imperfect, saturation for existential types in the semantic model is required. Thus, when working with Jonsson theories, the concept of independence can arrive at both through nonforking and Morley's rank. To study the behavior of the elements of a hole in the case of Jonsson sets, one can always consider the $\forall\exists$-consequences, which are true in the above-mentioned closures of the Jonsson set. In view of the above, it follows that the considered set of sentences will be a Jonsson theory. Within the framework of the newly introduced definitions, strongly minimal Jonsson sets were considered and described. To transfer from the apparatus of Model Theory, developed for complete theories, the basic concepts associated with the concept of strongly minimality for fixed formula subsets of the semantic model of the above Jonsson theory, we need the semantic model to be saturated in its power, i.e. the theory in question must be perfect.

Recall that Jonsson theory has a semantic model C of sufficiently large cardinality. The semantic models of a perfect Jonsson theory are uniquely determined by their cardinality. We will consider not all subsets of C, but only Jonsson subsets.

Let us define the Morley's rank for existentially definable subsets of the semantic model.

We want to assign to each Jonsson subset of X of the semantic model an ordinal (or perhaps -1 or ∞) – its Morley's rank, denoted by MR.

Let T be a fragment of some Jonsson set, and it is a perfect Jonsson theory, C be a semantic model, X be a definable set of C.

Definition 17. *$MR(X) \geq 0$ if and only if X is nonempty; $MR(X) \geq \lambda$ if and only if $MR(X) \geq \alpha$ for all $\alpha < \lambda$ (λ is limit ordinal); $MR(X) \geq \alpha + 1$ if and only if in X there is an infinite family X_i of pairwise disjoint $\exists$-definable subsets such that $MR(X_i) \geq \alpha$ for all i.*

Then Morley's rank of the set X is $MR(X) = sup\{\alpha \mid MR(X)\} \geq \alpha$.

Moreover, we will assume that $MR(X) = -1$ and $MR(X) = \infty$, if $MR(X) \geq \alpha$ for all α (in the latter case, we say that X has no rank).

Definition 18. *The Morley's degree $MD(X)$ of a Jonsson set X, having Morley's rank α, is the maximum length d of its decomposition $X = X_1 \cup ... \cup X_n$ into disjoint existentially definable subsets of rank α.*

In the case of rank 0, the degree of an existentially definable subset is simply the number of its elements. If an existentially definable subset has no rank, then its Morley's degree is not defined either.

Consider Jonsson minimal sets. Note that a strongly minimal set is a set of rank 1 and degree 1.

We will everywhere assume that the language L is countable. The considered theory T is an existentially complete perfect Jonsson theory in the countable language L.

Consider an example of an algebraic closure in Jonsson strongly minimal theories.

If K is an algebraically closed field and $A \subseteq K$, then $acl(A)$ is an algebraically closed subfield generated by A.

The following properties of the Jonsson algebraic closure are valid for any subset D of the semantic model of the Jonsson theory T.

Let M be some existentially closed submodel of the semantic model for a fixed theory in the language L, and $D \subseteq M$ be a Jonsson strongly minimal set.

Let $D \subseteq M^n$ be an infinite Δ-definable set, where $\Delta \subseteq L$ is the set of existential formulas of a given language.

Definition 19. *We say that D is Jonsson minimal in M if for any Δ-definable $Y \subseteq D$ either Y is finite or $D \backslash Y$ is finite.*

If $\phi(\overline{v}, \overline{a})$ is a formula that defines D, then we can say that $\phi(\overline{v}, \overline{a})$ is also Jonsson minimal.

Definition 20. *We say that D and φ are Jonsson strongly minimal if φ is Jonsson minimal in any existentially closed extension N from M.*

Definition 21. *We say that a theory T is Jonsson strongly minimal if the formula $v = v$ is Jonsson strongly minimal (that is, if $M \in ModE_T$, then M is Jonsson strongly minimal)*

Consider acl_D is an algebraic closure restricted to D. Recall that b is Jonsson algebraic over A if there is a formula $\varphi(\overline{v}, \overline{a}) \in \Delta$ with $\overline{a} \in A$ such that $\varphi(M, \overline{a})$ is finite.

For $A \subseteq D$ let $acl_D(A) = \{b \in D : b$ be a Jonsson algebraic over $A\}$.

The following well-known properties of the algebraic closure [9] are also valid for the Jonsson algebraic closure of any subset D of the semantic model of the theory T.

