S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том XX, стр. XX–XX (20XX) УДК ???.?

SOME RESULTS ON THE MAXIMAL GRAPH OF COMMUTATIVE RINGS

M. SOLEIMANI, F. MAHMUDI AND M. H. NADERI

Abstract. Let $(\Gamma(R))^c$ be the maximal graph of a commutative ring with identity R whose vertices are elements of R. Two distinct vertices a and b are adjacent if and only if $Ra+Rb \neq R$ or there exists a maximal ideal of R containing both them. In this paper, we study some properties of two (induced) subgraphs $(\Gamma_2(R))^c$ and $(G(R))^c$ of $(\Gamma(R))^c$.

Keywords: Commutative rings, Energy of a graph, Maximal graph, Nonseparable graph.

1. Introduction

Throughout this paper R is a finite commutative ring with identity and for every prime number p, K_p denotes the finite field of order p^2. For the set A, $|A|$ denotes the cardinality of A, also for the square matrix M, $|M|$ denotes the order of M. For any ring R, $\mathrm{Max}(R)$ denotes the set of all maximal ideals of R and $\mathrm{J}(R)$ is the Jacobson radical of R.

Now we mention some definitions of graph theory that will be used throughout the paper. For a graph G, by $V(G)$ and $E(G)$, we denote the

Naderi, M. H., Some results on the maximal graph of commutative rings.

set of all vertices and all edges, respectively. Let G be a simple graph. The *complement* G^c of G is defined by taking $V(G^c) = V(G)$ and two distinct vertices a and b are adjacent in G^c if and only if they are not adjacent in G. A *wheel* graph is a graph formed by connecting a single vertex to all vertices of a cycle, a wheel graph with $n \geq 4$ vertices also be defined as a graph with one vertex of degree $n-1$ and other vertices of degree 3. A *separation* of a connected graph is a decomposition of the graph into two nonempty connected subgraphs which have just one vertex in common. This common vertex is called a *separating vertex* of the graph. A graph is *nonseparable* graph if it is connected and has no separating vertices; otherwise it is separable. A connected graph is nonseparable if and only if any two of its edges lie on a common cycle. A *block* of a graph is a subgraph which is nonseparable and is maximal respect to this property. A nonseparable graph has just one block and it is itself.

Ivan Gutman and Bo Zhoui in [4] defined the Laplacian energy of the graph G. Let G be a simple graph with n vertices, its *Laplacian matrix* $L_{n\times n}$ is defined as: $L = D - A$, where D is the degree matrix and A is the adjacency matrix of the graph. Since G is a simple graph, A only contains 1s or 0s and its diagonal elements are all 0s.

The elements of L are given by

$$L_{ij} \;\; := \;\; \begin{cases} \deg(v_i) & if \quad i = j \\ -1 & if \quad i \neq j \; and \; v_i \; is \; adjacent \; to \; v_j \\ 0 & otherwise \end{cases}$$

where $\deg(v_i)$ is the degree of the vertex v_i. Let G be a graph with n vertices and m edges, $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of the adjacency matrix of G and $\mu_1, \mu_2, ..., \mu_n$ be the eigenvalues of the Laplacian matrix of G. The Laplacian energy of the graph G is

$$LE(G) = \Sigma_{i=1}^{n} |\mu_i - \frac{2m}{n}|.$$

Sharma and Bhatwadekar in [6] defined a graph $\Gamma(R)$, with vertices as elements of R, where two distinct vertices a and b are adjacent if and only if $Ra + Rb = R$. In [2], Gaur and Sharma defined a complement graph of $\Gamma(R)$; $(\Gamma(R))^c$; that two distinct vertices a and b of $(\Gamma(R))^c$ are adjacent if and only if $Ra + Rb \neq R$. By notation of [5], let $(\Gamma_1(R))^c$ be the subgraph of $(\Gamma(R))^c$, induced on the set of units of R, and $(\Gamma_2(R))^c$ be the subgraph of $(\Gamma(R))^c$ generated by non-unit elements. Also $(\Gamma_2(R) - \mathrm{J(R)})^c$ be a subgraph of $(\Gamma(R))^c$ induced on the set of nonunits of R which are not in Jacobson radical of R. For the sake of convenience, for any ring R with at least two maximal ideals, we denote

$\Gamma_2(R) - \mathrm{J(R)}$ by $G(R)$.

