ON RADIUS AND TYPICAL PROPERTIES OF n-VERTEX GRAPHS OF GIVEN DIAMETER

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ABSTRACT. A property of graphs from a class under consideration is *typical* if almost all graphs from this class have the given property. Typical properties of the class of n-vertex graphs of a fixed diameter k are studied. A family of embedded classes of typical n-vertex graphs of a given diameter $k \geq 3$, which possess a number of established metric properties, is constructed. Based on the typical properties of metric balls contained in the graph, the radius of almost all n-vertex graphs from the investigated classes is found. It is proved that for every fixed integer $k \geq 3$ almost all n-vertex graphs of diameter k have radius $\lceil \frac{k}{2} \rceil$, while the radius of almost all graphs of diameter $k = 1, 2$ is equal to the diameter. All found typical properties of n-vertex graphs of a fixed diameter $k \geq 2$ are also typical for connected graphs of diameter at least k, as well as for graphs (not necessarily connected) containing the shortest path of length at least k.

Keywords: graph, diameter, diametral vertices, radius, metric ball and sphere, typical graphs, almost all graphs.

INTRODUCTION

We study finite labeled ordinary n-vertex graphs. For a connected graph $G = (V, E)$, *the distance* $\rho_G(u, v)$ between its vertices $u, v \in V$ is defined as the length of the shortest path connecting these vertices. In this case, $e_G(v) = \max_{u \in V} \rho_G(v, u)$ is *the eccentricity* of the vertex v of the graph G, $d(G) = \max_{v \in V} e_G(v)$ — *the diameter* of the graph G, and $r(G) = \min_{v \in V} e_G(v)$ — *the radius* of the graph G.

Let us extend the notion of the metric $\rho_G : V^2 \to \mathbb{Z}_{\geq 0}$ (preserving all the metric axioms) to the case disconnected graphs in the standard way. We set $\rho_G(x, y) = \infty$ if a graph G does not have a path connecting the vertices $x, y \in V$ (in this case, we assume that $\infty + \infty = \infty$, $n + \infty = \infty$ и $\infty > n$ for every nonnegative integer n). For the extended metric, the diameter $d(G)$ and the radius $r(G)$ of a disconnected graph G will be equal to ∞.

When studying a given class Ω_n of graphs admitting a notion of dimension n, that is, the measure of their quantity (often the dimension of a graph is understood as the number of its vertices, of course, there are other approaches), questions of an asymptotic nature naturally arise. Under the asymptotic investigation of the class Ω_n, special attention is drawn to the topics around the following three questions. The first is a calculation of the asymptotically exact value of the number of such graphs of a given dimension (or to obtain good estimates for it). This makes it possible, with the specified accuracy, to easily calculate, as a rule, the difficult-to-calculate number $|\Omega_n|$. The second question is an extraction or a construction of a subclass of typical graphs $\Omega_n^* \subseteq \Omega_n$ for the given class Ω_n. And the third is the study of general, typical properties of the graphs under consideration. This approach essentially helps to understand the structure of graphs of the entire class, especially the graphs with a large number of vertices.

Let $\mathcal{J}_{n,\,d=k}$, $\mathcal{J}_{n,\,d\geq k}$, $\mathcal{J}_{n,\,d\geq k}^*$, $\mathcal{J}_{n,\,d\geq k}^{**}$ be the following classes of labeled n-vertex graphs: graphs of diameter k; connected graphs of diameter at least k; graphs (not necessarily connected) with the shortest path of length at least k; and graphs (not necessarily connected) of diameter at least k, respectively. Obviously, the following inclusions are fulfilled:

$$\mathcal{J}_{n,\,d=k} \subseteq \mathcal{J}_{n,\,d\geq k} \subseteq \mathcal{J}_{n,\,d\geq k}^* \subseteq \mathcal{J}_{n,\,d\geq k}^{**}.$$

J.W. Moon and L. Moser established that almost all graphs are graphs of diameter 2 [11]. This means that for $k = 2$ all four classes of graphs $\mathcal{J}_{n,\,d=2}$, $\mathcal{J}_{n,\,d\geq 2}$, $\mathcal{J}_{n,\,d\geq 2}^*$, $\mathcal{J}_{n,\,d\geq 2}^{**}$ have the same asymptotic cardinality, the number of graphs in these classes is asymptotically equal to the number of all n-vertex graphs $2^{\binom{n}{2}}$. The asymptotics of the number of labeled n-vertex graphs of diameter $k \geq 3$ is found in [7]. In [4, 6] it is independently proved that for $k \geq 3$ all three classes of graphs $\mathcal{J}_{n,\,d=k}$, $\mathcal{J}_{n,\,d\geq k}$, $\mathcal{J}_{n,\,d\geq k}^*$ have the same asymptotic cardinality, and the asymptotics of the number of graphs from these classes is established (see Section 1). Moreover, if we extend the class of connected graphs $\mathcal{J}_{n,\,d\geq k}$ to the class $\mathcal{J}_{n,\,d\geq k}^{**}$ (by adding all disconnected graphs) we also obtain an asymptotically equivalent class in case of $k = 3$ [4], but for $k \geq 4$, the class of larger asymptotic order is arisen [6].

In [5], when studying the variety of metric balls in graphs, the author constructed a class $\mathcal{F}_{n,k}$ of typical graphs for the class $\mathcal{J}_{n,\,d=k}$, $k \geq 3$ (and hence for the classes $\mathcal{J}_{n,\,d\geq k}$, $\mathcal{J}_{n,\,d\geq k}^*$) with a number of metric properties. It turned out that almost all n-vertex graphs of a given diameter $k \geq 3$ have a unique pair of diametral vertices (this is not the case for $k = 1, 2$) and a number of typical properties related to the variety of metric balls contained in the graph is fulfilled. So, in [5] for every fixed $k \geq 1$ a set of integer vectors $\Lambda_{n,k}$ of length $k + 1$ (consisting of $\lfloor \frac{k-1}{2} \rfloor$ different vectors for $k \geq 5$ and a single vector for $k < 5$) is found and it is proved that the diversity vector of balls (the ith component of the vector is equal to the number of different balls of radius i) of almost all n-vertex graphs of diameter k belongs to $\Lambda_{n,k}$, and this property is not valid after removing any vector from $\Lambda_{n,k}$.

