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SOME RESULTS ON THE MAXIMAL GRAPH OF COMMUTATIVE RINGS

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Abstract. Let $(\Gamma(R))^c$ be the maximal graph of a commutative ring with identity R whose vertices are elements of R. Two distinct vertices a and b are adjacent if and only if $Ra+Rb \neq R$ or there exists a maximal ideal of R containing both them. In this paper, we study some properties of two (induced) subgraphs $(\Gamma_2(R))^c$ and $(G(R))^c$ of $(\Gamma(R))^c$.

Keywords: Commutative rings, Energy of a graph, Maximal graph, Nonseparable graph.

1. Introduction

Throughout this paper R is a finite commutative ring with identity and for every prime number p, K_p denotes the finite field of order p^2. For the set A, $|A|$ denotes the cardinality of A, also for the square matrix M, $|M|$ denotes the order of M. For any ring R, $\mathrm{Max}(R)$ denotes the set of all maximal ideals of R and $\mathrm{J}(R)$ is the Jacobson radical of R.

Thus K_2 = GF(4), the Galois field ={0,1,x,x+1}

Now we mention some definitions of graph theory that will be used throughout the paper. For a graph G, by $V(G)$ and $E(G)$, we denote the

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set of all vertices and all edges, respectively. Let G be a simple graph. The *complement* G^c of G is defined by taking $V(G^c) = V(G)$ and two distinct vertices a and b are adjacent in G^c if and only if they are not adjacent in G. A *wheel* graph is a graph formed by connecting a single vertex to all vertices of a cycle, a wheel graph with $n \geq 4$ vertices also be defined as a graph with one vertex of degree $n-1$ and other vertices of degree 3. A *separation* of a connected graph is a decomposition of the graph into two nonempty connected subgraphs which have just one vertex in common. This common vertex is called a *separating vertex* of the graph. A graph is *nonseparable* graph if it is connected and has no separating vertices; otherwise it is separable. A connected graph is nonseparable if and only if any two of its edges lie on a common cycle. A *block* of a graph is a subgraph which is nonseparable and is maximal respect to this property. A nonseparable graph has just one block and it is itself.

Ivan Gutman and Bo Zhoui in [4] defined the Laplacian energy of the graph G. Let G be a simple graph with n vertices, its *Laplacian matrix* $L_{n\times n}$ is defined as:

$$L = D - A$$

Where D is the degree matrix and A is the adjacency matrix of the graph. Since G is a simple graph, A only contains 1s or 0s and its diagonal elements are all 0s.

The elements of L are given by

$$L_{ij} \ := \begin{cases} \deg(v_i) & if \quad i = j \\ -1 & if \quad i \neq j \ and \ v_i \ is \ adjacent \ to \ v_j \\ 0 & otherwise \end{cases}$$

where $\deg(v_i)$ is the degree of the vertex v_i. Let G be a graph with n vertices and m edges, $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of the adjacency matrix of G and $\mu_1, \mu_2, ..., \mu_n$ be the eigenvalues of the Laplacian matrix of G. The Laplacian energy of the graph G is

$$LE(G) = \Sigma_{i=1}^{n} |\mu_i - \frac{2m}{n}|.$$

Sharma and Bhatwadekar in [6] defined a graph $\Gamma(R)$, with vertices as elements of R, where two distinct vertices a and b are adjacent if and only if $Ra + Rb = R$. In [2], Gaur and Sharma defined a complement graph of $\Gamma(R)$; $(\Gamma(R))^c$; that two distinct vertices a and b of $(\Gamma(R))^c$ are adjacent if and only if $Ra + Rb \neq R$. By notation of [5], let $(\Gamma_1(R))^c$ be the subgraph of $(\Gamma(R))^c$, induced on the set of units of R, and $(\Gamma_2(R))^c$ be the subgraph of $(\Gamma(R))^c$ generated by non-unit elements.

Also $(\Gamma_2(R) - \mathrm{J(R)})^c$ be a subgraph of $(\Gamma(R))^c$ induced on the set of nonunits of R which are not in Jacobson radical of R. For the sake of convenience, for any ring R with at least two maximal ideals, we denote $\Gamma_2(R) - \mathrm{J(R)}$ by $G(R)$.