Lemma 7. (1) *$acl(acl(A)) = acl(A) \supseteq A$.*

(2) *If $A \subseteq B$, then $acl(A) \subseteq acl(B)$.*

(3) *If $a \in acl(A)$, then $a \in acl(A_0)$ for some finite $A_0 \subseteq A$.*

Lemma 8 (Exchange)**.** *Suppose that $D \subset M$ is Jonsson strongly minimal, $A \subseteq D$ and $a, b \in D$. If $a \in acl(A \cup \{b\}) \backslash acl(A)$, then $b \in acl(A \cup \{a\})$.*

The notion of independence, which generalizes linear independence in vector spaces, as well as in algebraically closed fields, algebraic independence can also be defined in a Jonsson strongly minimal set.

Let $M \in Mod\, E_T$, and D be a Jonsson strongly minimal set in M.

Definition 22. *We say that $A \subseteq D$ is Jonsson independent if $a \notin acl(A \backslash \{a\})$ for all $a \in A$. If $C \subset D$, we say that A is Jonsson independent over C if $a \notin acl(C \cup (A \backslash \{a\}))$ for all $a \in A$.*

Definition 23. *We say that A is a Jonsson basis for $Y \subseteq D$ if $A \subseteq Y$ is Jonsson independent and $acl(A) = acl(Y)$.*

It is clear that any maximal Jonsson independent subset of Y is a Jonsson basis for Y.

Definition 24. *If $Y \subseteq D$, then the Jonsson dimension of the set Y is the cardinality of the Jonsson basis for Y.*

Let $J \dim Y$ denote the Jonsson dimension of Y.

Note that if D is uncountable, then $J - \dim(D) =\mid D \mid$, since the language is countable and $acl(A)$ is countable for any countable $A \subseteq D$.

For Jonsson strongly minimal theories, each model is determined up to isomorphism by its own Jonsson dimension.

Theorem 11. *Let T be Jonsson strongly minimal theory. If $M, N \in ModE_T$, then $M \cong N$ if and only if $J \dim(M) = J \dim(N)$.*

Proof. Similar to the proof of Theorem 6.1.11 from [26]. □

To adapt the uncountable categoricity within the framework of the study of Jonsson theory, the concept of the central type was used. This concept is needed for additional information about Jonsson theory, and also technically the fact was used that the central type, after replacing a variable with a constant, turned into a theory, moreover, a complete one, and it was closely related to the original Jonsson theory in the sense that the class of existentially closed models of this new theory did not differ from the old one. It is well known from the work [2] that enrichment of a language with a unary predicate and a constant preserves the definability of the type in the enrichment, i.e. the enrichments discussed in this article are permissible. But at the same time, we know about the existence of a counterexample of Jonsson theory, which, when enriched with a unary predicate, loses the concept of Jonssonness. This example is the theory of fields of characteristic 0. It is well known that the field theory of characteristic 0 is not locally modular. In this regard, in order to exclude in advance such situations as the case of an example of field theory that are Jonsson teories, we assume that the operator cl, which defines the pregeometry on Jonsson theories, will be the algebraic closure operator, which is equal to the definable closure operator, i.e. $cl = acl = dcl$. An example of a Jonsson theory in which $dcl = acl$ is an example of vector spaces. In particular, for the operator cl, which defines the pregeometry in the language of the signature of the field theory of characteristic 0, it is true that $cl = acl$, but $cl \neq dcl$. Since the main result of this section on uncountable categoricity is proved for Jonsson strong minimality of central type, this assumption is correct because strongly minimal theories always admit acl as an operator cl.

For any automorphism of the semantic model, for any Jonsson subsets of the semantic model, the following result is true.

Theorem 12. *Let T be Jonsson theory, X be a Jonsson subset of the semantic model C of the theory T, $a \in C$. Then $a \in acl(X) \Leftrightarrow$ there are at most finitely many elements $f(a)$ when f runs through the automorphisms C that fix X pointwise.*

Proof. Take $a \in acl(X)$. Let $\varphi(v)$ be $L(X)$-formula such that $a \in \varphi(C)$ and $\varphi(C)$ is finite. For any automorphism f of the model C fixing X pointwise, $f(a) \in \varphi(C)$. Accordingly, the images of the element a under the automorphisms of C that fix X pointwise form a finite set.

Conversely, let $a \notin acl(X)$, so for any $L(X)$-formula $\varphi(v)$, if $a \in \varphi(C)$, then $\varphi(C)$ is infinite. Let $a_0, ..., a_n$ be different elements realizing $tp(a/X)$. For each formula $\varphi(v) \in tp(a/X)$ there is some element $b \in \varphi(C)$, $b \neq a_0, ..., a_n$. Since $tp(a/X)$ is closed under finite conjunction, $tp(a/X) \cup \{\neg(v = a_i) \colon i \leq n\}$ consistent and can be expanded to complete type over $X \cup \{a_0, ..., a_n\}$ so that any element a_{n+1} that implements this type satisfies the condition $a_{n+1} \models tp(a/X)$, $a_{n+1} \neq a_0, ..., a_n$. Thus, there are infinitely many realizations of type $tp(a/X)$ in C, and each of them is the image of a under some automorphism C fixing X pointwise.