In this paper, in the class $\mathcal{F}_{n,k}$, $k \geq 3$, we distinguish a family $\mathcal{F}_{n,k,p}$, $p \geq 1$ (Section 2) of nested subclasses of n–vertex graphs:

$$\ldots \subseteq \mathcal{F}_{n,k,p+1} \subseteq \mathcal{F}_{n,k,p} \subseteq \ldots \mathcal{F}_{n,k,1} \subseteq \mathcal{F}_{n,k}.$$

Each of these classes preserves the already established properties of graphs from $\mathcal{F}_{n,k}$; in addition, the graphs of the introduced classes have a property of metric spheres, which ensures the presence of a predetermined number of vertices in the intersection of spheres of radius 1. It is proved that $\mathcal{F}_{n,k,p}$ is the class of typical n-vertex graphs of the fixed diameter k (Theorem 4 and its corollaries), and hence of the classes $\mathcal{J}_{n,\,d \geq k}$, $\mathcal{J}^*_{n,\,d \geq k}$. The introduced condition on the spheres of the graph ensures the existence of vertices of a large degree and a wide variety of short shortest paths, provided, in a certain sense, of the "uniqueness" (up to a segment of two vertices) of the longest shortest path for almost all graphs of diameter k. This condition also turns out to be useful in studying typical properties of n-vertex graphs associated with various metric characteristics.

The relation $r(G) \leq d(G) \leq 2r(G)$ between the radius and the diameter of an arbitrary connected graph G is well known. Moreover, for every r and d satisfying the relation $r \leq d \leq 2r$ and $n \geq d+r$, F. Ostrand's theorem implies the existence of an n-vertex graph G with $r(G) = r$ and $d(G) = d$ [12]. From the result of J.W. Moon and L. Moser, it is easy to obtain that almost all n-vertex graphs have diameter and radius equal to 2 (see, for example, [2]). Yu.D. Burtin, considering a random n-vertex graph $G_p(n)$ with the probability $p = p(n)$ of the presence of an edge, showed that $L_p(n) + 1 \leq r(G_p(n)) \leq d(G_p(n)) \leq L_p(n) + 2$ under natural restrictions on the growth of $p(n)$ if $n \to \infty$, i.e. radius and diameter of the random graph $G_p(n)$ can take only two values $L_p(n) + 1$ and $L_p(n) + 2$, which are calculated in [1], with probability tending to 1 as $n \to \infty$. The question naturally arises about a possible radius of almost all n-vertex graphs of a fixed diameter k. In Section 3, based on the found typical properties of metric balls contained in a graph and Theorem 4, we establish the radius of almost all graphs of a given fixed diameter (Theorems 5 and 6). Moreover, almost all graphs of diameter $k \geq 3$ are not self-centered, while for $k = 1, 2$ have this property.

All obtained typical properties for n-vertex graphs of a fixed diameter $k \geq 2$ remain typical for connected graphs of diameter at least k, as well as for graphs (not necessarily connected) containing a shortest path of a length at least k (see, in particular, Corollary 8).

1. Preliminary information

The article uses the generally accepted concepts and notation of graph theory [2, 10], as well as the standard concepts of combinatorial analysis [9]. We consider only finite ordinary (i.e., without loops and multiple edges) graphs $G = (V, E)$ with set of vertices $V = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$. As usual, denote by $G \setminus v$ the graph obtained as a result of *removing a vertex v* and all edges incident to it, $G \setminus V'$ is the graph obtained by removing all vertices from a subset $V' \subseteq V$, $B_i^G(v) = \{u \in V \mid \rho_G(v, u) \leq i\}$ is *a ball of radius i centered at a vertex $v \in V$* in the metric space of the graph G with the metric ρ_G, $S_i^G(v) = \{u \in V \mid \rho_G(v, u) = i\}$ is *a sphere of radius i centered at a vertex $v \in V$*, K_n — *a complete n-vertex graph*. For a shortest path P with endpoints v_0 and v_n, sequentially passing through vertices $v_0, v_1, \ldots, v_n$, the notation $P = (v_0, v_1, \ldots, v_n)$ is used. A vertex is called *central*

if its eccentricity is equal to the radius, a vertex of degree 1 is *pendant,* a shortest path of length $d(G)$ is *the diametral path* of the graph G, and under by *a pair of diametral vertices* we mean an unordered sample of two vertices from the set V, the distance between which is equal to the diameter. A graph is *self-centered* if all its vertices are central.

We will write $\lceil x \rceil$ ($\lfloor x \rfloor$) to denote the smallest (largest) integer greater (less) or equal to a real nonnegative number x and further apply the following well-known combinatorial identity

$$\binom{n-m}{2} = \binom{n}{2} - nm + \frac{m(m+1)}{2}. \tag{1}$$

To denote *the asymptotic equality* of functions $f(n)$ and $g(n)$ as $n \to \infty$, we use the notation $f(n) \sim g(n)$, which by definition means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$ or, equivalently, $f(n) = g(n)(1 + r(n))$ for all large enough n, where $r(n) = o(1)$ is *the approximation error* of $g(n)$.

To estimate the measure of the number of graphs with a certain property, the concept of *almost all* is often used; in this approach, the studied property is considered for graphs with a large number of vertices. Let $\mathcal{J}_n$ be the class of labeled n-vertex graphs with the fixed set of vertices $V = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$. Consider some property $\mathcal{P}$, by which each graph may or may not possess. Through $\mathcal{J}_n^{\mathcal{P}}$ denote the set of all graphs from $\mathcal{J}_n$ that possess the property $\mathcal{P}$. *Almost all graphs possess the property* $\mathcal{P}$ if $\lim_{n\to\infty} \frac{|\mathcal{J}_n^{\mathcal{P}}|}{|\mathcal{J}_n|} = 1$, i.e. $|\mathcal{J}_n^{\mathcal{P}}| \sim |\mathcal{J}_n|$, and *there are almost no graphs with the property* $\mathcal{P}$, if $\lim_{n\to\infty} \frac{|\mathcal{J}_n^{\mathcal{P}}|}{|\mathcal{J}_n|} = 0$. As already noted, the class of graphs of diameter 2 turns out to be rich among all graphs in the sense of the above concept.

