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ON THE MAXIMALITY OF DEGREES OF METRICS UNDER COMPUTABLE REDUCIBILITY

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ABSTRACT. We study the semilattice $\mathcal{CM}_c(\mathbf{X})$ of degrees of computable metrics on a Polish space $\mathbf{X}$ under computable reducibility. It is proved that this semilattice does not have maximal elements if $\mathbf{X}$ is a noncompact space. It is also shown that the degree of the standard metric on the unit interval is maximal in the respective semilattice.

KEYWORDS: computable metric space, Cauchy representation, reducibility of representations, computable analysis

The present paper is a sequel to the article [1] in which the author started the investigation of the first-order properties of the degree structure of metrics on a Polish space under computable reducibility $\leq_c$. All necessary definitions, notations and preliminary results can be found there, below we briefly recall some of them.

We regard Polish spaces as structures of the kind $\mathbf{X} = (X, \tau, W, \nu)$, where (X, τ) is a Polish space in the classical sense (i. e., a separable completely metrizable space), W is some countable dense subset of X, and ν is a numbering of W. Under this approach, it is natural to study a reducibility of complete metrics on $\mathbf{X}$ induced by computable reducibility of their respective Cauchy representations.

Recall that the *Cauchy representation* of a Polish metric space (X, ρ, W, ν) with a distinguished dense subset W and its numbering ν is a partial mapping δ_ρ from the Baire space ω^ω onto X that puts points $x \in X$ into correspondence with sequences of elements of W quickly converging to x, in other words, for $f \in \omega^\omega$ and $x \in X$,

$$\delta_\rho(f) = x \text{ if } w_{f(n)} \to x \text{ and } \rho(w_{f(n)}, x) < 2^{-n} \text{ for all } n,$$

where $w_k = \nu k$ is the kth point in W. Every $f \in \omega^\omega$ with the above property is called a *Cauchy name* for x.

More generally, a *representation* of a set X is a partial surjection $\delta\colon \omega^\omega \to X$. This notion generalizes the notion of a numbering of a countable set. Theory of representations and computability theory on represented spaces were studied by Kreitz and Weihrauch (see [2, 3]). Computable reducibility of representations was introduced in [2], it is an analogue of the notion of reducibility of numberings. Representation δ_1 of a set X is said to be *computably reducible* to a representation δ_2 of X, $\delta_1 \leq_c \delta_2$, if there is a Turing functional Φ_e such that

$$\delta(f) = \delta' \circ \Phi_e(f) \text{ for } f \in \mathrm{dom}(\delta).$$

If δ_X and δ_Y are representations of sets X and Y, then we say that a partial function $F\colon X \to Y$ is (δ_X, δ_Y)-*computable* if there is a Turing functional Φ_e such that

$$F \circ \delta_X(f) = \delta_Y \circ \Phi_e(f) \text{ for } f \in \mathrm{dom}(F \circ \delta_X).$$

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It is clear that for representations δ_1, δ_2 of X it holds $\delta_1 \leq_c \delta_2$ iff the identity function id_X is (δ_1, δ_2)-computable.

Let $\mathbf{X} = (X, \tau, W, \nu)$ be a Polish space. The set of all complete metrics on X compatible with topology τ is denoted by $M(\mathbf{X})$. Having fixed W and ν, we can define the notion of computable reducibility of metrics on $\mathbf{X}$: for $\rho_1, \rho_2 \in M(\mathbf{X})$, we say that $\rho_1 \leq_c \rho_2$ if $\delta_{\rho_1} \leq_c \delta_{\rho_2}$; in other words, reducibility of metrics is induced by reducibility of their Cauchy representations. Computable reducibility of metrics in $M(\mathbf{X})$ leads to the degree structure $\mathcal{M}_c(\mathbf{X})$. The subordering of $\mathcal{M}_c(\mathbf{X})$ consisting of c-degrees of computable metrics on $\mathbf{X}$ is denoted by $\mathcal{CM}_c(\mathbf{X})$.

In [1] it was shown that $\mathcal{CM}_c(\mathbf{X})$ is a lower semilattice for any space $\mathbf{X}$. It was also proved that the degree of a metric ρ such that the computable space (X, ρ, W, ν) contains a computable limit point is not minimal in $\mathcal{CM}_c(\mathbf{X})$. Main result of the present article is dual to the second mentioned result: we show that the degree of any metric on a noncompact space $\mathbf{X}$ is not maximal in $\mathcal{CM}_c(\mathbf{X})$. In other words, our main result can be stated as follows.

