S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 18, №1, стр. 423–432 (2021) УДК 517.54
DOI 10.33048/semi.2021.18.029 MSC 30CX65

A VERSION OF SCHWARZ'S LEMMA FOR MAPPINGS WITH WEIGHTED BOUNDED DISTORTION

M.V. TRYAMKIN

Abstract. We consider the class of mappings generalizing qusiregular mappings. Every mapping from this class is defined in a domain of Euclidean n-space and possesses the following properties: it is open, continuous, and discrete, it belongs locally to the Sobolev class W_q^1, it has finite distortion and nonnegative Jacobian, and its function of weighted (p, q)-distortion is integrable to a certian power depending on p and q, where $n - 1 < q \leqslant p < \infty$. We obtain an analog of Schwarz's lemma for such mappings provided that $p \geqslant n$. The technique used is based on the spherical symmetrization procedure and the notion of Grötzsch condenser.

Keywords: capacitary estimates, Grötzsch condenser, mappings with weighted bounded distortion, Schwarz's lemma, spherical symmetrization.

1. Introduction

The maximum modulus principle known from the theory of functions of a complex variable has a number of consequences, one of which is the following.

Schwarz's lemma ([1, 3.5.5]). *Let $\mathbb{D}$ be the disk $\{z \in \mathbb{C} \colon |z| < 1\}$. If $f \colon \mathbb{D} \to \mathbb{D}$ is a holomorphic function such that $f(0) = 0$, then*
(a) $|f(z)| \leqslant |z|$ *for all* $z \in \mathbb{D}$;
(b) $|f'(0)| \leqslant 1$.

Tryamkin, M.V., A version of Schwarz's lemma for mappings with weighted bounded distortion.

The study was carried out within the framework of the state contract of the Sobolev Institute of Mathematics (Project no. 0314-2019-0006).

Received March, 2, 2021, published April, 18, 2021.

Schwarz's lemma implies that, under a conformal mapping of the disk $|z| < 1$ onto the disk $|w| < 1$ leaving the origin fixed, the image of every circle $|z| = t$ lies inside the disk $|w| < t$.

For quasiconformal mappings in space, similar estimates were obtained by Soviet mathematician B. V. Šabat [2] and Japanese mathematician K. Ikoma [3]. To write them out introduce some notation. Given $x_0 \in \mathbb{R}^n$ and $t > 0$, we let $B^n(x_0, t) = \{x \in \mathbb{R}^n : |x - x_0| < t\}$, $B^n(t) = B^n(0, t)$, $B^n = B^n(1)$; a closed ball will be denoted by putting a bar over B. Recall that a homeomorphism $f : D \to D'$ of domains D and D' in $\mathbb{R}^n$, $n \geqslant 2$, is called *K-quasiconformal*, $K \in [1, \infty)$, if the two-sided modulus estimate (the reader may consult [4, Ch. 4] for more information pertaining the modulus)

$$\frac{1}{K} \operatorname{mod}_n \Gamma \leqslant \operatorname{mod}_n f(\Gamma) \leqslant K \operatorname{mod}_n \Gamma$$

holds for every family Γ of curves lying in the domain D. A relation analogous to item (a) of Schwarz's lemma was deduced in [2]: if $f : B^3 \to B^3$ is a K-quasiconformal mapping with the property $f(0) = 0$, then for $0 < |x| < 1$

$$\left[\Psi_3\left(\frac{1}{|x|}\right)\right]^{1/\sqrt{K}} \leqslant \Psi_3\left(\frac{1}{|f(x)|}\right) \leqslant \left[\Psi_3\left(\frac{1}{|x|}\right)\right]^{\sqrt{K}},$$

where the function Ψ_3 is defined by the equality $\operatorname{mod}_3 R_G^3(a) = \ln \Psi_3(a)$. Here the symbol $R_G^3(a)$ means *the Grötzsch ring* corresponding to a, i. e., the ring domain whose complementary components are the closed ball $\overline{B}^3$ and the ray $\{(x_1, 0, 0) \in \mathbb{R}^3 : x_1 \geqslant a\}$, where $a > 1$. The quantity $\operatorname{mod}_3 R_G^3(a)$ is computed by the formula

$$\operatorname{mod}_3 R_G^3(a) = \sqrt{\frac{4\pi}{\operatorname{mod}_3 \Gamma_{R_G^3(a)}}},$$

in which $\Gamma_{R_G^3(a)}$ denotes the family of all curves joining the complementary components of the Grötzsch ring.