Theorem 1 [11]. *Almost all graphs have diameter* 2.

Moreover, the following theorem holds.

Theorem 2 (see, for example, [2]). *Almost all graphs have diameter and radius equal to* 2.

In the study and selection of almost all graphs in the class of graphs under consideration it is often useful to define not characteristic properties themselves for the notion of almost all, but directly select a subclass of typical graphs itself (in [4,5] a more general concept of a class of typical combinatorial objects and an abstract typical combinatorial object for a given class of objects admitting the concept of dimension is formulated). Further we will also use this formal concept for graphs (when the dimension of a graph is understood as the number of its vertices). Let Ω be an arbitrary class of graphs such that $\Omega_n \neq \varnothing$ for all large enough n, where $\Omega_n = \Omega \cap \mathcal{J}_n$. A subclass $\Omega^* \subseteq \Omega$ is *the class of typical graphs of the class* Ω if

$$\lim_{n\to\infty} \frac{|\Omega_n^*|}{|\Omega_n|} = 1.$$

In [5,6] the classes $\mathcal{F}_{n,k}$ and $\mathcal{T}_{n,\,d=k}$, $k \geq 3$ (the detailed definition of these classes are given in Section 2) of typical graphs for the classes $\mathcal{J}_{n,\,d=k}$, $\mathcal{J}_{n,\,d\geq k}$, $\mathcal{J}_{n,\,d\geq k}^*$ are constructed and asymptotically exact value $2^{\binom{n}{2}} \xi_{n,k}$ of the number of graphs in

these classes is found, here

$$\xi_{n,k} = q_k \ (n)_{k-1} \left(\frac{3}{2^{k-1}}\right)^{n-k+1}, \quad q_k = \frac{1}{2}\,(k-2)\,2^{-\binom{k-1}{2}},$$

$(n)_k = n(n-1)\cdots(n-k+1)$ is the number of order placements from n elements by k, and in this case we define $(n)_0 = (0)_0 = 1$ and $(n)_k = 0$ if $n < k$.

Theorem 3 [5,6]**.** *Let* $k \geq 3$ *and* $0 < \varepsilon < 1$ *do not depend on* n. *Then there exists a constant* $c > 0$ *independent of* n *and such that for every* $n \in \mathbb{N}$ *the following inequalities hold*

$$2^{\binom{n}{2}}\xi_{n,k}\left(1 - c\left(\frac{5+\varepsilon}{6}\right)^{n-k+1}\right) \leq |\mathcal{F}_{n,k}| \leq |\mathcal{T}_{n,\,d=k}| \leq |\mathcal{J}_{n,\,d=k}|$$

$$\leq |\mathcal{J}_{n,\,d\geq k}| \leq |\mathcal{J}^*_{n,\,d\geq k}| \leq 2^{\binom{n}{2}}\xi_{n,k}\left(1 + c\left(\frac{5+\varepsilon}{6}\right)^{n-k+1}\right).$$

Note that for $k = 3$ the upper bound in Theorem 3 takes the form $2^{\binom{n}{2}}\xi_{n,3}$ [4]. Moreover, this upper estimate is valid even for a class of graphs containing all disconnected graphs (which do not necessarily have a connected component with shortest path of length 3), namely, for the superclass $\mathcal{J}^{**}_{n,\,d\geq 3}$ of the class $\mathcal{J}^*_{n,\,d\geq 3}$.

Corollary 1 [4, 5] (case k=3)**.** *Let* $0 < \varepsilon < 1$ *be independent of* n. *Then there is a constant* $c > 0$ *independent of* n *and such that for every* $n \in \mathbb{N}$ *the following inequalities hold*

$$2^{\binom{n}{2}}\xi_{n,3}\left(1 - c\left(\frac{5+\varepsilon}{6}\right)^{n-2}\right) \leq |\mathcal{F}_{n,3}| \leq |\mathcal{T}_{n,\,d=3}| \leq |\mathcal{J}_{n,\,d=3}|$$

$$\leq |\mathcal{J}_{n,\,d\geq 3}| \leq |\mathcal{J}^*_{n,\,d\geq 3}| \leq |\mathcal{J}^{**}_{n,\,d\geq 3}| \leq 2^{\binom{n}{2}}\xi_{n,3}.$$

Note that $\mathcal{F}_{n,3}$ is the union of the subclasses $\mathcal{F}_{n,3}(x,y)$ over all different $x,y \in V$, and x,y is the unique pair of diametral vertices of every graph from $\mathcal{F}_{n,3}(x,y)$. In this article, we use the estimates of the number of graphs in the class $\mathcal{F}_{n,3}(x,y)$ obtained in [5], as well as of the following subclass of labeled n-vertex graphs:

$$a_n = |\{G \in \mathcal{J}_n \,|\, B_1^G(x) \cap B_1^G(y) = \varnothing\}| = 2^{\binom{n}{2}}\,\frac{8}{9}\left(\frac{3}{4}\right)^n, \text{ where } x,y \in V, \, x \neq y \ [4].$$

Lemma 1 [5]**.** *Let* x,y *be different vertices of* V *and* $c > 0$, $0 < \varepsilon < 1$ *are arbitrary constants independent of* n. *Then there is an integer* $N > 0$ *such that for every* $n > N$ *the inequality holds*

$$|\mathcal{F}_{n,3}(x,y)| \geq a_n\left(1 - c\left(\frac{5+\varepsilon}{6}\right)^{n-2}\right).$$