Similar reasoning shows that $a \in dcl(X) \Leftrightarrow f(a) = a$ for every automorphism f of the model C that fixes X pointwise. □

Hence it follows that the definable closure $dcl(A)$ of the Jonsson set A, that is, the set of all elements definable over A coincides with the set of elements invariant under all automorphisms over A.

Theorem 12 implies that an element b is algebraic over A if and only if it has only finitely many conjugates over A.

Further, using Jonsson strongly minimal sets, we consider some properties of the combinatorial geometry of the algebraic closure.

It is well known that in the proof of Morley's uncountable categoricity theorem, the properties of the algebraic closure on strongly minimal sets are essentially used. Using Jonsson strongly minimal sets, we study the combinatorial geometries of the algebraic closure.

Let X be a subset of the semantic model of some fixed Jonsson theory and let $cl : P(X) \to P(X)$ be an operator on the set of subsets X. We say that (X, cl) is a *J-pregeometry* if the following conditions are satisfied:

1) if $A \subseteq X$, then $A \subseteq cl(A)$ and $cl(cl(A)) = cl(A)$;

2) if $A \subseteq B \subseteq X$, then $cl(A) \subseteq cl(B)$;

3) (exchange) $A \subseteq X$, $a, b \in X$ and $a \in cl(A \cup \{b\})$, then $a \in cl(A)$, $b \in cl(A \cup \{a\})$;

4) (finite character) If $A \subseteq X$ and $a \in cl(A)$, then there is a finite $A_0 \subseteq A$ such that $a \in cl(A_0)$.

We say that $A \subseteq X$ is *closed* if $cl(A) = A$.

Note (by Theorem 12 and Lemma 7) that if D is Jonsson strongly minimal, one can define a Jonsson pregeometry by defining $cl(A) = acl(A) \cap D$ for $A \subseteq D$. We generalize basic ideas about independence and dimension from Jonsson strongly minimal sets to an arbitrary Jonsson pregeometries.

Definition 25. *If (X, cl) is a Jonsson pregeometry, we say that A is Jonsson independent if $a \notin cl(A \setminus \{a\})$ for all $a \in A$, and B is a J-basis for Y if $B \subseteq Y$ is J-independent and $Y \subseteq acl(B)$.*

If $A \subseteq X$, we also consider the localization $cl_A(B) = cl(A \cup B)$.

If (X, cl) is a J-pregeometry, then we say that $Y \subseteq X$ is *Jonsson independent over A*, if Y is Jonsson independent in (X, cl_A).

$\dim(Y/A)$ is the dimension of Y in the localization (X, cl_A), $\dim(Y/A)$ is called the dimension of Y over A.

Definition 26. *We say that a J-pregeometry (X, cl) is a J-geometry if $cl(\varnothing) = \varnothing$ and $cl(\{x\}) = \{x\}$ for any $x \in X$.*

We distinguish some properties of the pregeometry that will play an important role.

Definition 27. *Let (X, cl) be a J-pregeometry. We say that (X, cl) is trivial if $cl(A) = \bigcup_{a \in A} cl(\{a\})$ for any $A \subseteq X$. We say that (X, cl) is modular if, for any finite-dimensional closed sets $A, B \subseteq X$, holds $Jdim(A \cup B) = Jdim(A) + Jdim(B) - Jdim(A \cap B)$.*

(X, cl) is locally modular if (X, cl_a) is modular for some $a \in X$.

Theorem 13. *Let (X, cl) be a J-pregeometry. The following conditions are equivalent:*

(1) *(X, cl) is modular;*
(2) *if $A \subseteq X$ is closed and non-empty, $b \in X$, $x \in cl(A, b)$, then there exists $a \in A$ such that $x \in cl(a, b)$;*
(3) *if $A, B \subseteq X$ are closed and non-empty, $x \in cl(A, B)$, then there exist $a \in A$ and $b \in B$ such that $x \in cl(a, b)$.*

Proof. Similarly to the proof of Lemma 8.1.13 from [26]. □

5. Jonsson independence for $JSp(A)$

In Model Theory, one of the modern approaches to studying the properties of elements of the considered complete stable theory's models is forking. The concept of nonforking is analogous of the concept of independence between elements, and also, respectively, between the considered formula subsets of a fixed model. The notion of independence leads to geometric combinatorial connections between the considered formula subsets, which allows one to obtain the necessary results in specific posed model-theoretic problems. Within the framework of the study of complete theories, the concept of a simple theory was defined, and this concept significantly allowed the use of the properties of nonforking in wide classes of theories than stable theories. So, for example, the following theorem shows that in any simple theory a good system of independence necessarily arises from forking.