Theorem 1. *Let $\mathbf{X}$ be a noncompact Polish space. Then $\mathcal{CM}_c(\mathbf{X})$ has no maximal elements.*

It is worth noting that this result is a direct generalization of a result from [4] that shows that the degree of the standard metric $\rho_R(x, y) = |x - y|$ on the space $\mathbf{R}$ of real numbers is not maximal in the ordering of c-degrees of computable metrics on $\mathbf{R}$. Theorem 1 can be fully relativized to the case of metrics of arbitrary complexity. In other words, the following holds.

Theorem 2. *Let $\mathbf{X}$ be a noncompact Polish space. Then $\mathcal{M}_c(\mathbf{X})$ has no maximal elements.*

Although this result is obtained by a straightforward relativization, the very possibility of this relativization is a nontrivial fact which is based on a few important observations made throughout the proof of Theorem 1.

The following theorem is related to lattice embeddings into $\mathcal{CM}_c(\mathbf{X})$ and is again a direct generalization of a result from [4].

Theorem 3. *Let ρ be a computable metric on a noncompact space $\mathbf{X}$. Then the countable atomless Boolean algebra $\mathrm{Int}(1 + \eta)$ embeds into $\mathcal{CM}_c(\mathbf{X})$ above $\deg_c(\rho)$ with preservation of joins and meets.*

Theorem 1 does not necessarily hold in compact spaces, as the following counterexample shows: it turns out that c-degree of the standard metric on the unit interval $\mathbf{I}$ is maximal in $\mathcal{CM}_c(\mathbf{I})$. More formally, consider the Polish space $\mathbf{I} = (I, \tau_I, \mathbb{Q}, \nu_{\mathbb{Q}})$, where τ_I is the usual topology on the unit interval $I = [0, 1]$, $\mathbb{Q}$ is the set of rational numbers in I and $\nu_{\mathbb{Q}}$ is a Gödel numbering of $\mathbb{Q}$. Let ρ_R also denote the restriction of the usual real metric to the unit interval. Then the following result holds.

Proposition 1. $\deg_c(\rho_R)$ *is a maximal element of* $\mathcal{CM}_c(\mathbf{I})$.

However, the following proposition shows that $\deg_c(\rho_R)$ is not a maximal element of $\mathcal{M}_c(\mathbf{I})$, that is, there exists a noncomputable metric strictly above ρ_R.

Proposition 2. *There exists a $\mathbf{0}'$-computable metric ρ^* on $\mathbf{I}$ such that $\rho_R <_c \rho^*$.*

The paper is organized as follows. Section 1 contains the proofs of Theorems 1 and 2. Section 2 consists of the proof of Theorem 3. In the last section we prove the two results regarding the unit interval.

1. Proof of Theorem 1

1.1. **Proof idea.** Suppose $\mathbf{X} = (X, \tau, W, \nu)$ is a noncompact space and let $\rho \in M(\mathbf{X})$ be a computable metric on $\mathbf{X}$. We need to construct a computable metric $\rho' \in M(\mathbf{X})$ such that $\rho' >_c \rho$. More specifically, metric ρ' should satisfy the following requirements for $e \in \omega$:

$\mathcal{R}_e$: $\rho' \nleq_c \rho$ via Φ_e,

$\mathcal{S}$: $\rho \leq_c \rho'$.

Our proof is based on a generalization of the construction of Theorem 1 from [4]. In that theorem the analogues of the above requirements were satisfied to obtain a computable metric (indeed, countably many pairwise nonequivalent computable metrics) strictly above the standard metric ρ_R on the space of real numbers. Later, in Theorem 4.1 of [1], it was shown that the strategy for negative requirements $\mathcal{R}_e$ can be generalized to the case of an arbitrary computable metric space with a computable limit point.

Let us briefly discuss the strategy for an isolated requirement $\mathcal{R}_e$. Recall that a metric space is compact if and only if it is complete and totally bounded. Since we only work with complete metric spaces, it is clear that the noncompact space (X, ρ) must not be totally bounded, that is, there exists a rational number $r \in (0, \frac{1}{3})$ such that there is no finite $3r$-net in (X, ρ). Since we do not know this r in advance, our proof will be non-uniform. For the rest of the proof, fix such a number r. Fix also a $k \in \omega$ such that $2^{-k} < r$. For natural numbers a, b, n, denote

$$\bar{a} = (a, \dots, a, \dots) \in \omega^\omega,\ \ a^n\bar{b} = (a, \dots, a, b, b, \dots) \in \omega^\omega \ \ (a \text{ repeated } n \text{ times}).$$