2. Class of graphs $\mathcal{F}_{n,k,p}$

Let $p \geq 1$ be an integer. For every $k \geq 3$, the class $\mathcal{F}_{n,k}$ of typical graphs of the class $\mathcal{J}_{n,\,d=k}$ was constructed by author in [5]. In this section we define a subclass $\mathcal{F}_{n,k,p}$ of the class $\mathcal{F}_{n,k}$. To define this class, first consider special graphs of diameter 3 and their properties. Let $x,y \in V$ and $\mathcal{F}_{n,3,p}(x,y)$ be the class of all graphs $F \in \mathcal{J}_n$ with the following properties:

a) the vertices x,y are not pendant in F;

b) $\rho_F(z,x) = \rho_F(z,y) = 2$ for some vertex $z \in V$ (a *pole of graph*);

c) $d(F) = 3$, the graph F has the unique pair of diametral vertices x,y and does not contain shuttlecocks (subgraphs defined in [3]) or, equivalently, does not contain coinciding balls of radius 1 with centers at different vertices;

d) the following property of spheres holds:

$$|S_1^F(u) \cap S_1^F(v)| \geq p \ \ \forall u,v \in V \setminus \{x,y\} \text{ and } u \neq v,$$

$$|S_1^F(u) \cap S_1^F(v)| \geq p \ \ \forall v \in V \setminus \{x,y\} \ \ \forall u \in \{x,y\}.$$

Figure 1 shows an example of a graph from class $\mathcal{F}_{n,3,2}(x,y)$ for $n \geq 9$. Note that $\mathcal{F}_{n,3,2}(x,y) = \varnothing$ if $n < 9$. Indeed, let $F \in \mathcal{F}_{n,3,2}(x,y)$. Then F contains a 4-vertex diametral path P with endpoints x, y. Moreover, there is a vertex $z \notin V(P)$ due to property b). Further, F have a vertex $x_1 \notin V(P) \cup \{z\}$ adjacent to x and a vertex $y_1 \notin V(P) \cup \{x_1, z\}$ adjacent to y. Finally, there are vertices $x_2 \in S_1(x) \cap S_1(x_1) \setminus V(P)$ and $y_2 \in S_1(y) \cap S_1(y_1) \setminus V(P)$ due to property d). Thus, $n \geq 9$.
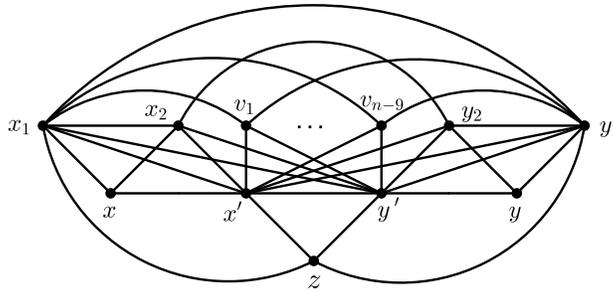

FIG. 1. A graph F_n in the class $\mathcal{F}_{n,3,2}(x,y)$

Note that graphs from $\mathcal{J}_n$ possessing only properties a), b), c) form the class $\mathcal{F}_{n,3}(x,y)$ [5], only properties a), c) — $\mathcal{T}_n^*(x,y)$ [6], and property c) — the class $\mathcal{T}_n(x,y)$ [4]. Therefore,

$$\mathcal{F}_{n,3,p+1}(x,y) \subseteq \mathcal{F}_{n,3,p}(x,y) \subseteq \mathcal{F}_{n,3}(x,y) \subseteq \mathcal{T}_n^*(x,y) \subseteq \mathcal{T}_n(x,y). \qquad (2)$$

In addition, it is easy to see that the existence condition of a diametral path of length 3 with endpoints x, y and property d) imply property a) of graphs of the class $\mathcal{F}_{n,3,p}(x,y)$. Estimate the number of graphs in $\mathcal{F}_{n,3,p}(x,y)$. For this we need the following lemmas.

Let x, y, u, v be different elements of V and $0 \leq s < p$. We define classes of n-vertex graphs as follows:

$$\mathcal{B}_n(x,y,u,v;s) = \{G \in \mathcal{J}_n \,|\, B_1^G(x) \cap B_1^G(y) = \varnothing \text{ and } |S_1^G(u) \cap S_1^G(v)| = s\},$$

$$\mathcal{C}_n(x,y,u;s) = \{G \in \mathcal{J}_n \,|\, B_1^G(x) \cap B_1^G(y) = \varnothing \text{ and } |S_1^G(x) \cap S_1^G(u)| = s\},$$

$$\mathcal{B}_{n,p}(x,y,u,v) = \bigcup_{0 \leq s < p} \mathcal{B}_n(x,y,u,v;s),$$

$$\mathcal{C}_{n,p}(x,y,u) = \bigcup_{0 \leq s < p} \mathcal{C}_n(x,y,u;s).$$

Lemma 2. *Let x, y, u, v be different vertices of V and $p \geq 1$. Then, for every $n \geq p$, the following inequality holds*

$$|\mathcal{B}_{n,p}(x,y,u,v)| \leq a_n b_p(n) \Big(\frac{3}{4}\Big)^n, \ \ \text{here } b_p(n) = 128 \sum_{s=0}^{p-1} \binom{n}{s} 3^{-(3+s)}.$$

Proof. From the definition of the class $\mathcal{B}_n(x, y, u, v; s)$, it is easy to understand that all graphs of this class are contained among graphs G constructed as follows:

1) choose an s-element subset $V_0 \subseteq V \setminus \{u, v\}$ and join each vertex from V_0 by an edge with the vertices u and v, there are $\binom{n-2}{s}$ possibilities;

2) choose an i-element subset $V_u \subseteq V \setminus (V_0 \cup \{u\})$, $0 \le i \le n - 1 - s$, and join each vertex from V_u by an edge with u, as a result we have $S_1^G(u) = V_0 \cup V_u$;

3) choose a j-element subset of $V_v \subseteq V \setminus (V_0 \cup V_u \cup \{u, v\})$, with $0 \le j \le n - 2 - s - i$ for $i \ne n - 1 - s$ and $V_u = \varnothing$ if $i = n - 1 - s$. Join each vertex from V_v by an edge with v, as a result we obtain $S_1^G(v) = V_0 \cup V_v$ and $V_u \cap V_v = \varnothing$. Note that the number of possibilities for choosing the j-element set V_v does not exceed $\binom{n-1-s-i}{j}$, and for the upper bound of the number of such graphs we can assume that $0 \le j \le n - 1 - s - i$;

4) choose a l-element subset $V_x \subseteq V \setminus \{x, y, u, v\}$, $0 \le l \le n - 4$, and join each vertex from V_x by an edge with x, as a result we have $S_1^G(x) \subseteq V_x \cup \{u, v\}$;

5) choose a m-element subset $V_y \subseteq V \setminus (V_x \cup \{x, y, u, v\})$, $0 \le m \le n - 4 - l$, and join each vertex from V_y by an edge with y, as a result we obtain $S_1^G(y) \subseteq V_y \cup \{u, v\}$ and $V_x \cap V_y = \varnothing$;

6) define an arbitrary graph on the $(n - 4)$-element set $V \setminus \{x, y, u, v\}$.