In this paper, we study some properties of the subgraphs $G(R)^c$ and $(\Gamma_2(R))^c$. In section 2, at first we characterize rings R such that the subgraphs $G(R)^c$ and $(\Gamma_2(R))^c$ are wheel, and we investigate rings R with $|\mathrm{Max}(R)| \geq 2$ such that $G(R)^c$ and $(\Gamma_2(R))^c$ are separable graphs. In section 3, we characterize rings $R = R_1$ and $R = R_1 \times R_2$ where R_1 and R_2 are quasi-local rings, such that $(\Gamma_2(R))^c$ and $(G(R))^c$ are monocyclic. In section 4 and section 5, we investigate the energy of the graphs $(\Gamma_2(R))^c$ and $(G(R))^c$ and the Laplacian energy of them where $R = R_1 \times R_2$ and $R = R_1 \times R_2 \times R_3$, R_i is a finite quasi-local ring for $i = 1, 2, 3$.

2. Monocyclic graph

We begin this section with the following lemma.

Lemma 1. *Let R be a ring. Then $(G(R))^c$ is not a wheel graph.*

Proof. We assume that there exists a vertex x of degree $n-1$, where n is the number of vertices. So $x \in \mathrm{J(R)}$, which is a contradiction. □

In the next theorem, we obtain a necessary and sufficient condition that $(\Gamma_2(R))^c$ be a wheel graph.

Theorem 1. *Let R be a ring. Then the following statements are equivalent:*
(1) $(\Gamma_2(R))^c$ *is a wheel graph.*
(2) (R, M) *is a quasi-local ring with* $|M| = 4$.

Proof. (1) → (2) We may distinguish the following cases:
Case 1. $|\mathrm{Max}(R)| = 1$. Thus $(\Gamma_2(R))^c = M$, where $\mathrm{Max}(R) = \{M\}$ is a complete graph. On the other hand, this graph is wheel so it has 4 vertices, as desired.
Case 2. $|\mathrm{Max}(R)| = 2$. Thus $R = R_1 \times R_2$, where (R_i, M_i) is a quasi-local ring for $1 \leq i \leq 2$, and $\mathrm{Max}(R) = \{N_1, N_2\}$, where $N_1 = M_1 \times R_2$ and $N_2 = R_1 \times M_2$ and $\mathrm{J}(R) = M_1 \times M_2$. Since $(\Gamma_2(R))^c$ is a wheel graph, there exists $x \in \mathrm{J}(R)$, where x is adjacent to other vertices also other vertices are a cycle. So there exists $y \in N_1 - \mathrm{J}(R)$ and $z \in N_2 - \mathrm{J}(R)$ such that y is adjacent to z, which is a contradiction.
Case 3. $|\mathrm{Max}(R)| \geq 3$. Clearly this case does not happen.
(2) → (1) By the definition of a wheel graph, the assertion is clear. □

Proposition 1. *Let R be a ring. Then*
(1) If $|\mathrm{Max}(R)| = 2$, *then* $(G(R))^c$ *is a separable graph.*
(2) If $|\mathrm{Max}(R)| \geq 3$, *then* $(G(R))^c$ *is a nonseparable graph.*

Proof. (1) Clearly $(G(R))^c$ is a disconnected graph so it is separable.

(2) $(G(R))^c$ is a connected graph and any two of its edges lie on a common cycle thus the assertion is obtained by Theorem 1.4. □

In the next theorem, we obtain a necessary and sufficient condition that $(\Gamma_2(R))^c$ be a seprable graph.

Theorem 2. *Let R be a ring. Then the following statements hold:*

(1) If $|\mathrm{Max}(R)| = 2$ and $R = F_1 \times F_2$, where F_1 and F_2 are fields, then $(\Gamma_2(R))^c$ is a separable graph, otherwise this graph is nonseparable.

(2) If $|\mathrm{Max}(R)| \geq 3$, then $(\Gamma_2(R))^c$ is a nonseparable graph.

Proof. (1) Suppose $|\mathrm{Max}(R)| = 2$ and $R = F_1 \times F_2$, so there exists one vertex in J(R) which is a separating vertex thus $(\Gamma_2(R))^c$ is separable. In other cases $|\mathrm{J(R)}| > 1$ thus we don't have separating vertex. On the other hand $(\Gamma_2(R))^c$ is connected, as follows.

(2) It is clear. □

At the end of this section, we determine the block of two subgraphs $(G(R))^c$ and $(\Gamma_2(R))^c$. By Proposition 2.3 and Theorm 2.4, it is enough to investigate $|\mathrm{Max}(R)| = 2$ for $(G(R))^c$ and $R = F_1 \times F_2$, where F_1 and F_2 are fields for $(\Gamma_2(R))^c$.

Remark 1. *(1) Let R be a ring with $|\mathrm{Max}(R)| = 2$ and $R = R_1 \times R_2$ where (R_1, M_1) and (R_2, M_2) are two quasi-local rings, $N_1 = M_1 \times R_2$ and $N_2 = R_1 \times M_2$ are two maximal ideals of R, then one of the two subgraphs $N_1 - \mathrm{J(R)}$ and $N_2 - \mathrm{J(R)}$ that has more vertices number is the block of $(G(R))^c$.*

(2) Let $R = F_1 \times F_2$, where F_1 and F_2 are fields, $|F_1| \geq |F_2|$, $N_1 = (0) \times F_2$ and $N_2 = F_1 \times (0)$ are two maximal ideals of R. Then N_2 is the block of $(\Gamma_2(R))^c$.