Now back to the strategy for $\mathcal{R}_e$. We pick a special point $y = w_a \in W$ and wait until $\Phi_e(\bar{a})(k+1) \downarrow = u$. When it happens, we pick another special point $x = w_b$ such that $\rho(x, w_u) > 3r$ (it is possible by the choice of r). Consider the sequence $a^p\bar{b} \in \omega^\omega$, where $p = \varphi_e(\bar{a})(k+1)$. By the Use Principle we have $\Phi_e(a^p\bar{b})(k+1) = \Phi_e(\bar{a})(k+1) = u$, thus, if we arrange our construction so that $a^p\bar{b}$ is a ρ'-name for w_b, then $\rho(w_u, w_b) > 3r > r$ implies that $\Phi_e(a^p\bar{b})$ is not a ρ-name for w_b, which means that $\mathcal{R}_e$ is satisfied. In order to make $a^p\bar{b}$ a ρ'-name for w_b, it is sufficient to ensure that $\rho'(x, y) < 2^{-p+1}$. This can be done by means of a continuous deformation of a small neighbourhood $B_\rho(x, r)$ of x that makes x close to y. A construction of such a deformation is described in [1]: we embed the metric space (X, ρ, W) into the Banach space ℓ^∞ via the Fréchet embedding F defined by

$$(F(x))_i = \rho(x, w_i) - \rho(w_i, w_0),$$

then the deformation in question is given by

$$\Gamma(z) = \left(\tfrac{r - \rho(x,z)}{r}\, h\right) ^\frown \left(F(y) - \tfrac{\rho(x,z)}{r}(F(y) - F(z))\right)$$

for $z \in B_\rho(x, r)$, where $h < 2^{-p+1}$ and $^\frown$ stands for the concatenation of a real and an infinite sequence of reals. Mapping Γ can be extended to a map $\gamma\colon X \to \ell^\infty$ by letting $\gamma(z) = 0 ^\frown F(z)$ for $z \notin B_\rho(x, r)$. Letting $\rho_1(z, v) = \|\gamma(z) - \gamma(v)\|$, we obtain a deformed metric ρ_1 carrying the topology τ (see Proposition 5.1 of [1]) and such that $\rho_1(x, y) = h$. Thus, ρ_1 satisfies $\mathcal{R}_e$. The following formulas express ρ_1 in terms of ρ; these are exactly Formulas (5.1)–(5.3) of [1].

Let z, v be points of X; assume that $\rho(x,z) \leqslant \rho(x,v)$. Then, if $\rho(x,z) < r$ and $\rho(x,v) \geqslant r$, we have

$$\rho_1(z,v) = \max\left(\tfrac{r-\rho(x,z)}{r}\, h,\ \tfrac{\rho(x,z)}{r}\rho(z,v) + \tfrac{r-\rho(x,z)}{r}\rho(v,y)\right); \tag{1.1}$$

if $\rho(x,v) < r$,

$$\rho_1(z,v) = \max\left(\tfrac{\rho(x,v)-\rho(x,z)}{r}\, h,\ \tfrac{\rho(x,z)}{r}\rho(z,v) + \tfrac{\rho(x,v)-\rho(x,z)}{r}\rho(v,y)\right); \tag{1.2}$$

and if $\rho(x,z) \geqslant r$, then

$$\rho_1(z,v) = \rho(z,v). \tag{1.3}$$

Metric ρ' satisfying all requirements $\mathcal{R}_e$ will be a pointwise limit of metrics ρ_s defined at stages of the construction. At each stage s we also output a finite set $A_s \subseteq W$ such that:

(1) $\rho_s(z,v) = \rho_{s+1}(z,v) = \ldots = \rho'(z,v)$ for all $z, v \in A_s$,
(2) $A_s \subseteq A_t$ for $s \leqslant t$,
(3) $\bigcup_{i\in\omega} A_s = W$.

This will imply that ρ' is computable: informally speaking, we can compute the distance $\rho'(z,v)$ between two given special points z and v as soon as they both are contained in a set A_s.

Strategy for the global positive requirement $\mathcal{S}$ is also the same as in [4]. In a nutshell, strategy from [4] is based on the fact that the space $(\mathbf{R}, \rho_R)$ is unbounded, thus we can choose balls of the same radius to be deformed in each $\mathcal{R}_n$-strategy, which gives us an algorithm of c-reduction of ρ_R to the metric under construction. The same idea works here: by the choice of r, there exists an infinite sequence of nonintersecting balls B_n of the same radius r. These balls can be used in the respective $\mathcal{R}_n$-strategies. The fact that the balls B_n possess the same radius will permit us to prove Lemma 4 that asserts that $\rho'(z,v) \leqslant C \cdot \rho(z,v)$ for all $z, v \in X$ and some real number C. From the following basic lemma it then follows that $\rho \leq_c \rho'$.