An inequality in the spirit of item (b) of Schwarz's lemma was derived in [3]: if $f : B^3 \to B^3$ is a K-quasiconformal mapping with the property $f(0) = 0$, then

$$\varliminf_{x \to 0} \frac{|f(x)|}{|x|^{1/\sqrt{K}}} \leqslant 1. \tag{1}$$

For mappings with bounded distortion in the sense of Yu. G. Reshetnyak [5], a version of Schwarz's lemma is due to S. Rickman [6, Ch. III, Theorem 1.10]: if $f : B^n \to B^n$ is a mapping with bounded distortion and $f(0) = 0$, then for every $x \in B^n$

$$|f(x)| \leqslant \nu_n^{-1}(K_I(f)\nu_n(|x|)),$$

where ν_n^{-1} is the inverse of the function $\nu_n : [0, 1) \to \mathbb{R}$ defined by the equality $\nu_n(t) = \operatorname{cap}_n E_G(t)$, in which the symbol $E_G(t)$ indicates to the ring condenser $(B^n, \{(x_1, 0, \ldots, 0) \in \mathbb{R}^n : 0 \leqslant x_1 \leqslant t\})$ called *the Grötzsch condenser* corresponding to t; the number $K_I(f)$ is the inner dilatation of the mapping f.

Mappings with bounded distortion have a multitude of generalizations, for some of which there exist statements akin to Schwarz's lemma — for instance, Ukrainian mathematician E. A. Sevost'yanov [7, Theorem 2.9.1] proved a relation similar to inequality (1) for open discrete ring mappings.

The purpose of this paper is to establish an analog of Schwarz's lemma for the class of mappings with weighted bounded (p,q)-distortion that was introduced by S. K. Vodop'yanov [12]. This class serves as a natural generalization of the class of mappings with bounded distortion in the sense of Yu. G. Reshetnyak; it was to a certain extent studied by the author in the papers [8], [9], [10], [11], where modulus estimates were found, some results about asymptotic values were produced, and the boundary correspondence problem was considered.

Definition 1. Let $n \geqslant 2$, and let $\theta, \hat{\theta}\colon \mathbb{R}^n \to [0,\infty]$ be locally summable functions (called *weight* functions) such that $0 < \theta < \infty$ and $0 < \hat{\theta} < \infty$ almost everywhere. A mapping $f\colon D \to \mathbb{R}^n$ of a domain $D \subset \mathbb{R}^n$ is said to be *a mapping with* $(\theta,\hat{\theta})$-*weighted bounded* (p,q)-*distortion*, $n-1 < q \leqslant p < \infty$, provided that

1) f is continuous, open, and discrete;
2) f belongs to the Sobolev class $W^1_{q,\mathrm{loc}}(D;\mathbb{R}^n)$;
3) the determinant $J(x,f)$ of the matrix $Df(x)$ of the partial derivatives of the mapping f is nonnegative for almost all $x \in D$;
4) the mapping f has finite distortion, i. e., for almost all $x \in D$, the equality $J(x,f) = 0$ implies that $Df(x) = 0$;
5) the function of $(\theta,\hat{\theta})$-weighted (p,q)-distortion

$$D \ni x \mapsto \mathcal{K}^{\theta,\hat{\theta}}_{q,p}(x,f) = \begin{cases} \frac{\theta^{1/q}(x)|Df(x)|}{\hat{\theta}^{1/p}(f(x))J(x,f)^{1/p}} & \text{if} \quad J(x,f) \neq 0, \\ 0 & \text{if} \quad J(x,f) = 0, \end{cases}$$

belongs to the class $L_\varkappa(D)$, where $\varkappa$ is given by the condition $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$ (for $p = q$, we set $\varkappa = \infty$).

Denote the quantity $\|\mathcal{K}^{\theta,\hat{\theta}}_{q,p}(\cdot,f)\|_{L_\varkappa(D)}$ by $K^{\theta,\hat{\theta}}_{q,p}(f;D)$. Henceforth, $\hat{\theta} \equiv 1$.

2. Results

Statements generalizing item (a) and item (b) of Schwarz's lemma for mappings from Definition 1 are contained in Theorem 3 and Theorem 4. To discuss these theorems, we require some preliminaries.

Let $\mathcal{H}^k$ stand for the k-dimensional Hausdorff measure in $\mathbb{R}^n$. Set $\Omega_n = \mathcal{H}^n(B^n)$, $\sigma_n = \mathcal{H}^{n-1}(\partial B^n)$. Given points $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, we employ $[x,y]$ to designate the line segment joining them.

Definition 2. Assume that D is a nonempty subset of $\mathbb{R}^n$, and F_0, F_1 are nonempty subsets of $\overline{D}$. Let D be open in $\mathbb{R}^n$, and let F_0 and F_1 be closed in $\overline{D}$. The triple $E = (F_0, F_1; D)$ will be referred to as a *condenser*. Let $1 < p < \infty$. The ω-*weighted* p-*capacity* of the condenser E is the number

$$\operatorname{cap}^\omega_p E = \inf \int\limits_D |\nabla u(x)|^p \omega(x)\, dx,$$

the infimum being taken over all functions $u \in C(D) \cap W^1_\infty(D) \cap L^1_p(D,\omega)$ such[1] that $u \geqslant 1$ (respectively, $u \leqslant 0$) in some neighborhood of the set F_1 (respectively,

[1] A locally summable function $u\colon D \to \mathbb{R}$ belongs to the space $L^1_p(D,\omega)$ provided that it has the generalized gradient ∇u such that $\int_D |\nabla u(x)|^p \omega(x)\, dx < \infty$. For $\omega \equiv 1$, we use the symbol $L^1_p(D)$ istead of $L^1_p(D,1)$.