Theorem 14 (Kim-Pillay [9], p. 233). *Let T be a simple theory, I be a good independence system of the theory T. Then, for each tuple $\bar{a} \in \Omega$ and $A \subseteq B$ of small subsets of Ω, $(\bar{a}, B, A) \in I$ if and only if $tp(\bar{a}/B)$ does not fork over A.*

In this section, a point of view is given that makes it possible to transfer the concept of stability, simplicity of a theory and nonforking of types, and, accordingly, independence to a class of theories that, generally speaking, are not complete, and thus this approach is useful in the study of inductive classes of algebras, the theory which, generally speaking, are not complete, but satisfy the natural algebraic conditions of joint embedding and amalgam.

Let T be an arbitrary Jonsson theory of the signature σ, E_T be the class of all existentially closed models of the theory T. Let $[T] \in JSp(A)/_{\bowtie}$, then $E_{[T]} = \bigcup_{\Delta\in[T]} E_\Delta$.

Let us call the factor spectrum $JSp(A)/_{\bowtie}$ *homogeneous*, if for any $[T]_1$, $[T]_2 \in JSp(A)/_{\bowtie}$ it follows that $E_{[T]_1} \cap E_{[T]_2} = \varnothing$.

Let $X \subseteq C_T$, where C_T be the semantic model of the considered Jonsson theory T. We say that the set X is a ∇-*cl-subset* of the model C_T if X satisfies the following conditions:

(1) X is a ∇-definable set (this means that there is a ∇-formula in the language L, whose solution in C_T is the set X, where ∇ is a kind of formula, for example $\exists, \forall, \forall\exists$);
(2) $cl(X) = M$, $M \in E_T$, where cl is some closure operator that defines a J-geometry over C_T in the sense of Definition 26 (for example, $cl = acl$ or $cl = dcl$).

Let T be a Jonsson theory, $S^\nabla(X)$ be the set of all ∇-complete n-types over X, which consistent with T for every finite n. We say that a Jonsson theory T is a ∇-λ-*stable* if for every T-existentially closed model of A, for every subset X of the set A, from the fact that $|X| \leq \lambda$ it follows that $|S^\nabla(X)| \leq \lambda$. We will call a Jonsson theory ∇-*stable* if it is a ∇-λ-stable for some λ.

The class $[T] \in JSp(A)/_{\bowtie}$ is called a ∇-*stable* if each theory $\Delta \in [T]$ is ∇-stable.

Let $[T] \in JSp(A)/_{\bowtie}$, $\mathcal{X}$ be the class of all ∇-cl-subsets of the semantic model $C_{[T]}$ and $\mathcal{P}$ be the class of all ∇-types (not necessarily complete), let $J^\nabla NF$ (Jonsson ∇-nonforking) $\subseteq \mathcal{P} \times \mathcal{X}$ be some binary relation. Let us write in the form of axioms some conditions imposed on $J^\nabla NF$.

Axiom 1. If $(p, X) \in J^\nabla NF$, $f : X \to Y$ are isomorphic embeddings, then $(f(p), f(X)) \in J^\nabla NF$.

Axiom 2. If $(p, X) \in J^\nabla NF$ and $q \subseteq p$, then $(q, X) \in J^\nabla NF$.

Axiom 3. If $X \subseteq Y \subseteq Z$ and $p \in S^\nabla(Z)$, then $(p, X) \in J^\nabla NF$ if and only if $(p, Y) \in J^\nabla NF$ and $(p \upharpoonright Y, X) \in J^\nabla NF$.

Axiom 4. If $X \subseteq Y$, $dom(p) \subseteq Y$ and $(p, X) \in J^\nabla NF$, then there exist $q \in S^\nabla(Y)$ such that $p \subseteq q$ and $(q, X) \in J^\nabla NF$).

Axiom 5. There is a cardinal μ such that if $X \subseteq Y \subseteq Z$, $p \in S^\nabla(Y)$ and $(p, X) \in J^\nabla NF$, then $\left|\{q \in S^\nabla(Z) : p \subseteq q, (q, X) \in J^\nabla NF\}\right| < \mu$.

Axiom 6. There is a cardinal $\varkappa$, that for every $p \in \mathcal{P}$ and every $X \in \mathcal{X}$, if $(p, X) \in J^{\nabla}NF$, then there exist $X_1 \subseteq X$ such that $|X_1| < \varkappa$ and $(p, X_1) \in J^{\nabla}NF$.

Axiom 7. If $p \in S^{\nabla}(X)$, then $(p, X) \in J^{\nabla}NF$.

Let $[T] \in JSp(A)/_{\bowtie}$. Let's introduce the notation:

$[T_{\nabla}] = \{\Delta \in [T] : \Delta$ is a ∇-λ-stable, ∇-complete$\}$.