Lemma 1 ([5]). *Let $d, d' \in M(\mathbf{X})$. If there is a $C > 0$ such that $d'(x,y) \leqslant C \cdot d(x,y)$ for all $x, y \in X$, then $d \leq_c d'$.*

1.2. **Analytical reasoning.** In this subsection we explain the analytical reasoning behind the construction of ρ' that is very similar to the one in Section 5 of [1]. Throughout the subsection, we often refer to Section 5 of [1] and use various results from there.

First of all, without loss of generality we will assume that the metric ρ is bounded by 1: if this is not the case, then letting $\hat{\rho}(x,y) = \min(\rho(x,y), 1)$ produces a computable metric $\hat{\rho}$ that is c-equivalent to ρ. This is an important detail. Later on we will see why it is convenient to bound the metric from above.

By the choice of the number r, there exists an infinite sequence $(\tilde{x}_n)_{n\in\omega}$ such that $\rho(\tilde{x}_n, \tilde{x}_m) > 3r$ for $n \neq m$. Moreover, such a sequence can be found effectively, given r as a parameter. We have $B_\rho(\tilde{x}_n, r) \cap B_\rho(\tilde{x}_m, r) = \varnothing$ for $m \neq n$. Denote $x_n = \tilde{x}_{2n}$, $y_n = \tilde{x}_{2n+1}$. Points x_n and y_n are used in the strategy for requirement $\mathcal{R}_n$. Let h_n be a suitable small positive number defined in the strategy. Similarly to the proof of Theorem 4.1 of [1], we build a sequence of metrics ρ_n as follows. Let $\rho_0 = \rho$. Suppose that the metric ρ_n has already been defined. Define a map

$\Gamma_{n+1}\colon B_{\rho_n}(x_n, r) \to \ell^\infty$ by

$$\Gamma_{n+1}(z) = \big(\tfrac{r-\rho_n(x_n,z)}{r}\, h_n\big) \frown \big(F_n(y_n) - \tfrac{\rho_n(x_n,z)}{r}(F_n(y_n) - F_n(z))\big),$$

where F_n is the Fréchet embedding of (X, ρ_n, W) into ℓ^∞. Extend Γ_{n+1} to a mapping $\gamma_{n+1}\colon X \to \ell^\infty$ by putting $\gamma_{n+1}(z) = 0 \frown F_n(z)$ for $z \notin B_{\rho_n}(x_n, r)$. Let $\rho_{n+1}(z, v) = \|\gamma_{n+1}(z) - \gamma_{n+1}(v)\|$.

Metric ρ_{n+1} can be expressed in terms of ρ_n by Formulas (1.1)–(1.3). The following lemma is an analogue of Proposition 5.4 of [1] and is proved in the same way.

Lemma 2. *For all $n \in \omega$, the following hold:*

(1) *ρ_n is a well-defined complete metric inducing topology τ on X;*
(2) *$\rho_n(z, v) = \rho_{n-1}(z, v)$ for all $z, v \notin B_{\rho_{n-1}}(x_{n-1}, r)$;*
(3) *The identity map id_X induces a series of surjective isometries $B_{\rho_0}[x_n, r] \to B_{\rho_1}[x_n, r] \to \ldots \to B_{\rho_n}[x_n, r]$.*

Proof. Proceed as in the proof of Proposition 5.4 of [1]. The last clause follows from Propositions 5.2 and 5.3 of [1] and the fact that $\rho_0(x_n, \tilde{x}_m) > 3r$ for all $\tilde{x}_m \neq x_n$, thus $\min(\rho_0(x_n, x_i) - r, \rho_0(x_n, y_i)) > 2r > r$ for $i < n$. □

Corollary 1. *For all $z, v \in X$, there are at most two $n \in \omega$ such that $\rho_{n+1}(z, v) \neq \rho_n(z, v)$.*

Proof. If $z, v \notin \bigcup_{n\in\omega} B_{\rho_0}(x_n, r)$, then $\rho_n(z, v) = \rho_0(z, v)$ for all n. If $z \in B_{\rho_0}(x_n, r)$ for some n, then we may have $\rho_{n+1}(z, v) \neq \rho_n(z, v)$. A similar change may happen if $v \in B_{\rho_0}(x_m, r)$ for some m. Other than that, no changes will occur. □

Corollary 1 implies that there exists a pointwise limit ρ' of metrics ρ_n that also is a metric on X.