F_0). Here the weight function $\omega\colon \mathbb{R}^n \to [0,\infty]$ is locally summable and $0<\omega<\infty$ almost everywhere.

If an open set A contains a compact set C, then the condenser $(\partial A, C; A)$ is denoted simply by (A, C).

If $\omega \equiv 1$, then, in the definition of capacity, the infimum is taken over the functions of a wider class, namely $C(D)\cap L^1_p(D)$. We write $\operatorname{cap}_p E$ in place of $\operatorname{cap}^1_p E$.

Theorem 1 ([12]). *Let D be a domain in $\mathbb{R}^n$. Suppose that $f\colon D\to\mathbb{R}^n$ is a mapping with $(\theta,1)$-weighted bounded (p,q)-distortion, $n-1<q\leqslant p<\infty$, and the weight function $\omega=\theta^{-\frac{n-1}{q-(n-1)}}$ is locally summable. Set $s=p/(p-(n-1))$, $r=q/(q-(n-1))$. If $E=(A,C)$ is a condenser, where $A\Subset D$ and the set $C\subset A$ is compact*[2]*, then*

$$(\operatorname{cap}_s f(E))^{1/s}\leqslant K^{\theta,1}_{q,p}(f;D)^{n-1}(\operatorname{cap}^{\omega}_r E)^{1/r}.$$

The proof of Theorem 3 makes use of a special geometric transformation that gives the condenser a symmetry of a certain kind while changing its capacity monotonically. This transformation goes back to Gehring's works (see, for example, [13]).

Definition 3. Let a set $A\subset\mathbb{R}^n$ be open or closed. Its *spherical symmetrization* with respect to the ray $\{(-t,0,\dots,0)\in\mathbb{R}^n\colon t\geqslant 0\}$ is called the set $\operatorname{Sym} A$ that is defined as follows:

1) $\operatorname{Sym} A\cap\partial B^n(t)=\partial B^n(t)$ if and only if $\partial B^n(t)\subset A$;

2) $\operatorname{Sym} A\cap\partial B^n(t)=\varnothing$ if and only if $\partial B^n(t)\cap A=\varnothing$;

3) if A is open (closed) and $\varnothing\neq A\cap\partial B^n(t)\neq\partial B^n(t)$, then $\operatorname{Sym} A\cap\partial B^n(t)$ is an open (closed) spherical cap in $\partial B^n(t)$ centered at $(-t,0,\dots,0)$ satisfying the property $\mathcal{H}^{n-1}(\operatorname{Sym} A\cap\partial B^n(t))=\mathcal{H}^{n-1}(A\cap\partial B^n(t))$.

Under the spherical symmetrization of a condenser $E=(A,C)$ is understood the condenser $\operatorname{Sym} E=(\operatorname{Sym} A,\operatorname{Sym} C)$.

Theorem 2 ([14]). *Let E be a condenser and $p>1$. The spherical symmetrization procedure does not increase the p-capacity of E:*

$$\operatorname{cap}_p E\geqslant\operatorname{cap}_p \operatorname{Sym} E.$$

Let e_1 denote the vector $(1,0,\dots,0)\in\mathbb{R}^n$, and let $0\leqslant t<1$. The condenser $E_G(t)=(B^n,[0,te_1])$ is called *the Grötzsch condenser* corresponding to t. We write $\nu^{\omega}_p(t)=\operatorname{cap}^{\omega}_p E_G(t)$, $\nu_p(t)=\operatorname{cap}_p E_G(t)$.

It is known that the quantity $\nu_n(\cdot)$ strictly increases (see [6, Ch. III, Lemma 1.2]). Our next task is to demonstrate that this property is preserved in some other cases.

Lemma 1. *For $1<s\leqslant n$, the function $\nu_s\colon[0,1)\to\mathbb{R}$ is strictly increasing.*

Proof. Let $0<t_1<t_2<1$. Consider the three condensers:

$$E_1=(B^n(t_2/t_1),[0,t_2e_1]),\ E_2=(B^n,[0,t_2e_1]),\ E_3=(B^n(t_2/t_1),\overline{B}^n).$$

It is readily seen that the following inclusions hold:

$$[0,t_2e_1]\subset B^n\subset\overline{B}^n\subset B^n(t_2/t_1).$$

[2]The notation $A\Subset D$ means that the closure $\overline{A}$ is compact and $\overline{A}\subset D$.