3. The energy of the graphs $(\Gamma_2(R))^c$ and $(G(R))^c$

In this section we obtain adjacency matrix of two graphs $(\Gamma_2(R))^c$ and $(G(R))^c$, where $R = R_1 \times R_2$ or $R = R_1 \times R_2 \times R_3$, R_i is a finite quasi-local ring for every $i = 1, 2, 3$. Also we investigate the energy of them.

In the following, the matrices 0, 1 and -1 are the matrices in which all entries are 0, 1 and -1, respectively, and A_i, for every positive integer i, denotes the matrix in which all entries on the main diagonal are 1 and other entries are 0.

Let $R = R_1 \times R_2$, where (R_i, M_i) is a finite quasi-local ring and $|R_i| = r_i$, $|M_i| = m_i$, for $i = 1, 2$. Then

$$A((\Gamma_2(R))^c) = \begin{pmatrix} A_1 & 0 & 1 \\ 0 & A_2 & 1 \\ 1 & 1 & A3 \end{pmatrix},$$

,

$$A(G(R)^c) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

in which $|A_1| = m_1r_2 - m_1m_2$, $|A_2| = r_1m_2 - m_1m_2$ and $|A_3| = m_1m_2$.

The following table shows the energy of graphs $(\Gamma_2(R))^c$ and $(G(R))^c$ for some rings:

R	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_9$	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_9$
$E(A(\Gamma_2(R))^c)$	2.82	4.95	9.64	22.88	23.3	49.76
$E(G(R)^c)$	0	2	4	14	12	32

Let $R = R_1 \times R_2 \times R_3$ be a ring, where (R_i, M_i) is a finite quasi-local ring and $|R_i| = r_i$, $|M_i| = m_i$, for $i = 1, 2, 3$. Then

$$A(\Gamma_2(R)^c) = \begin{pmatrix} A_1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & A_2 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & A_3 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & A_4 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & A_5 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & A_6 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & A_7 \end{pmatrix},$$

$$A(G(R)^c) = \begin{pmatrix} A_1 & 0 & 0 & 1 & 1 & 0 \\ 0 & A_2 & 0 & 1 & 0 & 1 \\ 0 & 0 & A_3 & 0 & 1 & 1 \\ 1 & 1 & 0 & A_4 & 1 & 1 \\ 1 & 0 & 1 & 1 & A_5 & 1 \\ 0 & 1 & 1 & 1 & 1 & A_6 \end{pmatrix},$$

in which $|A_1| = m_1r_2r_3 - m_1m_2r_3 - m_1r_2m_3 + m_1m_2m_3$,
$|A_2| = r_1m_2r_3 - r_1m_2m_3 - m_1m_2r_3 + m_1m_2m_3$,
$|A_3| = r_1r_2m_3 - r_1m_2m_3 - m_1r_2m_3 + m_1m_2m_3$,
$|A_4| = m_1m_2r_3 - m_1m_2m_3$,
$|A_5| = m_1r_2m_3 - m_1m_2m_3$,
$|A_6| = r_1m_2m_3 - m_1m_2m_3$,
$|A_7| = m_1m_2m_3$.

The next table shows the energy of graphs $(\Gamma_2(R))^c$ and $(G(R))^c$ for some rings:

R	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$
$E(A(\Gamma_2(R)^c))$	11.45	18.73	28.95
$E(G(R)^c)$	8.9	16.05	23.86

4. The Laplacian energy of the graphs $(\Gamma_2(R))^c$ and $(G(R))^c$

Similar to the previous section, we survey the Laplacian matrix and Laplacian energy of the graphs $(\Gamma_2(R))^c$ and $(G(R))^c$, where $R = R_1 \times R_2$ or $R = R_1 \times R_2 \times R_3$, R_i is a finite quasi-local ring, for $i = 1, 2, 3$.

Let $R = R_1 \times R_2$, where (R_i, M_i) is a quasi-local finite ring and $|R_i| = r_i$ and $|M_i| = m_i$ for $i = 1, 2$. Then

$$L((\Gamma_2(R))^c) = \begin{pmatrix} L_1 & 0 & -1 \\ 0 & L_2 & -1 \\ -1 & -1 & L_3 \end{pmatrix},$$

$$L(G(R)^c) = \begin{pmatrix} {L_1}' & 0 \\ 0 & {L_2}' \end{pmatrix},$$

in which $|L_i| = |A_i|$ for $i = 1, 2, 3$ and entries on the main diagonal of L_1, L_2 and L_3 are $m_1r_2 - 1$, $r_1m_2 - 1$ and $m_1r_2 + r_1m_2 - m_1m_2 - 1$, respectively, and other entries are -1. Also $|L_i'| = |A_i|$, for $i = 1, 2$, and entries on the main diagonal of L_1' and L_2' are $m_1r_2 - m_1m_2 - 1$ and $r_1m_2 - m_1m_2 - 1$, respectively, and other entries are -1.