Lemma 3. *ρ' is a complete metric inducing topology τ on X.*

Proof. Repeat the proof of Proposition 5.5 of [1]. □

Lemma 4. *There exists a constant $C > 0$ such that $\rho'(z, v) \leqslant C \cdot \rho_0(z, v)$ for all $z, v \in X$.*

Proof. First of all we show that $\rho_1(z, v) \leqslant \frac{2}{r} \cdot \rho_0(z, v)$ for all $z, v \in X$. Fix arbitrary points $z, v \in X$. By symmetry we will assume that $\rho_0(x_0, z) \leqslant \rho_0(x_0, v)$. The following three possibilities can occur.

Case 1. $\rho_0(x_0, z) < r$ and $\rho_0(x_0, v) \geqslant r$. Then

$$\begin{aligned}
\rho_1(z, v) &= \max\big(\tfrac{r-\rho_0(x_0,z)}{r}\, h_0,\ \tfrac{\rho_0(x_0,z)}{r}\rho_0(z, v) + \tfrac{r-\rho_0(x_0,z)}{r}\rho_0(y_0, v)\big) \\
&\leqslant \max\big(\tfrac{\rho_0(x_0,v)-\rho_0(x_0,z)}{r}\, h_0,\ \tfrac{\rho_0(x_0,z)}{r}\rho_0(z, v) + \tfrac{\rho_0(x_0,v)-\rho_0(x_0,z)}{r}\rho_0(y_0, v)\big) \\
&\leqslant \max\big(\tfrac{\rho_0(z,v)}{r}\, h_0,\ \tfrac{\rho_0(x_0,z)}{r}\rho_0(z, v) + \tfrac{\rho_0(z,v)}{r}\rho_0(y_0, v)\big) \\
&= \max\big(\tfrac{h_0}{r},\ \tfrac{\rho_0(x_0,z)+\rho_0(y_0,v)}{r}\big) \cdot \rho_0(z, v).
\end{aligned}$$

Case 2. $\rho_0(x_0, v) < r$. One can obtain the same boundary in the same manner.

Case 3. $\rho_0(x_0, z) \geqslant r$. In this case, $\rho_1(z, v) = \rho_0(z, v)$.

By construction, $h_0 \leqslant 1$. We also have $\rho_0(x_0, z) + \rho_0(y_0, v) \leqslant 2$; this is the moment when we see why it was important to bound the metric ρ_0 from above. Since $r < 1$, the claim follows.

To prove the statement of the lemma, fix arbitrary $z, v \in X$ and consider the following possibilities.

Case 1. $z, v \notin \bigcup_{n\in\omega} B_{\rho_0}(x_n, r)$. Then $\rho'(z,v) = \rho_0(z,v)$.

Case 2. $z \in B_{\rho_0}(x_n, r)$ and $v \in B_{\rho_0}(x_m, r)$, $n \neq m$ (say $n < m$). We have

$$\rho_0(z,v) = \rho_1(z,v) = \ldots = \rho_n(z,v),$$
$$\rho_0(z,x_n) = \rho_1(z,x_n) = \ldots = \rho_n(z,x_n),$$
$$\rho_0(y_n,v) = \rho_1(y_n,v) = \ldots = \rho_n(y_n,v).$$

Then, arguing as above, we obtain $\rho_{n+1}(z,v) \leqslant \frac{2}{r} \cdot \rho_n(z,v)$. Similarly, we have $\rho_{n+1}(v,x_m) \leqslant \frac{2}{r} \cdot \rho_n(v,x_m)$ and $\rho_{n+1}(y_m,z) \leqslant \frac{2}{r} \cdot \rho_n(y_m,z)$. Then

$$\rho_{n+1}(z,v) = \rho_{n+2}(z,v) = \ldots = \rho_m(z,v),$$
$$\rho_{n+1}(v,x_m) = \rho_{n+2}(v,x_m) = \ldots = \rho_m(v,x_m),$$
$$\rho_{n+1}(y_m,z) = \rho_{n+2}(y_m,z) = \ldots = \rho_m(y_m,z),$$

and after several careful substitutions we have

$$\begin{aligned}
\rho_{m+1}(z,v) &\leqslant \max\left(\tfrac{h_m}{r},\ \tfrac{\rho_m(v,x_m)+\rho_m(y_m,z)}{r}\right) \cdot \rho_m(z,v) \\
&\leqslant \max\left(\tfrac{h_m}{r},\ \tfrac{1}{r}\left(\tfrac{2}{r}\rho_0(v,x_m) + \tfrac{2}{r}\rho_0(y_m,z)\right)\right) \cdot \rho_m(z,v) \\
&\leqslant \max\left(\tfrac{1}{r},\ \tfrac{4}{r^2}\right) \cdot \tfrac{2}{r} \cdot \rho_0(z,v) \\
&= \tfrac{8}{r^3} \cdot \rho_0(z,v).
\end{aligned}$$

After that, the distance between these points will never change, and $\rho'(z,v) = \rho_{m+1}(z,v)$.