In light of Theorem 2.6 in [15], these inclusions lead to

$$\operatorname{cap}_s E_1 \leqslant \left((\operatorname{cap}_s E_2)^{\frac{1}{1-s}} + (\operatorname{cap}_s E_3)^{\frac{1}{1-s}} \right)^{1-s},$$

whence

$$(\operatorname{cap}_s E_1)^{\frac{1}{1-s}} \geqslant \nu_s(t_2)^{\frac{1}{1-s}} + (\operatorname{cap}_s E_3)^{\frac{1}{1-s}}. \tag{2}$$

Define the transformation $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ by setting $\varphi(x) = (t_1/t_2)x$ for every $x \in \mathbb{R}^n$. Then $\varphi(E_1) = (B^n, [0, t_1 e_1])$. Invoking the well-known result about the behaviour of the modulus under similarity transformations (see Theorem 4.2.18 in [4][3]), we obtain that

$$\nu_s(t_1) = \operatorname{cap}_s \varphi(E_1) = \left(\frac{t_1}{t_2}\right)^{n-s} \operatorname{cap}_s E_1.$$

Consequently, (2) takes the form

$$\left(\frac{t_2}{t_1}\right)^{\frac{n-s}{1-s}} \nu_s(t_1)^{\frac{1}{1-s}} \geqslant \nu_s(t_2)^{\frac{1}{1-s}} + (\operatorname{cap}_s E_3)^{\frac{1}{1-s}}.$$

Remind the reader that the inequalities $0 < t_1 < t_2 < 1$ and $1 < s \leqslant n$ hold. Hence,

$$\nu_s(t_1)^{\frac{1}{1-s}} \geqslant \nu_s(t_2)^{\frac{1}{1-s}} + (\operatorname{cap}_s E_3)^{\frac{1}{1-s}},$$

and therefore $\nu_s(t_1)^{\frac{1}{1-s}} > \nu_s(t_2)^{\frac{1}{1-s}}$. Since $s > 1$, we conclude that $\nu_s(t_1) < \nu_s(t_2)$, which is what we sought to prove. □

The next result serves as an analog of item (a) of Schwarz's lemma for mappings described in Definition 1.

Theorem 3. *Let $f\colon B^n \to B^n$ be a mapping with $(\theta, 1)$-weighted bounded (p, q)-distortion, $n-1 < q \leqslant p < \infty$, the weight function $\omega = \theta^{-\frac{n-1}{q-(n-1)}}$ locally summable, and $f(0) = 0$. Then for $p \geqslant n$*

$$|f(x)| \leqslant \nu_s^{-1}\left(K_{q,p}^{\theta,1}(f; B^n)^{s(n-1)} \nu_r^{\omega}(|x|)^{s/r} \right),$$

where $s = p/(p-(n-1))$, $r = q/(q-(n-1))$.

Proof. Fix an arbitrary point $x \in B^n$. Consider the condenser $E = (B^n, [0, x])$. Because the mapping f is continuous and $f(0) = 0$, the set $f([0, x])$ is a curve joining 0 and $f(x)$. Performing the symmetrization procedure for the condenser $f(E)$ (see Definition 3), we discover that

$$[-|f(x)|e_1, 0] \subset \operatorname{Sym} f([0, x]).$$

Since f transforms the ball B^n into itself, it certainly follows that $\operatorname{Sym} f(B^n) \subset B^n$. Now apply to the capacities of the condensers $\operatorname{Sym} f(E)$ and $(B^n, [-|f(x)|e_1, 0])$ the monotonicity property, and apply Theorem 2 to the condensers $f(E)$ and $\operatorname{Sym} f(E)$:

$$\operatorname{cap}_s f(E) \geqslant \operatorname{cap}_s \operatorname{Sym} f(E) \geqslant \nu_s(|f(x)|).$$

Estimating the quantity $\operatorname{cap}_s f(E)$ by Theorem 1, we arrive at

$$\nu_s(|f(x)|) \leqslant K_{q,p}^{\theta,1}(f; B^n)^{s(n-1)} \left(\operatorname{cap}_r^{\omega} E\right)^{s/r} = K_{q,p}^{\theta,1}(f; B^n)^{s(n-1)} \nu_r^{\omega}(|x|)^{s/r}.$$

[3]To be precise, Theorem 4.2.18 in [4] says about how the modulus of a curve family changes under a similarity transformation. However, the capacity of a condenser behaves in the same way because of the known result which states that the modulus of a curve family agrees with the capacity of the corresponding condenser.