Remark. *In the above-described construction of graphs, a situation is admissible when $(V_0 \cup V_u \cup V_v) \cap \{x, y\} \ne \varnothing$ and, therefore, additionally there arise graphs G such that $\varnothing \ne B_1^G(x) \cap B_1^G(y) \subseteq \{u, v\}$, i.e. not belonging to the class $\mathcal{B}_n(x, y, u, v; s)$.*

Thus, using the Newton's Binomial Theorem and the identity (1), we obtain

$$
\begin{aligned}
|\mathcal{B}_n(x, y, u, v; s)| &\le \binom{n-2}{s} \sum_{i=0}^{n-1-s} \binom{n-1-s}{i} \sum_{j=0}^{n-1-s-i} \binom{n-1-s-i}{j} \\
&\quad \sum_{l=0}^{n-4} \binom{n-4}{l} \sum_{m=0}^{n-4-l} \binom{n-4-l}{m} 2^{\binom{n-4}{2}} \\
&\le 2^{\binom{n-4}{2}} \binom{n}{s} \sum_{i=0}^{n-1-s} \binom{n-1-s}{i} \sum_{j=0}^{n-1-s-i} \binom{n-1-s-i}{j} 3^{n-4} \\
&= 2^{\binom{n}{2}} 2^{10} \binom{n}{s} 3^{-(5+s)} \left(\frac{9}{16}\right)^n = a_n 128 \binom{n}{s} 3^{-(3+s)} \left(\frac{3}{4}\right)^n.
\end{aligned}
$$

Hence
$$
|\mathcal{B}_{n,p}(x, y, u, v)| \le \sum_{0 \le s < p} |\mathcal{B}_n(x, y, u, v; s)| = a_n b_p(n) \left(\frac{3}{4}\right)^n.
$$

$\square$

Lemma 3. *Let x, y, u be different vertices of V and $p \ge 1$. Then, for every $n \ge p$, the following inequality holds*

$$
|\mathcal{C}_{n,p}(x, y, u)| \le a_n c_p(n) \left(\frac{5}{6}\right)^n, \quad \text{here } c_p(n) = 72 \sum_{s=0}^{p-1} \binom{n}{s} 5^{-(1+s)}.
$$

Proof. Similarly to the proof of Lemma 2, construct graphs G forming a superclass of the class $\mathcal{C}_n(x, y, u; s)$ as follows:

1) choose an s-element subset $V_0 \subseteq V \setminus \{x, u\}$ and join each vertex from V_0 by an edge with the vertices x and u, there are $\binom{n-2}{s}$ possibilities;

2) choose an i-element subset $V_x \subseteq V \setminus (V_0 \cup \{x\})$, $0 \le i \le n-1-s$, and join each vertex from V_x by an edge with x, as a result we have $S_1^G(x) = V_0 \cup V_x$;

3) choose a j-element subset of $V_u \subseteq V \setminus (V_0 \cup V_x \cup \{x, u\})$, with $0 \le j \le n-2-s-i$ for $i \ne n-1-s$ and $V_u = \varnothing$ if $i = n-1-s$. Join each vertex from V_u by an edge with u, as a result we obtain $S_1^G(u) = V_0 \cup V_u$ and $V_x \cap V_u = \varnothing$. Note that the number of possibilities for choosing the j-element set V_u does not exceed $\binom{n-1-s-i}{j}$, and for the upper bound of the number of such graphs we can assume that $0 \le j \le n-1-s-i$;

4) choose a l-element subset of $V_y \subseteq V \setminus (V_0 \cup V_x \cup \{x, y, u\})$, with $0 \le l \le n-3-s-i$ for $i \le n-3-s$ and $V_y = \varnothing$ if $n-2-s \le i \le n-1-s$. Join each vertex from V_y by an edge with y, as a result we obtain $S_1^G(y) \subseteq V_y \cup \{x, u\}$. The estimation of the number of possibilities for choosing the set V_y is similar to the item 3;

5) define an arbitrary graph on the $(n-3)$-element set $V \setminus \{x, y, u\}$.

Thus, using the Newton's Binomial Theorem and the identity (1), we obtain

$$
\begin{aligned}
|\mathcal{C}_n(x,y,u;s)| &\le \binom{n-2}{s} \sum_{i=0}^{n-1-s} \binom{n-1-s}{i} \sum_{j=0}^{n-1-s-i} \binom{n-1-s-i}{j} \\
&\quad \sum_{l=0}^{n-1-s-i} \binom{n-1-s-i}{l} 2^{\binom{n-3}{2}} \\
&\le 2^{\binom{n-3}{2}} \binom{n}{s} \sum_{i=0}^{n-1-s} \binom{n-1-s}{i} 4^{n-1-s-i} \\
&= 2^{\binom{n}{2}} 64 \binom{n}{s} 5^{-(1+s)} \left(\frac{5}{8}\right)^n = a_n 72 \binom{n}{s} 5^{-(1+s)} \left(\frac{5}{6}\right)^n.
\end{aligned}
$$

Hence

$$
|\mathcal{C}_{n,p}(x,y,u)| \le \sum_{0 \le s < p} |\mathcal{C}_n(x,y,u;s)| = a_n c_p(n) \left(\frac{5}{6}\right)^n.
$$

$\square$

Lemma 4. *Let x, y be different vertices of V and $p \ge 1$, $\lambda > 0$, $0 < \varepsilon < 1$ are arbitrary constants independent of n. Then there is an integer $N > 0$ such that for every $n > N$ the inequality holds*

$$
|\mathcal{F}_{n,3,p}(x,y)| \ge a_n \left(1 - \lambda \left(\frac{5+\varepsilon}{6}\right)^{n-2}\right). \tag{3}
$$