Theorem 15. *Let $JSp(A)/_{\bowtie}$ be a homogeneous factor spectrum, $[T] \in JSp(A)/_{\bowtie}$, then in the class $[T_{\nabla}]$ the relation $J^{\nabla}NF$ satisfies Axioms 1-7.*

Proof. Due to the homogeneity of the factor spectrum, it follows that each class $E_{[T]}$ is elementary with respect to all spectrum. From this follows the perfection of the spectrum itself. Next we use the proof of Theorem 9 modulo the change of ∇, since the perfectness of spectrum, $[T]^*$ is a model complete theory and, accordingly, any formula in the language of this theory is equivalent to $\forall \cap \exists$. $\square$

We now define the abstract concept of Jonsson independence.

Let $[T] \in JSp(A)/_{\bowtie}$, $I = \{(\bar{a}, Y, X) | \bar{a} \in C_{[T]},\ X,\ Y$ be ∇-*cl*-subsets of the semantic model $C_{[T]}$, such that $X \subseteq Y\}$. We will call a set I *a Jonsson independence system of the class* $[T]$ if the following 6 properties are satisfied:

1^0 (invariance) for every triple $(\bar{a}, Y, X) \in I$ and automorphism f of the model $C_{[T]}$, $(f(\bar{a}), f(Y), f(X)) \in I$;

2^0 (local character) for every $\bar{a}$ and Y, there is a countable subset $X \subseteq Y$ such that $(\bar{a}, Y, X) \in I$;

3^0 (finite character) for each $\bar{a}$, X and Y, $(\bar{a}, Y, X) \in I$ if and only if, for all finite tuples $\bar{b}$ in Y, $(\bar{a}, X \cup \bar{b}, X) \in I$;

4^0 (extension) for each $\bar{a}$, X and Y, there exists a tuple $\bar{a}'$, that has the same length and type over X, as $\bar{a}$, so that $(\bar{a}', Y, X) \in I$;

5^0 (symmetry) for each $\bar{a}$, $\bar{b}$ and X, $(\bar{a}, X \cup \bar{b}, X) \in I$ if and only if $(\bar{b}, X \cup \bar{a}, X) \in I$;

6^0 (transitivity) for each $\bar{a}$ and $X \subseteq Y \subseteq Z$, $(\bar{a}, Z, X) \in I$ if and only if $(\bar{a}, Y, X) \in I$ и $(\bar{a}, Z, Y) \in I$.

It follows from the properties of 3^0 and 5^0 that if Y, $Y' \supseteq X$ ∇-*cl*-subsets of the semantic model $C_{[T]}$, then $(\bar{b}, Y', X) \in I\ \forall \bar{b} \in Y$ if and only if $(\bar{b}', Y, X) \in I$ $\forall \bar{b}' \in Y'$. In this case, we say that Y and Y' *are Jonsson independent* over X.

Note also that if $\bar{a}'$ is a subsequence of $\bar{a}$ and $(\bar{a}, Y, X) \in I$, then $(\bar{a}', Y, X) \in I$ also. Indeed, according to the property 3^0 it suffices to check that $(\bar{a}', X \cup \bar{b}, X) \in I$ for all $\bar{b} \in Y$. We know that $(\bar{a}, X \cup \bar{b}, X) \in I$. By the symmetry property 5^0, $(\bar{b}, X \cup \bar{a}, X) \in I$, whence $(\bar{b}, X \cup \bar{a}', X) \in I$, according to property 3^0 and again by property 5^0 we have $(\bar{a}', X \cup \bar{b}, X) \in I$.

A Jonsson independent system I of class $[T]$ is called *good* if it satisfies the following property:

7^0 (amalgamation) let X be an existentially closed model of the theory T, Y, $Y' \supseteq X$ is Jonsson independent over X, $\bar{b}$, $\bar{b}'$ are tuples in $C_{[T]}$, that have the same type p over X and satisfy $(\bar{b}, Y, X) \in I$, $(\bar{b}', Y', X) \in I$, respectively; then there is some tuple $\bar{c}$ in $C_{[T]}$, that implements the same type as $\bar{b}$ over Y and $\bar{b}'$ over Y', and satisfying $(\bar{c}, Y \cup Y', X) \in I$.