Case 3. There is only one n such that z or v (or both) is contained in $B_{\rho_0}(x_n, r)$. A similar boundary can be obtained in this case as well.

Now, $C = \frac{8}{r^3}$ can serve as the constant from the statement of the lemma. □

We will also need the following observation.

Lemma 5. *The following hold:*

(1) *If* $z \notin \left(\bigcup_{n\in\omega} B_{\rho_0}[x_n, r]\right) \cup \{y_n \mid n \in \omega\}$, *then for all* $n \in \omega$

$$\rho'(z, B_{\rho_0}[x_n,r]) = \inf_{v\in B_{\rho_0}[x_n,r]} \rho'(z,v) \geqslant \min(\rho_0(x_n,z) - r, \rho_0(y_n,z)).$$

(2) *Let* $B_1 = B_{\rho_0}[x_n, r]$, $B_2 = B_{\rho_0}[x_m, r]$ *for* $m \neq n$. *Then*

$$\rho'(B_1, B_2) = \inf_{z\in B_1, v\in B_2} \rho'(z,v) \geqslant r.$$

Proof. (1). For all $v \in B_{\rho_0}[x_n, r]$ we have $\rho_0(z,v) = \rho_1(z,v) = \ldots = \rho_n(z,v)$, and after that $\rho_{n+1}(z,v) \geqslant \min(\rho_n(x_n,z) - r, \rho_n(y_n,z)) = \min(\rho_0(x_n,z) - r, \rho_0(y_n,z))$ by the consideration before Proposition 5.2 of [1]. After that, distance between these points never changes, and $\rho'(z,v) = \rho_{n+1}(z,v)$.

(2). Fix $m < n$ and fix two points $z \in B_{\rho_0}[x_n, r], v \in B_{\rho_0}[x_m, r]$. As above, we have $\rho_0(x_n,v) = \rho_1(x_n,v) = \ldots = \rho_m(x_n,v)$, and after that $\rho_{m+1}(x_n,v) \geqslant \min(\rho_m(x_m,x_n) - r, \rho_m(y_m,x_n)) = \min(\rho_0(x_m,x_n) - r, \rho_0(y_m,x_n)) > 2r$. Similarly, $\rho_{m+1}(y_n,v) \geqslant \min(\rho_0(x_m,y_n) - r, \rho_0(y_m,y_n)) > 2r$. Afterwards, we have $\rho_{m+1}(z,v) = \ldots = \rho_n(z,v)$ and $\rho_{n+1}(z,v) \geqslant \min(\rho_n(x_n,v) - r, \rho_n(y_n,v)) > r$. □

1.3. **Construction.** At the start of the construction, we are given a computable metric $\rho \in M(\mathbf{X})$. We have fixed a rational $r \in (0, \frac{1}{3})$ such that the space (X, ρ) does not have a finite $3r$-net and have chosen a $k \in \omega$ such that $2^{-k} < r$.

Stage 0. Let $A_0 = \varnothing$, $\rho_0 = \rho$.

Stage $s+1$. Appoint a follower to requirement $\mathcal{R}_s$ as follows. Since A_s is not a $3r$-net in (X, ρ_0), we can find a special point y_s such that $\rho_0(y_s, w) > 3r$ for all $w \in A_s$. Let $A = A_s \cup \{w_s, y_s\}$. Let $e \leqslant s$ be the least number such that the requirement $\mathcal{R}_e$ has not been met yet and $\Phi_e(\bar{a})(k+1) \downarrow= u$ in $s+1$ steps, where $w_a = y_e$ is the follower of $\mathcal{R}_e$. If no such e exists, end the stage. Otherwise, choose an element $x_e = w_b$ such that $\rho_0(x_e, w) > 3r$ for all $w \in A \cup \{w_u\}$. Let $\rho_{s'}$ be the metric most recently defined in the construction. Proceed as described in Subsection 1.1 with $\rho_{s'}$ in place of ρ and obtain a new metric ρ_{s+1}. Let $A_{s+1} = A \cup \{w_u, x_e\}$.

1.4. **Verification.** Define ρ' to be the pointwise limit of metrics defined in the construction. As in Subsection 6.2 of [1] it can be shown that ρ' is a computable metric. By construction, $\rho_0(x_i, x_j) > 3r$ for all $i \neq j$ and $\rho_0(x_i, y_j) > 3r$ for all i, j. From the discussion in Subsection 1.1 it is not hard to see that all requirements $\mathcal{R}_e$ are satisfied. By Lemmas 1 and 4, $\rho \leq_c \rho'$.