The condition $p \geqslant n$ implies $s \leqslant n$. It remains to make use of the fact that by Lemma 1 the function $\nu_s(\cdot)$ strictly increases and therefore the inverse $\nu_s^{-1}(\cdot)$ exists and also strictly increases. □

Remark 1. Theorem 3 tells us nothing about the case $n-1 < q \leqslant p < n$. The reason is that we can ensure the strict monotonicity of the function ν_s only for $s \leqslant n$: it is not hard to see where the proof of Lemma 1 "breaks down" if $s > n$.

We will preface the discussion of a theorem analogous to item (b) of Schwarz's lemma with necessary information. The fact that is the content of Lemma 2 can be found in [7]. It is not singled out there as a separate statement: it appears as part of the reasoning in the proof of some lemma. For convenience, we formulate this fact and prove it appealing to Proposition 1 and Proposition 2 (it is these statements that the author of the monograph [7] is referred to when justifying the fact). Although Proposition 2 is a known topological result, we expound on it because we could not find its proof in the literature.

Lemma 2. *Let $f\colon B^n \to B^n$ be a continuous open mapping and $f(0)=0$. Then for every $t \in (0,1)$*

$$B^n(m_f(t)) \subset f(B^n(t)), \tag{3}$$

where $m_f(t) = \min\{|f(x)|\colon |x| = t\}$.

Proof. Assume the contrary and suppose that for some $t \in (0,1)$ there exists $y \in B^n(m_f(t)) \setminus f(B^n(t))$. Since $0 \in B^n(m_f(t)) \cap f(B^n(t))$, we infer that

$$B^n(m_f(t)) \cap f(B^n(t)) \neq \varnothing \neq B^n(m_f(t)) \setminus f(B^n(t)).$$

According to Proposition 1 there exists $y' \in B^n(m_f(t)) \cap \partial f(B^n(t))$. Then in view of Proposition 2 there exists a point $x' \in \partial B^n(t)$ such that $y' = f(x')$. As a result we have: $|f(x')| < \min\{|f(x)|\colon |x| = t\}$, where $|x'| = t$. This is a contradiction, so the proof is complete. □

Proposition 1 ([16, Ch. 5, § 46, I, Theorem 1])**.** *Let A and C be subsets of a topological space X. If the set C is connected and $C \cap A \neq \varnothing \neq C \setminus A$, then $C \cap \partial A \neq \varnothing$.*

Proposition 2. *Let D be a domain in $\mathbb{R}^n$. If $f\colon D \to \mathbb{R}^n$ is a continuous open mapping, then $\partial f(A) \subset f(\partial A)$ whenever $A \Subset D$.*

Proof. Fix arbitrarily $y \in \partial f(A)$. Then there exists a sequence $(x_k) \subset A$ such that $f(x_k) \to y$ as $k \to \infty$. Since $A \Subset D$, there exist a subsequence (x_{k_l}) and a point $x \in \overline{A}$ such that $x_{k_l} \to x$ as $l \to \infty$. By virtue of the continuity of f we have: $f(x_{k_l}) \to f(x)$ as $l \to \infty$. But $f(x_k) \to y$ as $k \to \infty$, which implies that $f(x) = y$. We will be finished with the proof if we establish that $x \in \partial A$. Because $\overline{A} = \operatorname{int} A \cup \partial A$, either $x \in \operatorname{int} A$ or $x \in \partial A$. Assume that $x \in \operatorname{int} A$. Then $y = f(x) \in f(\operatorname{int} A)$. But f is open, and so $f(\operatorname{int} A) \subset \operatorname{int} f(A)$. Therefore, $y \in \operatorname{int} f(A)$, which contradicts the inclusion $y \in \partial f(A)$. Thus, $x \in \partial A$, whence $y = f(x) \in f(\partial A)$. □

The next result is the key ingredient in the proof of Theorem 4.

Proposition 3 ([17, 2.2.3]). *Let a compact set C be contained in an open set $A \subset \mathbb{R}^n$ and let $p > 1$. The capacity of the condenser (A, C) can be estimated as follows:*

$$\operatorname{cap}_p(A,C) \geqslant \begin{cases} \sigma_n^{\frac{p}{n}} \left|\frac{p-n}{p-1}\right|^{p-1} \left|\mathcal{H}^n(A)^{\frac{p-n}{n(p-1)}} - \mathcal{H}^n(C)^{\frac{p-n}{n(p-1)}}\right|^{1-p} & \text{for } p \neq n, \\ n^{n-1}\sigma_n \left(\ln \frac{\mathcal{H}^n(A)}{\mathcal{H}^n(C)}\right)^{1-n} & \text{for } p = n. \end{cases}$$

We will also need one more statement.

Lemma 3. *Let $t_0 > 0$. Suppose that we are given mappings $f\colon B^n(t_0) \to \mathbb{R}^n$ and $\Phi\colon (0, t_0) \to (0, \infty)$. If the mapping f is continuous, then*

$$\varliminf_{x\to 0} \frac{|f(x)|}{\Phi(|x|)} = \varliminf_{t\to 0} \frac{m_f(t)}{\Phi(t)},$$

where $m_f(t) = \min\{|f(x)|\colon |x| = t\}$.