In this paper, we study some properties of the subgraphs $G(R)^c$ and $(\Gamma_2(R))^c$. In section 2, at first we characterize rings R such that the subgraphs $G(R)^c$ and $(\Gamma_2(R))^c$ are wheel, and we investigate rings R with $|\mathrm{Max}(R)| \geq 2$ such that $G(R)^c$ and $(\Gamma_2(R))^c$ are separable graphs. In section 3, we characterize rings $R = R_1$ and $R = R_1 \times R_2$ where R_1 and R_2 are quasi-local rings, such that $(\Gamma_2(R))^c$ and $(G(R))^c$ are monocyclic. In section 4 and section 5, we investigate the energy of the graphs $(\Gamma_2(R))^c$ and $(G(R))^c$ and the Laplacian energy of them where $R = R_1 \times R_2$ and $R = R_1 \times R_2 \times R_3$, R_i is a finite quasi-local ring for $i = 1, 2, 3$.

2. Monocyclic graph

We begin this section with the following lemma.

Lemma 1. *Let R be a ring. Then $(G(R))^c$ is not a wheel graph.*

Proof. We assume that there exists a vertex x of degree $n - 1$, where n is the number of vertices. So $x \in \mathrm{J(R)}$, which is a contradiction. □

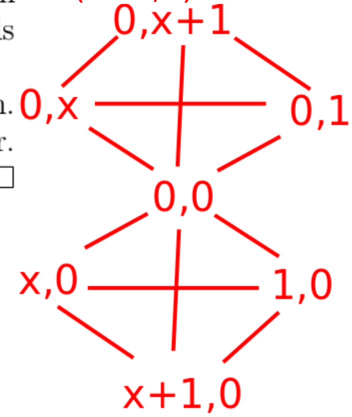
In the next theorem, we obtain a necessary and sufficient condition that $(\Gamma_2(R))^c$ be a wheel graph.

Theorem 1. *Let R be a ring. Then the following statements are equivalent:*
(1) $(\Gamma_2(R))^c$ is a wheel graph.
(2) (R, M) is a quasi-local ring with $|M| = 4$ or $R \cong K_2 \times K_2$.

Non-units in R:
(0,a), (a,0),
where a\in GF(4)

Proof. (1) → (2) We may distinguish the following cases:
Case 1. $|\mathrm{Max}(R)| = 1$, thus $(\Gamma_2(R))^c = M$; where $\mathrm{Max}(R) = \{M\}$; is a complete graph. On the other hand, this graph is wheel so it has 4 vertices, as desired.
Case 2. $|\mathrm{Max}(R)| = 2$, thus $R = R_1 \times R_2$ where (R_i, M_i) is a quasi-local ring for $1 \leq i \leq 2$. $\mathrm{Max}(R) = \{N_1, N_2\}$ where $N_1 = M_1 \times R_2$ and $N_2 = R_1 \times M_2$ and $\mathrm{J(R)} = M_1 \times M_2$. Since $(\Gamma_2(R))^c$ is a wheel graph we should have $|\mathrm{J(R)}| = 1$ and $|N_1| = |N_2| = 4$ so R_1 and R_2 are fields and $|R_1| = |R_2| = 4$, as follows.
Case 3. $|\mathrm{Max}(R)| \geq 3$, by the assumption, this case doesn't happen.
(2) → (1) By the definition of a wheel graph, the assertion is clear. □

Γ_2:
vertices
(0,0), (0,1), (0,x)
(0,x+1)
(1,0), (x,0),
(x+1,0)



Not a weel?

Proposition 1. *Let R be a ring. Then*

(1) If $|\mathrm{Max}(R)| = 2$, then $(G(R))^c$ is a separable graph.

(2) If $|\mathrm{Max}(R)| \geq 3$, then $(G(R))^c$ is a nonseparable graph.

Proof. (1) Clearly $(G(R))^c$ is a disconnected graph so it is separable.

(2) $(G(R))^c$ is a connected graph and any two of its edges lie on a common cycle thus the assertion is obtained by Theorem 1.4. □

In the next theorem, we obtain a necessary and sufficient condition that $(\Gamma_2(R))^c$ be a seprable graph.

Theorem 2. *Let R be a ring. Then the following statements hold:*

(1) If $|\mathrm{Max}(R)| = 2$ and $R = F_1 \times F_2$, where F_1 and F_2 are fields, then $(\Gamma_2(R))^c$ is a separable graph, otherwise this graph is nonseparable.

(2) If $|\mathrm{Max}(R)| \geq 3$, then $(\Gamma_2(R))^c$ is a nonseparable graph.

Proof. (1) Suppose $|\mathrm{Max}(R)| = 2$ and $R = F_1 \times F_2$, so there exists one vertex in J(R) which is a separating vertex thus $(\Gamma_2(R))^c$ is separable. In other cases $|\mathrm{J(R)}| > 1$ thus we don't have separating vertex. On the other hand $(\Gamma_2(R))^c$ is connected, as follows.