1.5. **Proof of Theorem 2.** Suppose that $\rho \in M(\mathbf{X})$ is an arbitrary metric on $\mathbf{X}$. We can carry out the construction from Theorem 1 to obtain a metric ρ' satisfying all negative requirements $\mathcal{R}_e$, then Lemmas 1 and 4 guarantee that $\rho \leq_c \rho'$. It remains to note that Lemma 1 holds in arbitrary metric spaces, not necessarily computable.

2. Proof of Theorem 3

As mentioned in the introduction, this theorem is a generalization of a result from [4], namely of Theorem 3 of that paper. The idea of the proof is the same as in [4]: we embed the Boolean algebra $\mathcal{P}_{\mathrm{Comp}}(\omega)$ of computable subsets of ω into $\mathcal{CM}_c(\mathbf{X})$ and use the fact that $\mathrm{Int}(1+\eta)$ is embeddable into $\mathcal{P}_{\mathrm{Comp}}(\omega)$. Below we explain how to embed the 4-element Boolean algebra into $\mathcal{CM}_c(\mathbf{X})$, and then we give a hint how to generalize this construction to obtain the embedding of $\mathcal{P}_{\mathrm{Comp}}(\omega)$.

To show that the 4-element Boolean algebra embeds into $\mathcal{CM}_c(\mathbf{X})$, we construct metrics d_0, d_1 and d_{01} such that $\deg_c(d_{01}) = \sup(\deg_c(d_0), \deg_c(d_1))$ and $\deg_c(\rho) = \inf(\deg_c(d_0), \deg_c(d_1))$ in $\mathcal{CM}_c(\mathbf{X})$. Metrics d_0 and d_1 will be built in a single construction and will satisfy the following requirements:

$\mathcal{R}_{ie}$: $d_i \not\leq_c d_{1-i}$ via Φ_e, for $i = 0, 1$ and $e \in \omega$,

$\mathcal{S}$: $\rho \leq_c d_0, d_1$.

Fix a rational number $r \in (0, \frac{1}{3})$ such that there is no finite $3r$-net in (X, ρ). Requirements $\mathcal{R}_{ie}$ and $\mathcal{S}$ are satisfied in the exact same manner as in Theorem 1.

Metric d_{01} contains all deformations defined in the course of the construction of both d_0 and d_1. More formally, at stage 0 we initially define $d_{0,0}, d_{1,0}, d_{01,0} = \rho$. Suppose that at stage s we finish the $\mathcal{R}_{0e}$-strategy and introduce a deformation $\Gamma\colon B_{d_{0,s'}}(x_{0e}, r) \to \ell^\infty$:

$$\Gamma(z) = \left(\tfrac{r - d_{0,s'}(x_{0e}, z)}{r}\, h_{0e}\right) ^\frown \left(F_{s'}(y_{0e}) - \tfrac{d_{0,s'}(x_{0e}, z)}{r}(F_{s'}(y_{0e}) - F_{s'}(z))\right),$$

where $d_{0,s'}$ is the most recently defined approximation of the metric d_0 and $F_{s'}$ is the Fréchet embedding of $(X, d_{0,s'}, W)$ into ℓ^∞. We add an analogue of Γ to d_{01} as follows. Let $\tilde{F}_{s''}$ be the Fréchet embedding of $(X, d_{01,s''}, W)$ into ℓ^∞, where $d_{01,s''}$ is the most recently defined approximation of the metric d_{01}. Let

$$\tilde{\Gamma}(z) = \big(\tfrac{r - d_{0,s''}(x_{0e},z)}{r}\, h_{0e}\big) \frown \big(\tilde{F}_{s''}(y_{0e}) - \tfrac{d_{0,s''}(x_{0e},z)}{r}(\tilde{F}_{s''}(y_{0e}) - \tilde{F}_{s''}(z))\big),$$

Extend $\tilde{\Gamma}$ to a mapping $\tilde{\gamma}\colon X \to \ell^\infty$ in the usual way and let $d_{01,s}(z,v) = \|\tilde{\gamma}(z) - \tilde{\gamma}(v)\|$. In a similar way we incorporate into d_{01} every deformation defined for the metric d_1.