Proof. We must establish the equality

$$\lim_{\varepsilon\to 0} \inf_{x\in B^n(\varepsilon)\setminus\{0\}} \frac{|f(x)|}{\Phi(|x|)} = \lim_{\varepsilon\to 0} \inf_{0<t<\varepsilon} \frac{m_f(t)}{\Phi(t)}.$$

To this end, fix arbitrarily $\varepsilon > 0$ and show that

$$\inf_{x\in B^n(\varepsilon)\setminus\{0\}} \frac{|f(x)|}{\Phi(|x|)} = I,$$

where

$$I = \inf_{0<t<\varepsilon} \frac{m_f(t)}{\Phi(t)}. \tag{4}$$

It suffices to check the two assertions:

(i) for every point $x \in B^n(\varepsilon) \setminus \{0\}$

$$\frac{|f(x)|}{\Phi(|x|)} \geqslant I;$$

(ii) for each $\delta > 0$ there exists a point $x_\delta \in B^n(\varepsilon) \setminus \{0\}$ such that

$$\frac{|f(x_\delta)|}{\Phi(|x_\delta|)} < I + \delta.$$

Let us first verify assertion (i). Take any $x \in B^n(\varepsilon) \setminus \{0\}$. Denote $t = |x|$. Then

$$\frac{|f(x)|}{\Phi(|x|)} \geqslant \frac{\min\{|f(x)|\colon |x| = t\}}{\Phi(t)} = \frac{m_f(t)}{\Phi(t)} \geqslant I.$$

We now proceed to verify assertion (ii). Fix arbitrarily $\delta > 0$. Relation (4) means that there exists $t_\delta \in (0, \varepsilon)$ satisfying the inequality

$$\frac{m_f(t_\delta)}{\Phi(t_\delta)} < I + \delta.$$

In view of the continuity of the mapping f, the minimum in the definition of $m_f(t_\delta)$ is attained for some x_δ, i. e., $m_f(t_\delta) = |f(x_\delta)|$, where $|x_\delta| = t_\delta$. Hence,

$$\frac{|f(x_\delta)|}{\Phi(|x_\delta|)} = \frac{m_f(t_\delta)}{\Phi(t_\delta)} < I + \delta.$$

$\square$

We are now in a position to discuss a statement resembling item (b) of Schwarz's lemma for mappings described in Definition 1.

Theorem 4. *Let* $f\colon B^n \to B^n$ *be a mapping with* $(\theta, 1)$*-weighted bounded* (p, q)*-distortion,* $n-1 < q \leqslant p < \infty$*, the weight function* $\omega = \theta^{-\frac{n-1}{q-(n-1)}}$ *locally summable, and* $f(0) = 0$*. Set* $s = p/(p-(n-1))$*,* $r = q/(q-(n-1))$*. Denote by* $C_r^\omega(t)$ *the capacity* $\operatorname{cap}_r^\omega E(t)$ *of the spherical condenser* $E(t) = (B^n, \overline{B}^n(t))$*,* $0 < t < 1$*. Then for* $p = n$

$$\varlimsup_{x\to 0} \frac{|f(x)|}{\Phi_1(|x|)} \leqslant 1,$$

where

$$\Phi_1(t) = \frac{\mathcal{H}^n(f(B^n))^{\frac{1}{n}}}{\Omega_n^{\frac{1}{n}}} \exp\left[-\frac{\sigma_n^{\frac{1}{n-1}} C_r^\omega(t)^{\frac{n}{r(1-n)}}}{K_{q,p}^{\theta,1}(f; B^n)^n}\right];$$

for $p > n$

$$\varlimsup_{x\to 0} \frac{|f(x)|}{\Phi_2(|x|)} \leqslant 1,$$

where

$$\Phi_2(t) = \frac{1}{\Omega_n^{\frac{1}{n}}} \left[\frac{(n-s)\sigma_n^{\frac{s}{n(s-1)}} C_r^\omega(t)^{\frac{s}{r(1-s)}}}{(s-1) K_{q,p}^{\theta,1}(f; B^n)^{\frac{s(n-1)}{s-1}}} + \mathcal{H}^n(f(B^n))^{\frac{s-n}{n(s-1)}}\right]^{\frac{s-1}{s-n}}.$$

Proof. The measures of the sets involved in (3) are related by the inequality $\Omega_n m_f^n(t) \leqslant \mathcal{H}^n(f(B^n(t)))$, whence

$$m_f(t) \leqslant \frac{\mathcal{H}^n(f(B^n(t)))^{\frac{1}{n}}}{\Omega_n^{\frac{1}{n}}}. \tag{5}$$

We are going to treat the two cases.