Definition 28. *We will say that $\bar{a} \in C_{[T]}$ does not depend on Y over X and write $\bar{a} \perp^J_X Y$, if $(\bar{a}, Y, X) \in I$, where I is a Jonsson good independence system.*

Definition 29. *We will call a Jonsson theory T Jonsson simple (J-simple) if for any ∇-cl-set of B, for any $p \in S^{\nabla}(B)$ there exists a ∇-cl-set $A_0 \subset B$: $|A_0| \leq |T|$,* *such that p does not fork over A_0. A class $[T] \in JSp(A)/_{\bowtie}$ is called J-simple if every theory $\Delta \in [T]$ is J-simple.*

We have a Jonsson version of the Kim-Pillay theorem 14:

Theorem 16. *Let the class $[T] \in JSp(A)/_{\bowtie}$ be J-simple, perfect, ∇-complete. Then for each tuple $\bar{a} \in C_{[T]}$ and ∇-cl-Jonsson sets $X \subseteq Y$ of the model $C_{[T]}$ the following conditions are equivalent:*

(1) *$\bar{a} \perp^J_X Y$ in the language of theory Δ for each $\Delta \in [T]$;*
(2) *$(tp(\bar{a}/Y), X) \in J^{\nabla}NF$ in the language of theory Δ for each $\Delta \in [T]$;*
(3) *for all types $p \in \mathcal{P}$, consistent with $[T]^*$, $X \in \mathcal{X}$ the type p does not fork over X (in the classical sense of S. Shelah [8]), where $[T]^*$ is the center of the class $[T]$.*

Proof. The equivalence of conditions (1) and (2) follows from Theorem 14. The equivalence of conditions (2) and (3) follows from the following implications: since the class $[T]$ is perfect, then from Theorem 15 we have that for a given ∇ in the class $[T]$ the relation $J^{\nabla}NF$ satisfies axioms 1-7, and then by Theorem 8 we have a transition to classical stability in the sense of S. Shelah. □

6. Uncountably categorical central type of the Robinson spectrum

In this section, we study the model-theoretical properties of the Robinson spectrum of an arbitrary signature's arbitrary model. The interest of specialists in Model Theory and universal algebra in the study of ω-categorical universals ([10], sec. 5 from the appendix) is well known. We will consider a more general situation: the Robinson theory is a special case of the Jonsson theory, namely, it is the Jonsson universal. As an additional tool, the technique of working with the central types of the fixed spectrum is used, the elements of which are the Jonsson universals. The central type is obtained by enriching the language with additional constants and a unary predicate. A criterion for uncountable categoricity is obtained for the class of Robinson spectrum in the language of central types.

Definition 30. *A theory T is called Robinson theory if it satisfies the following conditions:*

(1) *T has at least one infinite model;*
(2) *T is universally axiomatizable;*
(3) *T admits the joint embedding property*
(4) *T admits the amalgamation property.*

Let T be a Robinson theory, A be an arbitrary model of signature σ. The Robinson spectrum of the model A is the set:

$$RSp(A) = \{T | T \text{ is Robinson theory in the language of signature } \sigma \text{ and } A \in Mod(T)\}.$$

Consider $RSp(A)/_{\bowtie}$ the factor set of the Jonsson spectrum of the model A with respect to $\bowtie$.

If T is an arbitrary Robinson theory in the language of signature σ, then $E_{[T]} = \bigcup\limits_{\Delta \in [T]} E_{\Delta}$ is the class of all existentially closed models of class $[T] \in RSp(A)/_{\bowtie}$.

Let A be an arbitrary model of signature σ. Let $|RSp(A)/_{\bowtie}| = |K|$, K be some index set. We say that the class $[T] \in RSp(A)/_{\bowtie}$ is a $\aleph$-categorical if any theory

$\Delta \in [T]$ is a $\aleph$-categorical and, respectively, the class $RSp(A)/_{\bowtie}$ will be called a $\aleph$-categorical if for each $j \in K$ the class $[T]_j$ is a $\aleph$-categorical.

Definition 31 ([28], p. 93). *An enrichment $\tilde{T}$ is called admissible if the ∇-type (this means that the ∇ subset of the language L_σ and any formula from this type belongs to ∇) in this enrichment is definable within the framework of $\tilde{T}_\Gamma$-stability, where Γ is the enrichment of the signature σ.*

Definition 32 ([28], p. 93). *A Robinson theory T is called hereditary if in any of its admissible enrichments, any extension is a Robinson theory. The class $[T] \in RSp(A)/_{\bowtie}$ will be called hereditary if each theory $\Delta \in [T]$ is hereditary.*

Consider the general scheme for obtaining the central type for an arbitrary Jonsson theory. Let's denote it by ($\sharp$).