The following table shows the Laplacian energy of graphs $(\Gamma_2(R))^c$ and $(G(R))^c$ for some rings:

R	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_9$	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_9$
$LE(A(\Gamma_2(R)^c))$	3.33	6	10.67	34	29.33	106
$LE(G(R)^c)$	0	2.67	4	20	12	66

Let $R = R_1 \times R_2 \times R_3$, where (R_i, M_i) is a quasi-local finite ring, for $i = 1, 2, 3$, and $|R_i| = r_i$ and $|M_i| = m_i$. Then

$$L(\Gamma_2(R)^c) = \begin{pmatrix} L_1 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & L_2 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & L_3 & 0 & -1 & -1 & -1 \\ -1 & -1 & 0 & L_4 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & L_5 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 & L_6 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & L_7 \end{pmatrix},$$

$$L(G(R)^c) = \begin{pmatrix} L_1' & 0 & 0 & -1 & -1 & 0 \\ 0 & L_2' & 0 & -1 & 0 & -1 \\ 0 & 0 & L_3' & 0 & -1 & -1 \\ -1 & -1 & 0 & L_4' & -1 & -1 \\ -1 & 0 & -1 & -1 & L_5' & -1 \\ 0 & -1 & -1 & -1 & -1 & L_6' \end{pmatrix},$$

$|L_i| = |A_i|$, entries on the main diagonal of $L_1, L_2, ..., L_7$ are $m_1r_2r_3-1$, $r_1m_2r_3-1$, $r_1r_2m_3-1$, $m_1r_2r_3+r_1m_2r_3-m_1m_2r_3-1$, $m_1r_2r_3+r_1r_2m_3-$

$m_1r_2m_3-1$, $r_1m_2r_3+r_1r_2m_3-r_1m_2m_3-1$, $m_1r_2r_3+r_1m_2r_3+r_1r_2m_3-m_1m_2r_3-m_1r_2m_3-r_1m_2m_3+m_1m_2m_3-1$, respectively, and other entries of them are -1.

Also $|L'_i|=|A_i|$ and entries on the main diagonal of L'_i are equal to the entries on the main diagonal of L_i minus $m_1m_2m_3$, for $i=1,...,6$, and other entries of them are -1.

Finally, we conclude the next table:

R	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4$
$LE(A(\Gamma_2(R))^c)$	14.94	24.77	45.82
$LE(G(R))^c$	11.22	20.06	33.34

References

[1] J. A. Bondy, U. S. R. Murty, *Graph theory*, vol 244 of Graduate Texts in Mathematics, Springer, New York, 2008.
https://www.springer.com/gp/book/9781846289699

[2] A. Gaur, A. Sharma, *Maximal graph of a commutative ring*, Int. J. Algebra, **7**(2013), no. 9-12, 581-588.
http://www.m-hikari.com/ija/ija-2013/ija-9-12-2013/sharmaIJA9-12-2013-2.pdf

[3] I. Gutman, *The energy of a graph*, Ber. Math. Statist. Sekt. Forsch-ungszentram Graz, **103**(1978), 1-22.

[4] I. Gutman, B, Zhou, *Laplacian energy of a graph*, Linear Algebra Appl **414** no. 1 (2006), 29-37.
https://www.sciencedirect.com/science/article/pii/S0024379505004465

[5] H. R. Maimani, M. Salimi, A. Sattari, S. Yassemi, *Comaximal graph of commutative rings*, J. Algebra **319** no. 4 (2008), 1801-1808.
https://www.sciencedirect.com/science/article/pii/S0021869307000828

[6] P. K. Sharma, S. M. Bhatwadekar, *A note on graphical representation of rings*, J. Algebra, **176** no. 1 (1995), 124-127.
https://www.sciencedirect.com/science/article/pii/S0021869385712367

M. Soleimani
Department of Mathematics, Faculty of Science,
University of Qom,
87317-51167, Qom, Iran
Email address: m.soleimani@gmail.com

F. Mahmudi
Department of Mathematics, Faculty of Science,
University of Qom,
87317-51167, Qom, Iran
Email address: mahmodi.fateme64@gmail.com

M. H. Naderi
Department of Mathematics, Faculty of Science,
University of Qom,
87317-51167, Qom, Iran
Email address: mh.naderi@qom.ac.ir