(2) It is clear. □

At the end of this section, we determine the block of two subgraphs $(G(R))^c$ and $(\Gamma_2(R))^c$. By Proposition 2.3 and Theorm 2.4, it is enough to investigate $|\mathrm{Max}(R)| = 2$ for $(G(R))^c$ and $R = F_1 \times F_2$, where F_1 and F_2 are fields for $(\Gamma_2(R))^c$.

Remark 1. *(1) Let R be a ring with $|\mathrm{Max}(R)| = 2$ and $R = R_1 \times R_2$ where (R_1, M_1) and (R_2, M_2) are two quasi-local rings, $N_1 = M_1 \times R_2$ and $N_2 = R_1 \times M_2$ are two maximal ideals of R, then one of the two subgraphs $N_1 - \mathrm{J(R)}$ and $N_2 - \mathrm{J(R)}$ that has more vertices number is the block of $(G(R))^c$.*

(2) Let $R = F_1 \times F_2$, where F_1 and F_2 are fields, $|F_1| \geq |F_2|$, $N_1 = (0) \times F_2$ and $N_2 = F_1 \times (0)$ are two maximal ideals of R. Then N_2 is the block of $(\Gamma_2(R))^c$.

3. The energy of the graphs $(\Gamma_2(R))^c$ and $(G(R))^c$

In this section we obtain adjacency matrix of two graphs $(\Gamma_2(R))^c$ and $(G(R))^c$, where $R = R_1 \times R_2$ or $R = R_1 \times R_2 \times R_3$, R_i is a finite quasi-local ring for every $i = 1, 2, 3$. Also we investigate the energy of them.

In the following, the matrices 0, 1 and -1 are the matrices in which all entries are 0, 1 and -1, respectively, and A_i, for every positive integer i, denotes the matrix in which all entries on the main diagonal are 1 and other entries are 0.

Let $R = R_1 \times R_2$, where (R_i, M_i) is a finite quasi-local ring and $|R_i| = r_i$, $|M_i| = m_i$, for $i = 1, 2$. Then

$$A((\Gamma_2(R))^c) = \begin{pmatrix} A_1 & 0 & 1 \\ 0 & A_2 & 1 \\ 1 & 1 & A3 \end{pmatrix},$$

,

$$A(G(R)^c) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

in which $|A_1| = m_1r_2 - m_1m_2$, $|A_2| = r_1m_2 - m_1m_2$ and $|A_3| = m_1m_2$.

The following table shows the energy of graphs $(\Gamma_2(R))^c$ and $(G(R))^c$ for some rings:

R	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_9$	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_9$
$E(A(\Gamma_2(R))^c)$	2.82	4.95	9.64	22.88	23.3	49.76
$E(G(R)^c)$	0	2	4	14	12	32

Let $R = R_1 \times R_2 \times R_3$ be a ring, where (R_i, M_i) is a finite quasi-local ring and $|R_i| = r_i$, $|M_i| = m_i$, for $i = 1, 2, 3$. Then

$$A(\Gamma_2(R)^c) = \begin{pmatrix} A_1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & A_2 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & A_3 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & A_4 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & A_5 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & A_6 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & A_7 \end{pmatrix},$$

$$A(G(R)^c) = \begin{pmatrix} A_1 & 0 & 0 & 1 & 1 & 0 \\ 0 & A_2 & 0 & 1 & 0 & 1 \\ 0 & 0 & A_3 & 0 & 1 & 1 \\ 1 & 1 & 0 & A_4 & 1 & 1 \\ 1 & 0 & 1 & 1 & A_5 & 1 \\ 0 & 1 & 1 & 1 & 1 & A_6 \end{pmatrix},$$

in which $|A_1| = m_1r_2r_3 - m_1m_2r_3 - m_1r_2m_3 + m_1m_2m_3$,
$|A_2| = r_1m_2r_3 - r_1m_2m_3 - m_1m_2r_3 + m_1m_2m_3$,
$|A_3| = r_1r_2m_3 - r_1m_2m_3 - m_1r_2m_3 + m_1m_2m_3$,
$|A_4| = m_1m_2r_3 - m_1m_2m_3$,
$|A_5| = m_1r_2m_3 - m_1m_2m_3$,
$|A_6| = r_1m_2m_3 - m_1m_2m_3$,
$|A_7| = m_1m_2m_3$.