Case $p = n$. Here $s = n$. Estimating the capacity $\operatorname{cap}_s f(E)$ with the aid of Proposition 3, and applying Theorem 1, one deduces that

$$\left(\ln \frac{\mathcal{H}^n(f(B^n))}{\mathcal{H}^n(f(\overline{B}^n(t)))}\right)^{1-n} \leqslant \frac{1}{n^{n-1}\sigma_n} \operatorname{cap}_n f(E) \leqslant \frac{K_{q,p}^{\theta,1}(f; B^n)^{n(n-1)}}{n^{n-1}\sigma_n} C_r^\omega(t)^{\frac{n}{r}}.$$

Therefore,

$$\mathcal{H}^n(f(B^n(t))) \leqslant \mathcal{H}^n(f(\overline{B}^n(t))) \leqslant \mathcal{H}^n(f(B^n)) \exp\left[-K_1 C_r^\omega(t)^{\frac{n}{r(1-n)}}\right],$$

where

$$K_1 = \left[\frac{K_{q,p}^{\theta,1}(f; B^n)^{n(n-1)}}{n^{n-1}\sigma_n}\right]^{\frac{1}{1-n}},$$

i.e., $\mathcal{H}^n(f(B^n(t))) \leqslant \Omega_n \Phi_1^n(t)$. Appealing to inequality (5) and Lemma 3, we conclude that

$$\varlimsup_{x\to 0} \frac{|f(x)|}{\Phi_1(|x|)} = \varlimsup_{t\to 0} \frac{m_f(t)}{\Phi_1(t)} \leqslant \varlimsup_{t\to 0} \frac{\mathcal{H}^n(f(B^n(t)))^{\frac{1}{n}}}{\Omega_n^{\frac{1}{n}}} \frac{1}{\Phi_1(t)} \leqslant 1.$$

Case $p > n$. Here $s < n$. As in the preceding case, we make use of Proposition 3 and Theorem 1:

$$\left(\mathcal{H}^n(f(\overline{B}^n(t)))^{\frac{s-n}{n(s-1)}} - \mathcal{H}^n(f(B^n))^{\frac{s-n}{n(s-1)}}\right)^{1-s} \leqslant$$

$$\leqslant \sigma_n^{-\frac{s}{n}} \left(\frac{n-s}{s-1}\right)^{1-s} \operatorname{cap}_s f(E) \leqslant \frac{(s-1)^{s-1} K_{q,p}^{\theta,1}(f;B^n)^{s(n-1)}}{\sigma_n^{\frac{s}{n}}(n-s)^{s-1}} C_r^{\omega}(t)^{\frac{s}{r}}.$$

Consequently,

$$\mathcal{H}^n(f(B^n(t))) \leqslant \mathcal{H}^n(f(\overline{B}^n(t))) \leqslant \left[K_2 C_r^{\omega}(t)^{\frac{s}{r(1-s)}} + \mathcal{H}^n(f(B^n))^{\frac{s-n}{n(s-1)}}\right]^{\frac{n(s-1)}{s-n}},$$

where

$$K_2 = \left[\frac{(s-1)^{s-1} K_{q,p}^{\theta,1}(f;B^n)^{s(n-1)}}{\sigma_n^{\frac{s}{n}}(n-s)^{s-1}}\right]^{\frac{1}{1-s}},$$

i. e., $\mathcal{H}^n(f(B^n(t))) \leqslant \Omega_n \Phi_2^n(t)$. It remains to apply inequality (5) and Lemma 3 in the same way as in the case $p = n$. □

Remark 2. The proof of Theorem 4 heavily relies on the possibility to deliver an upper bound for the measure $\mathcal{H}^n(f(B^n(t)))$ of the image of the ball $B^n(t)$ in terms of the capacity $C_r^{\omega}(t)$ of the spherical condenser $(B^n, \overline{B}^n(t))$. The case $n-1 < q \leqslant p < n$ was left untouched because for $p < n$ Proposition 3 gives a lower bound for $\mathcal{H}^n(f(B^n(t)))$.

Remark 3. The generality of the class of mappings described in Definition 1 makes it difficult to apply Theorem 3 and Theorem 4 in practice. To be more specific, the obstacle is that we usually do not know explicit expressions for the capacity $\nu_r^{\omega}(t)$ of the Grötzsch condenser $(B^n, [0, te_1])$ and for the capacity $C_r^{\omega}(t)$ of the spherical condenser $(B^n, \overline{B}^n(t))$, unless the weight ω is identically equal to one. There are no problems when the weight is, for example, a power function. According to item 2.22 of the monograph [15], for the weight $\omega(x) = |x|^{\lambda}$, where $\lambda > -n$, the capacity $C_r^{\omega}(t)$ is computed as follows:

$$C_r^{\omega}(t) = \begin{cases} c(n,r,\lambda)\left(1 - t^{\frac{r-n-\lambda}{r-1}}\right)^{1-r} & \text{for } r-n-\lambda \neq 0, \\ \sigma_n \left(\ln \frac{1}{t}\right)^{1-n} & \text{for } r-n-\lambda = 0. \end{cases}$$