Proof. Consider the following classes of n-vertex graphs:

$$
\mathcal{B}_{n,p}(x,y) = \bigcup_{\substack{u,v \in V \setminus \{x,y\} \\ u \ne v}} \mathcal{B}_{n,p}(x,y,u,v),
$$

$$
\mathcal{C}_{n,p}(x,y) = \bigcup_{u \in V \setminus \{x,y\}} \mathcal{C}_{n,p}(x,y,u).
$$

It is not hard to understand that

$$
\mathcal{F}_{n,3}(x,y) \setminus (\mathcal{B}_{n,p}(x,y) \cup \mathcal{C}_{n,p}(x,y) \cup \mathcal{C}_{n,p}(y,x)) \subseteq \mathcal{F}_{n,3,p}(x,y). \tag{4}
$$

Because p is a constant, by Lemmas 2 and 3 there exists an integer $N^* > 0$ such that for every $n > N^*$ the following inequalities hold

$$|\mathcal{B}_{n,p}(x,y)| \leq n^2 a_n b_p(n) \left(\frac{3}{4}\right)^n \leq \frac{\lambda}{3} \, a_n \left(\frac{5+\varepsilon}{6}\right)^{n-2}, \tag{5}$$

$$2|\mathcal{C}_{n,p}(x,y)| \leq 2n a_n c_p(n) \left(\frac{5}{6}\right)^n \leq \frac{\lambda}{3} \, a_n \left(\frac{5+\varepsilon}{6}\right)^{n-2}. \tag{6}$$

Therefore, by Lemma 1 and the relations (4),(5),(6) there is an integer $N \geq N^*$ such that for every $n > N$ the following inequalities hold

$$\begin{aligned} |\mathcal{F}_{n,3,p}| &\geq& |\mathcal{F}_{n,3}(x,y)| - |\mathcal{B}_{n,p}(x,y)| - 2\,|\mathcal{C}_{n,p}(x,y)| \\ &\geq& a_n\left(1 - \frac{\lambda}{3}\left(\frac{5+\varepsilon}{6}\right)^{n-2}\right) - a_n\frac{2\lambda}{3}\left(\frac{5+\varepsilon}{6}\right)^{n-2}. \end{aligned}$$

$\square$

Now, for $k \geq 3$, we define a subclass of the class $\mathcal{F}_{n,k}$ as follows. Let $u = (u_0, u_1, \ldots, u_{k-2})$ be an arbitrary ordered sequence of different vertices from the set V. Fix an arbitrary pair of neighboring elements u_s and u_{s+1}. On the set $V \setminus \{u_0, \ldots, u_{s-1}, u_{s+2}, \ldots, u_{k-2}\}$ of $n - k + 3$ vertices, define an arbitrary graph F from the class $\mathcal{F}_{n-k+3,3,p}(u_s, u_{s+1})$. Finally, join by edges the vertices u_i, u_{i+1} for $i \neq s$ and $0 \leq i < k - 2$. Denote the so-obtained graph by $G(u, s, F)$. Let $\mathcal{F}_{n,k,p}$ be the class of all graphs $G(u, s, F)$ constructed under condition $0 \leq s \leq \lfloor \frac{k-3}{2} \rfloor$, and let $\mathcal{F}_{n,k,p}^s$ denote the class of all graphs $G(u, s, F)$ for fixed $s \leq k - 3$. In what follows, we will use the notation $G(u, s, F)$ for the graph constructed for given u, s and F, without detailing the properties $u = (u_0, u_1, \ldots, u_{k-2})$, $0 \leq s \leq \lfloor \frac{k-3}{2} \rfloor$ and $F \in \mathcal{F}_{n-k+3,3,p}(u_s, u_{s+1})$, unless otherwise specified.

Note that if, in defining the graphs $G(u, s, F)$, instead of the class of graphs $\mathcal{F}_{n-k+3,3,p}(u_s, u_{s+1})$, we use the class $\mathcal{F}_{n-k+3,3}(u_s, u_{s+1})$ [5] (the class of graphs $\mathcal{T}_{n-k+3}^*(u_s, u_{s+1})$ [6]), then we arrive at the definition of the class $\mathcal{F}_{n,k}$ [5] (the class $\mathcal{T}_{n,d=k}$ [6]). Hence, by (2), we have

$$\mathcal{F}_{n,k,p+1} \subseteq \mathcal{F}_{n,k,p} \subseteq \mathcal{F}_{n,k} \subseteq \mathcal{T}_{n,d=k}. \tag{7}$$

Therefore, all properties of graphs $G(u, s, F)$ obtained earlier in [5,6] will also hold for graphs of the class $\mathcal{F}_{n,k,p}$. In particular, the properties stated in Lemmas 5 and 6 are valid (see, respectively, Lemmas 5 and 7 in [5]).

Lemma 5 (properties of graphs $G(u, s, F)$). *Let $k \geq 3$, $p \geq 1$ and $G = G(u, s, F) \in \mathcal{F}_{n,k,p}$. Then the following properties hold:*

(i) *$G \in \mathcal{J}_{n,d=k}$;*

(ii) *the vertices u_s, u_{s+1} are not pendant in F;*

(iii) *u_0, u_{k-2} is the unique pair of diametral vertices of the graph G and $u_0, u_1, \ldots, u_{k-2} \in V(P)$ for every diametral path P.*

Lemma 6 (balls properties of $G(u, s, F)$). *Let $k \geq 3$, $p \geq 1$, $G = G(u, s, F) \in \mathcal{F}_{n,k,p}$ and $P = (u_0, \ldots, u_s, u_s', u_{s+1}', u_{s+1}, \ldots, u_{k-2})$ is an arbitrary diametral path of the graph G. Then the following properties hold:*

(i) *if $x \in V(F)$ and $\rho_F(x, u_s) = 1$ then $B_i^G(x) = B_i^G(u_s')$ for every $i \geq 2$;*

(ii) *if $x \in V(F)$ and $\rho_F(x, u_{s+1}) = 1$ then $B_i^G(x) = B_i^G(u_{s+1}')$ for every $i \geq 2$;*

(iii) *if $x \in V(F)$ and $\rho_F(x, u_s) = \rho_F(x, u_{s+1}) = 2$ then $B_i^G(x) = B_i^G(u_s')$ for every $i \geq s + 2$.*

Using Lemma 5, as in [5,6] one can count the number of graphs of the class $\mathcal{F}_{n,k,p}$.