The next table shows the energy of graphs $(\Gamma_2(R))^c$ and $(G(R))^c$ for some rings:

R	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4$
$E(A(\Gamma_2(R)^c))$	11.45	18.73	28.95
$E(G(R)^c)$	8.9	16.05	23.86

4. The Laplacian energy of the graphs $(\Gamma_2(R))^c$ and $(G(R))^c$

Similar to the previous section, we survey the Laplacian matrix and Laplacian energy of the graphs $(\Gamma_2(R))^c$ and $(G(R))^c$, where $R = R_1 \times R_2$ or $R = R_1 \times R_2 \times R_3$, R_i is a finite quasi-local ring, for $i = 1, 2, 3$.

Let $R = R_1 \times R_2$, where (R_i, M_i) is a quasi-local finite ring and $|R_i| = r_i$ and $|M_i| = m_i$ for $i = 1, 2$. Then

$$L((\Gamma_2(R))^c) = \begin{pmatrix} L_1 & 0 & -1 \\ 0 & L_2 & -1 \\ -1 & -1 & L_3 \end{pmatrix},$$

$$L(G(R)^c) = \begin{pmatrix} {L_1}' & 0 \\ 0 & {L_2}' \end{pmatrix},$$

in which $|L_i| = |A_i|$ for $i = 1, 2, 3$ and entries on the main diagonal of L_1, L_2 and L_3 are $m_1r_2 - 1$, $r_1m_2 - 1$ and $m_1r_2 + r_1m_2 - m_1m_2 - 1$, respectively, and other entries are -1. Also $|L_i'| = |A_i|$, for $i = 1, 2$, and entries on the main diagonal of L_1' and L_2' are $m_1r_2 - m_1m_2 - 1$ and $r_1m_2 - m_1m_2 - 1$, respectively, and other entries are -1.

The following table shows the Laplacian energy of graphs $(\Gamma_2(R))^c$ and $(G(R))^c$ for some rings:

R	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_3$	$\mathbb{Z}_2\times\mathbb{Z}_4$	$\mathbb{Z}_2\times\mathbb{Z}_9$	$\mathbb{Z}_4\times\mathbb{Z}_4$	$\mathbb{Z}_4\times\mathbb{Z}_9$
$LE(A(\Gamma_2(R)^c))$	3.33	6	10.67	34	29.33	106
$LE(G(R)^c)$	0	2.67	4	20	12	66

Let $R = R_1 \times R_2 \times R_3$, where (R_i, M_i) is a quasi-local finite ring, for $i = 1, 2, 3$, and $|R_i| = r_i$ and $|M_i| = m_i$. Then

$$L(\Gamma_2(R)^c) = \begin{pmatrix} L_1 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & L_2 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & L_3 & 0 & -1 & -1 & -1 \\ -1 & -1 & 0 & L_4 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & L_5 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 & L_6 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & L_7 \end{pmatrix},$$

$$L(G(R)^c) = \begin{pmatrix} L'_1 & 0 & 0 & -1 & -1 & 0 \\ 0 & L'_2 & 0 & -1 & 0 & -1 \\ 0 & 0 & L'_3 & 0 & -1 & -1 \\ -1 & -1 & 0 & L'_4 & -1 & -1 \\ -1 & 0 & -1 & -1 & L'_5 & -1 \\ 0 & -1 & -1 & -1 & -1 & L'_6 \end{pmatrix},$$

$|L_i| = |A_i|$, entries on the main diagonal of $L_1, L_2, ..., L_7$ are $m_1r_2r_3-1$, $r_1m_2r_3-1$, $r_1r_2m_3-1$, $m_1r_2r_3+r_1m_2r_3-m_1m_2r_3-1$, $m_1r_2r_3+r_1r_2m_3-m_1r_2m_3-1$, $r_1m_2r_3+r_1r_2m_3-r_1m_2m_3-1$, $m_1r_2r_3+r_1m_2r_3+r_1r_2m_3-m_1m_2r_3-m_1r_2m_3-r_1m_2m_3+m_1m_2m_3-1$, respectively, and other entries of them are -1.

Also $|L'_i| = |A_i|$ and entries on the main diagonal of L'_i are equal to the entries on the main diagonal of L_i minus $m_1m_2m_3$, for $i = 1, ..., 6$, and other entries of them are -1.

Finally, we conclude the next table:

R	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$
$LE(A(\Gamma_2(R))^c)$	14.94	24.77	45.82
$LE(G(R))^c$	11.22	20.06	33.34

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