For instance, if we deal with a mapping f with $(|x|^{\alpha}, 1)$-weighted bounded (n,n)-distortion, when $\alpha \neq 0$ and $\alpha < n/(n-1)$, then Theorem 4 yields

$$\Phi_1(t) = \frac{\mathcal{H}^n(f(B^n))^{\frac{1}{n}}}{\Omega_n^{\frac{1}{n}}} e^{-K_1(1-t^{\alpha})}, \text{ where } K_1 = \frac{\sigma_n^{\frac{1}{n-1}} c(n,\alpha)^{\frac{1}{1-n}}}{K_{n,n}^{|x|^{\alpha},1}(f,B^n)^n}.$$

If f is a usual mapping with bounded distortion, i. e., $p = q = n$ and $\theta \equiv 1$, then

$$\Phi_1(t) = \frac{\mathcal{H}^n(f(B^n))^{\frac{1}{n}}}{\Omega_n^{\frac{1}{n}}} t^{1/K_{n,n}^{1,1}(f,B^n)^n}.$$

In particular, when $f\colon B^n \to B^n$ is a K-quasiconformal mapping leaving the origin fixed, we have:

$$f(B^n) = B^n, \quad K = K_{n,n}^{1,1}(f,B^n)^{n(n-1)},$$

whence $\Phi_1(t) = t^{1/K^{\frac{1}{n-1}}}$, which for $n = 3$ leads to (1).

References

[1] J.R. Muir, *Complex analysis: a modern first course in function theory*, John Wiley & Sons, 2015. MR3328716

[2] B.V. Šabat, *On the theory of quasiconformal mappings in space*, Soviet Math. Dokl., **1** (1960), 730–733. MR0130370

[3] K. Ikoma, *On a theorem of Schwarz type for quasiconformal mappings in space*, Nagoya Math. J., **29** (1967), 19–30. MR0213551

[4] F.W. Gehring, G.J. Martin, B.P. Palka, *An Introduction to the Theory of Higher-Dimensional Quasiconformal Mappings*, Math. Surveys Monogr., 216, Amer. Math. Soc., Providence, RI, 2017. MR3642872

[5] Yu.G. Reshetnyak, *Space Mappings with Bounded Distortion*, Amer. Math. Soc., Providence, 1989. MR0994644

[6] S. Rickman, *Quasiregular Mappings*, Springer-Verlag, Berlin, 1993. MR1238941

[7] E.A. Sevost'yanov, *Investigation of space mappings by the geometric method*, (in Russian), Kiev, Naukova dumka, 2014.

[8] M.V. Tryamkin, *Modulus inequalities for mappings with weighted bounded (p, q)-distortion*, Sib. Math. J., **56**:6 (2015), 1114–1132 . MR3492915

[9] M.V. Tryamkin, *On asymptotic curves and values in the theory of mappings with weighted bounded distortion*, Sib. Elektron. Mat. Izv., **12** (2015), 688–697. MR3493798

[10] M.V. Tryamkin, *Asymptotic curves and asymptotic values for mappings with weighted bounded (p, q)-distortion*, Russian Math. (Iz. VUZ), **60**:1 (2016), 76–80. MR3586666

[11] M.V. Tryamkin, *Boundary correspondence for homeomorphisms with weighted bounded (p, q)-distortion*, Math. Notes, **102**:3-4 (2017), 591–595 . MR3706881

[12] A.N. Baykin, S.K. Vodop'yanov, *Capacity estimates, Liouville's theorem, and singularity removal for mappings with bounded (p, q)-distortion*, Sib. Math. J., **56**:2 (2015), 237–261. MR3381241

[13] F.W. Gehring, *Symmetrization of rings in space*, Trans. Amer. Math. Soc., **101** (1961), 499-519. MR0132841

[14] J. Sarvas, *Symmetrization of condensers in n-space*, Ann. Acad. Sci. Fenn. Ser A. I. Math., **522** (1972), 1–44. MR0348108

[15] J. Heinonen, T. Kilpeläinen, O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Dover Publications, Inc., Mineola, NY, 2006. MR2305115

[16] K. Kuratowski, *Topology. Volume II*, Academic Press, 1968. MR0259835

[17] V.G. Maz'ya, *Sobolev Spaces*, Springer-Verlag, 1985. MR0817985

Maxim Vladimirovich Tryamkin
Sobolev Institute of Mathematics,
4, Koptyuga ave.,
Novosibirsk, 630090, Russia
Email address: maxtryamkin@math.nsc.ru