Lemma 7 (number of graphs in $\mathcal{F}_{n,k,p}$). *Let $k \geq 3$, $p \geq 1$ and $0 \leq s \leq k-3$. Then the following properties hold:*

(i) $|\mathcal{F}_{n,k,p}| = \frac{1}{2}(k-2)(n)_{k-1}|\mathcal{F}_{n-k+3,3,p}(x,y)|$, *where $x \neq y$;*

(ii) $|\mathcal{F}_{n,k,p}^s| = \frac{\sigma(s)}{k-2}|\mathcal{F}_{n,k,p}|$, *where $\sigma(s) = 1$ for $s = \frac{k-3}{2}$ and $\sigma(s) = 2$ if $s \neq \frac{k-3}{2}$.*

Proof. Similarly to counting graphs of the classes $\mathcal{T}_{n,\,d=k}$ [6] and $\mathcal{F}_{n,k}$, $\mathcal{F}_{n,k}^s$ (Lemma 6 [5]). $\square$

Lemma 8 (lower bound). *Let $k \geq 3$, $0 < \varepsilon < 1$ and $p \geq 1$ are constants independent of n. Then there is a constant $c > 0$ independent of n and k and such that for every $n \in \mathbb{N}$ the following inequality holds*

$$|\mathcal{F}_{n,k,p}| \geq 2^{\binom{n}{2}}\xi_{n,k}\Big(1 - c\Big(\frac{5+\varepsilon}{6}\Big)^{n-k+1}\Big).$$

Proof. By Lemma 4, there exists an integer $N > 0$ independent of n and k such that the inequality (3) holds for all $n > N$ and $\lambda = 1$. Put $c = \Big(\frac{6}{5+\varepsilon}\Big)^{N-1}$. Thus, reckoning with Lemma 7, inequality (3) for $\lambda = 1$ and identity (1), for every $n > N + k - 3$ we obtain

$$
\begin{aligned}
|\mathcal{F}_{n,k,p}| &\geq \frac{1}{2}(k-2)(n)_{k-1}2^{\binom{n-k+3}{2}}\frac{8}{9}\Big(\frac{3}{4}\Big)^{n-k+3}\Big(1-\Big(\frac{5+\varepsilon}{6}\Big)^{n-k+1}\Big) \\
&= 2^{\binom{n}{2}}q_k(n)_{k-1}\Big(\frac{3}{2^{k-1}}\Big)^{n-k+1}\Big(1-\Big(\frac{5+\varepsilon}{6}\Big)^{n-k+1}\Big) \\
&\geq 2^{\binom{n}{2}}\xi_{n,k}\Big(1 - c\Big(\frac{5+\varepsilon}{6}\Big)^{n-k+1}\Big).
\end{aligned}
$$

Now, let $n \leq N + k - 3$. Then $c\Big(\frac{5+\varepsilon}{6}\Big)^{n-k+1} > 1$. Therefore, the required inequality holds. $\square$

The following theorem follows directly from Lemma 8, relation (7) and Theorem 3.

Theorem 4 (asymptotics). *Let $k \geq 3$, $0 < \varepsilon < 1$ and $p \geq 1$ do not depend on n. Then there is a constant $c > 0$ independent of n and such that for every $n \in \mathbb{N}$ the following inequalities hold*

$$2^{\binom{n}{2}}\xi_{n,k}\Big(1 - c\Big(\frac{5+\varepsilon}{6}\Big)^{n-k+1}\Big) \leq |\mathcal{F}_{n,k,p}|$$
$$\leq |\mathcal{F}_{n,k}| \leq |\mathcal{T}_{n,\,d=k}| \leq |\mathcal{J}_{n,\,d=k}| \leq |\mathcal{J}_{n,\,d\geq k}|$$
$$\leq |\mathcal{J}_{n,\,d\geq k}^*| \leq 2^{\binom{n}{2}}\xi_{n,k}\Big(1 + c\Big(\frac{5+\varepsilon}{6}\Big)^{n-k+1}\Big).$$

In view of Corollary 1, for $k = 3$ the upper bound in Theorem 4 takes the following form.

Corollary 2 (case $k = 3$). *Let $0 < \varepsilon < 1$ and $p \geq 1$ do not depend on n. Then there is a constant $c > 0$ independent of n and such that for every $n \in \mathbb{N}$ the following inequalities hold*

$$2^{\binom{n}{2}}\xi_{n,3}\Big(1 - c\Big(\frac{5+\varepsilon}{6}\Big)^{n-2}\Big) \leq |\mathcal{F}_{n,3,p}| \leq |\mathcal{F}_{n,3}| \leq |\mathcal{T}_{n,\,k=3}| \leq |\mathcal{J}_{n,\,d=3}|$$
$$\leq |\mathcal{J}_{n,\,d\geq 3}| \leq |\mathcal{J}_{n,\,d\geq 3}^*| \leq |\mathcal{J}_{n,\,d\geq 3}^{**}| \leq 2^{\binom{n}{2}}\xi_{n,3}.$$

Emphasize a role of the properties of the graphs F that define the graphs $G(u, s, F)$ and, as a result, the classes $\mathcal{T}_{n,\,d=k}, \mathcal{F}_{n,k}, \mathcal{F}_{n,k,p}$. The condition on the degree of diametral vertices of the graph F (property a) is essential for justifying the lower bound in Theorem 4, otherwise additional repetitions of the graphs $G(u, s, F)$ arise (see [5, 6]). The existence of a pole (property b) and the absence condition of shuttlecocks (property c) are necessary to describe the diversity vector of balls of almost all n-vertex graphs of a given diameter [5], and the uniqueness condition of a pair of diametral vertices (property c) underlies the justification of the typicality of the widest class $\mathcal{T}_{n,\,d=k}$ among the above. The condition on spheres of the graph F (property d) introduced in this paper ensures the presence of a predetermined number of vertices in the intersection of spheres, as well as a wide variety of short shortest paths in graphs of the class $\mathcal{F}_{n,k,p}$ under the condition, in a certain sense, of "uniqueness" (up to a segment of two vertices from F) of the diametrical path.