Let C be a semantic model of the theory T, $A \subseteq C$. Let $\sigma_\Gamma = \sigma \cup \Gamma$, where $\Gamma = \{P\} \cup \{c\}$. Let $\bar{T} = Th_\forall(C, a)_{a \in P(C)} \cup Th_\forall(E_T) \cup \{P(c)\} \cup \{"P \subseteq"\}$, where $P(C)$ is the existentially closed submodel of C, $\{"P \subseteq"\}$ is an infinite set of sentences expressing the fact that the interpretation of the symbol P is an existentially closed submodel in the language of the signature σ_Γ. That is, the interpretation of the symbol P is a solution of the equation $P(C) = M \subseteq E_T$ in the language σ_Γ. Due to the heredity of the theory T, the theory $\bar{T}$ is a Jonsson theory. Consider all completions of the theory $\bar{T}$ in the language of signature σ_Γ. Since $\bar{T}$ is a Jonsson theory, then it has its own center, let us denote it by $\bar{T}^*$, this center is one of the above completions of the theory $\bar{T}$. When the signature σ_Γ is restricted to $\sigma \cup P$, due to the laws of first-order logic, since the constant c no longer belongs to this signature, you can replace this constant with a variable symbol, for example x. And then the theory $\bar{T}$ will be a complete 1-type for the variable x.

Using this schema for a class, we get the central type of the class. Consider the class $[T] \in JSp(A)/_{\bowtie}$. Let $Th(C)_{[T]}) = [T]^*$. For any $\Delta \in [T] \in JSp(A)/_{\bowtie}$ denote the theory obtained by the scheme ($\sharp$), by $\bar{\Delta}$. Consider the class $[\bar{T}]$ and then class $[\bar{T}]^*$ after restriction according to the scheme ($\sharp$) becomes the center type of the class $[\bar{T}]$ and is denoted by $P^c_{[T]}$.

We say that a model $M \in E_T$ is *Jonsson minimal* if for any definable $X \subseteq M$ either X is finite or $M \backslash X$ is finite. We say that a theory T *Jonsson strongly minimal*, if every model $M \in E_T$ is minimal. A non-algebraic type containing a Jonsson strongly minimal formula is called *Jonsson strongly minimal.*

Theorem 17 ([29], p. 298). *Let T be universal theory, complete for existential sentences, having a countably algebraically universal model. Then T has an algebraically prime model, which is (Σ, Δ)-atomic.*

Definition 33 ([29], p. 304). *A model A is called the Δ-good algebraically prime model of the theory T if A is a countable model of T and for each model B of the theory T, each $n \in \omega$ and all $a_0, \dots, a_{n-1} \in A$, $b_0, \dots, b_{n-1} \in B$ if*

$$(A, a_0, \dots, a_{n-1}) \equiv_\Delta (B, b_0, \dots, b_{n-1}),$$

then for each $a_n \in A$ there is some $b_n \in B$ such that $(A, a_0, \dots, a_n) \equiv_\Delta (B, b_0, \dots, b_n)$.

Theorem 18 ([29], p. 309). *Let T be $\forall\exists$-theory, complete for existential sentences, admitting R_1. Then the following conditions are equivalent:*

(1) *T has an algebraically prime model;*
(2) *T has $(\exists, \Delta)$-atomic model;*

(3) *T has $(\Delta, \exists)$-atomic model;*
(4) *T has a Δ-good algebraically prime model;*
(5) *T has a single algebraically prime model.*

Theorem 19. *Let $[T]$ be hereditary class from $RSp(A)/_{\bowtie}$, then the following conditions are equivalent:*

(1) *any countable model from $E_{[\overline{T}]}$ has an algebraically prime model extension in $E_{[\overline{T}]}$;*
(2) *$P^c_{[\overline{T}]}$ is the strongly minimal type, where $P^c_{[\overline{T}]}$ is the central type of $[\overline{T}]$.*

Proof. (1)⇒(2). Consider a semantic model $C_{[\overline{T}]}$ of the class $[\overline{T}]$. From Definitions 2 and 3 it follows that the model $C_{[\overline{T}]}$ is ω -universal. Since its cardinality is greater than countable, consider its countable elementary submodel D. Since the model $C_{[\overline{T}]}$ is existentially closed (see Lemma 1), its elementary submodel D is also existentially closed. Hence we have that it is countably algebraically universal. Hence it remains to apply Theorem 17, according to which every theory $\overline{\Delta} \in [\overline{T}]$ has an algebraically prime model A_0. We define by induction $A_{\delta+1}$, which will be an algebraically prime model extension of A_δ and $A_\lambda = \bigcup\{A_\delta \mid \delta < \lambda\}$. Then let $\overline{A} = \bigcup\{A_\delta | \delta < \omega_1\}$. Suppose that $B \models \Delta$ and $cardB = \omega_1$. To show that $B \approx A$, let us expand B into a chain $\{B_\delta | \delta < \omega_1\}$ of countable models. Due to the Jonssonness of the theory $\overline{\Delta}$, this is possible. Define a function $g : \omega_1 \to \omega_1$ and a chain $\{f_\delta : A_{g\delta} \to B_\delta \mid 0 < \delta < \omega_1\}$ of the isomorphisms by induction on δ:

1) $g0 = 0$ and $f_0 : A_0 \to B_0$;
2) $g\lambda = \bigcup\{g\delta|\delta < \lambda\}$ and $f_\lambda = \bigcup\{f_\delta|\delta < \lambda\}$;
3) $f_{\delta+1}$ is equal to the union of the chain $\{f^\gamma_\delta|\gamma \le \rho\}$, which is determined by induction on γ;
4) $f^0_{\delta+1} = f_\delta$, $f^\lambda_{\delta+1} = \bigcup\{|f^\gamma_{\delta+1}|\gamma < \lambda\}$;
5) suppose that $f^\gamma_1 : A_{g\delta+\gamma} \to B_{\delta+1}$. If $f^\gamma_{\delta+1}$ is mapping onto, then $\rho = \gamma$. Otherwise, by virtue of the algebraic primeness of $A_{g\delta+\gamma+1}$, we can extend $f^\gamma_{\delta+1}$ to $f^{\gamma+1}_{\delta+1} : A_{g\delta+\gamma+1} \to B_{\delta+1}$;
6) $g(\delta + 1) = g\delta + \rho$.

It is clear that $f = \bigcup\{| f_\delta \mid \delta < \omega_1\}$ maps isomorphically $\overline{A}$ to B. Now it remains to apply Theorem 18. Since B is an arbitrary model of the theory Δ, and $\overline{A}$ is the single algebraic prime and existentially closed model by virtue of the condition and construction, it follows that E_Δ for each $\overline{\Delta} \in [\overline{T}]$ in uncountable cardinality has a single model, which means that the semantic model $C_{[\overline{T}]}$ is saturated, that is, the class $[\overline{T}]$ will be perfect. It follows that $Mod[\overline{T}]^* = E_{[\overline{T}]}$. Therefore, $[\overline{T}]^*$ is a ω_1-categorical. By virtue of the Lachlan-Baldwin theorem, in the theory $[\overline{T}]^*$ there exists a strongly minimal formula. Passing to the central type, we get a nonprinciple type that contains a Jonsson strongly minimal formula, therefore, the type is Jonsson strongly minimal.

(2) ⇒ (1). Since $P^c_{[\overline{T}]}$ is a strongly minimal type, when returning to the signature $\sigma_\Gamma = \sigma \cup \Gamma$ this type becomes $[\overline{T}]^*$ theory. Since this theory is the center of the class $[\overline{T}]$, then it is complete. Let us show that $[\overline{T}]^*$ is ω_1-categorical. By inductance, for any models $A, B \in Mod[\overline{T}]^*$ there exists models $A', B' \in E_{[\overline{T}]}$ and isomorphic embeddings $f : A \to A'$, $g : B \to B'$. Without loss of generality, we can assume that $|A'| = |B'| = \omega_1$. Suppose $A \ncong B$, then $A' \ncong B'$. Therefore, there is $\varphi(x) \in B(At)$

such that $A' \models \varphi(x)$ and $B' \models \neg\varphi(x)$. Since $[T]$ is a hereditary class, then $[\overline{T}] \in RSp(A)/_{\bowtie}$ and $A' \in Mod([\overline{T}]^*)$ due to the universal axiomatizability of this class and the fact that A', as an existentially closed model, is isomorphically embedded in the semantic model C of the class $[T]$. Since $[\overline{T}]^* = Th(C)$, which means it is complete, then $[\overline{T}]^* \vdash \exists x\varphi(x)$. Since A' and B' are Jonsson minimal, then either $\varphi(A')$ is finite, or $A' \setminus \varphi(A')$ is finite. Let $\varphi(A')$ be finite, then there is a $\forall\exists$-sentence ψ that shows that $\varphi(A')$ is finite and $[\overline{T}]^* \vdash \forall\exists(\varphi \& \psi)$, hence $B' \models \psi(x)$, but $B' \models \psi(x) \& \neg\varphi(x)$, but at the same time, since $A', B' \in E_{[\overline{T}]}$, $A' \equiv_{\forall\exists} B'$, then we got a contradiction with a strongly minimality.

If the definable complement of the formula is finite in the model A', the proof of the contradiction is similar to the above. That is, we have shown that $[\overline{T}]^*$ is a ω_1-categorical.

Since the theory $[\overline{T}]^*$ is a ω_1-categorical, then by Morley's uncountable categoricity theorem, it is perfect. Then $[\overline{T}]^*$ is a model complete theory and $Mod[\overline{T}]^* = E_{\overline{\Delta}}$ for each $\overline{\Delta} \in [\overline{T}]$ (by the criterion of the perfectness of Jonsson theory), i.e. $Mod[\overline{T}]^* = E_{[\overline{T}]}$. If $[\overline{T}]^*$ is a model complete, then any isomorphic embedding is elementary. Since $[\overline{T}]^*$ is a complete theory, by virtue of Morley's theorem, we obtain what is required. □

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