Corollary 3. *Let $k \geq 3$ and $p \geq 1$ be independent of n. Then for $n \to \infty$*

$$|\mathcal{F}_{n,k,p}| \sim |\mathcal{F}_{n,k}| \sim |\mathcal{T}_{n,\,d=k}| \sim |\mathcal{J}_{n,\,d=k}| \sim |\mathcal{J}_{n,\,d\geq k}| \sim |\mathcal{J}^*_{n,\,d\geq k}| \sim 2^{\binom{n}{2}} \xi_{n,k}.$$

Corollary 4. *Let $k \geq 3$ and $p \geq 1$ be independent of n. Then $\mathcal{F}_{n,k,p}$ is the class of typical graphs of the class of n-vertex graphs of diameter k.*

3. Radius of almost all graphs from $\mathcal{J}_{n,\,d=k}$

Note that K_n is the unique n-vertex graph of diameter 1 and $r(K_n) = 1$. Therefore, almost all graphs of diameter $k = 1$ have a radius equal to the diameter. A similar fact for graphs of diameter 2 also trivially follows from well-known theorems.

Theorem 5. *Almost all graphs of diameter 2 have radius 2.*

Proof. Put $\mathcal{J}_{n,\,d=r=2} = \{G \in \mathcal{J}_n \mid d(G) = r(G) = 2\}$. Then, using Theorems 1 and 2, for $n \to \infty$ we infer

$$\frac{|\mathcal{J}_{n,\,d=r=2}|}{|\mathcal{J}_{n,\,d=2}|} = \frac{|\mathcal{J}_{n,\,d=r=2}|}{|\mathcal{J}_n|} \, \frac{|\mathcal{J}_n|}{|\mathcal{J}_{n,\,d=2}|} \; \longrightarrow \; 1.$$

$\square$

Consider the general case: graphs of diameter $k \geq 3$. We will need the following simple observation.

Lemma 9 (see, for example, [8]). *The radius of a simple path of length k is equal to $\lceil \frac{k}{2} \rceil$ and the central vertices of the path are at a distance $\lceil \frac{k}{2} \rceil$ u $\lfloor \frac{k}{2} \rfloor$ from its endpoints.*

Now, investigate a radius of graphs of the class $\mathcal{F}_{n,k,p}$, $k \geq 3$.

Lemma 10. *If $k \geq 3$, $p \geq 1$ and $G \in \mathcal{F}_{n,k,p}$ then $r(G) = \lceil \frac{k}{2} \rceil$.*

Proof. Let $G = G(u, s, F)$, here $u = (u_0, u_1, \ldots, u_{k-2})$,

$$0 \leq s \leq \left\lfloor \frac{k-3}{2} \right\rfloor \tag{8}$$

and $F \in \mathcal{F}_{n-k+3,3,p}(u_s, u_{s+1})$. By Lemma 5, we have $d(G) = d(P) = k$, where P is an arbitrary diametral path of the graph G. Therefore, the following relation hold

$$r(G) \geq \left\lceil \frac{d(G)}{2} \right\rceil = \left\lceil \frac{k}{2} \right\rceil. \tag{9}$$

Note that u_0, u_{k-2} is the unique pair of diametral vertices of the graph G, and this vertices are the endpoints of the path P by Lemma 5. Consider a vertex $w \in V(P)$ such that $\rho_G(u_0, w) = \lceil \frac{k}{2} \rceil$ (or $\rho_G(u_{k-2}, w) = \lceil \frac{k}{2} \rceil$). Then, by Lemma 9, we obtain

$$e_P(w) = r(P) = \left\lceil \frac{k}{2} \right\rceil. \tag{10}$$

Let v be an arbitrary vertex from $F \setminus P$. By (8), we have $\lceil \frac{k}{2} \rceil \geq s + 2$. In view of Lemma 6 and properties of the graph F, there is a vertex $v^* \in V(P)$ such that $B^G_{\lceil k/2 \rceil}(v) = B^G_{\lceil k/2 \rceil}(v^*)$. Reckoning with (10), we obtain

$$w \in B^G_{\lceil \frac{k}{2} \rceil}(v^*) = B^G_{\lceil \frac{k}{2} \rceil}(v).$$

Consequently, $\rho_G(w, v) \leq \lceil \frac{k}{2} \rceil$. Therefore, due to the arbitrariness of the choice of the vertex v and the equality (10), infer

$$e_G(w) = \left\lceil \frac{k}{2} \right\rceil. \tag{11}$$

Thus, by (9) and (11), we conclude $r(G) = \lceil \frac{k}{2} \rceil$. $\square$

Lemmas 5(iii), 9 and the proof of Lemma 10 imply the following corollary.

Corollary 5. *For each graph $G \in \mathcal{F}_{n,k,p}$, every central vertex of its arbitrary diametral path is the central vertex of the graph G.*

From Corollary 3 and Lemma 10, we obtain the following theorem and its corollaries.

Theorem 6. *For every fixed integer $k \geq 3$, almost all n-vertex graphs of diameter k have radius $\lceil \frac{k}{2} \rceil$.*

Corollary 6. *There are almost no self-centered n-vertex graphs of fixed diameter $k \geq 3$, while almost all graphs of diameter $k = 1, 2$ are self-centered.*

Proof. Reckoning with Theorem 5, we obtain that all vertices of almost all graphs of diameter $k = 1, 2$ are central. And for $k \geq 3$, it is enough notice that if $G \in \mathcal{F}_{n,k,p}$ then $d(G) > r(G)$. $\square$

Corollary 7. *Almost all n-vertex graphs of odd fixed diameter have at least two central vertices.*

Proof. All vertices of a complete graph are central. And for odd $k \geq 3$, the diametral path of any graph from the class of typical graphs $\mathcal{F}_{n,k,p}$ contains two central vertices of the entire graph by Lemma 9 and Corollary 5. $\square$

Corollary 8. *For every fixed $k \geq 3$, almost all n-vertex graphs of each of the following classes $\mathcal{J}_{n,\, d \geq k}$, $\mathcal{J}^*_{n,\, d \geq k}$ are connected and have radius $\lceil \frac{k}{2} \rceil$, diameter k.*

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