To see that the $\mathcal{R}_{0e}$-strategy succeeds, let $y_{0e} = w_a$ and $x_{0e} = w_b$ be the points used in the strategy. When $\mathcal{R}_{0e}$ requires attention, say at stage s, let $u = \Phi_e(\bar{a})(k+1)$. Points y_{0e} and x_{0e} uniquely correspond to the requirement $\mathcal{R}_{0e}$, thus y_{0e} is not used in the construction of d_1. In particular, y_{0e} does not belong to any ball $B_\rho(x_{1e'}, r)$, and by Lemma 5 we have $d_1(y_{0e}, w_u) > r$ regardless of where w_u is located. This guarantees the success of diagonalization against Φ_e.

Observe that $\deg_c(\rho)$ is the greatest lower bound of $\deg_c(d_0)$ and $\deg_c(d_1)$. To prove this, it suffices to show that, given a d_0-name and a d_1-name for $z \in X$, we can effectively construct a ρ-name for z. Fix a d_0-name f_0 and a d_1-name f_1 for z. Let $S = \{x_{ie}, y_{ie} \mid i = 0,1, e \in \omega\}$ and $T = \{x_{ie} \mid i = 0,1, e \in \omega\}$. The construction appoints followers $v \in S$ in such a way that for each n it holds $\rho(v, w_n) > 3r$ for all followers v appointed after stage $n+1$. Among the remaining (finitely many) followers, we are able to find such a v^* that $\rho(v, w_n) > \frac{3r}{2}$ for all $v \neq v^*$. Clearly, this v^* may be defined ambiguously, but there is a uniform computable way to find this element for each w_n. Consider the point $w_{f_0(k+2)}$ and find the corresponding v^*. We split the proof into the following three cases.

Case 1. $v^* = x_{1e^*}$ or $v^* = y_{1e^*}$ for some $e^* \in \omega$. Then by Lemma 5(1) we have $d_0(w_{f_0(k+2)}, B_\rho[x_{0e}, r]) > \frac{r}{2}$ for all e. Since $d_0(w_{f_0(k+2)}, w_{f_0(n)}) < \frac{r}{2}$ for $n \geqslant k+2$, then $w_{f_0(n)} \notin \bigcup_{e\in\omega} B_\rho[x_{0e}, r]$ for $n \geqslant k+2$, which means that $d_0(w_{f_0(m)}, w_{f_0(n)}) = \rho(w_{f_0(m)}, w_{f_0(n)})$ for all $m, n \geqslant k+2$, thus

$$\big(f_0(k+2), f_0(k+3), \dots\big)$$

is a ρ-name for z.

Case 2. $v^* = y_{0e^*}$ for some $e^* \in \omega$. Then $d_0(w_{f_0(k+2)}, B_\rho[x_{0e}, r]) > \frac{r}{2}$, thus $d_0(z, B_\rho[x_{0e}, r]) > \frac{r}{4}$ for all $e \neq e^*$. The following possibilities can then hold:

(1) $z \in B_\rho[x_{0e^*}, r]$. Then clearly $\rho(z, v) > 2r$ for all $v \in S - \{x_{0e^*}\}$.
(2) $z \notin \bigcup_{e\in\omega} B_\rho[x_{0e}, r]$. In this case, $\rho(z, w_{f_0(k+2)}) = d_0(z, w_{f_0(k+2)}) < \frac{r}{4}$ by (1.3) and $\rho(z, v) > \frac{5r}{4}$ for all $v \in S - \{x_{0e^*}\}$.

By Lemma 5(1), in both of these cases for all $e \in \omega$ we have $d_1(z, B_\rho[x_{1e}, r]) > \frac{r}{4}$, i.e., z is far from all balls used in the construction of d_1. Then, similarly to the above, $d_1(w_{f_1(m)}, w_{f_2(n)}) = \rho(w_{f_1(m)}, w_{f_1(n)})$ for all $m, n \geqslant k+2$, and

$$\big(f_1(k+2), f_1(k+3), \dots\big)$$

is a ρ-name for z.

Case 3. $v^* = x_{0e^*}$, $e^* \in \omega$. We break this case into the following subcases:

(1) $w_{f_0(k+2)} \notin B_\rho[x_{0e^*}, r]$. As above, we have $d_0(z, B_\rho[x_{0e}, r]) > \frac{r}{4}$ for all $e \neq e^*$, thus $z \in B_\rho[x_{0e^*}, r]$ or $z \notin \bigcup_{e\in\omega} B_\rho[x_{0e}, r]$.
(2) $w_{f_0(k+2)} \in B_\rho[x_{0e^*}, r]$. Since $d_0(w_{f_0(k+2)}, z) < \frac{r}{4}$, by Lemma 5 we must have $\rho(z, x_{0e^*}) < \frac{5r}{4}$ or $\rho(z, y_{0e^*}) < \frac{r}{4}$.

In any of these subcases, arguing as in Case 2, we obtain that $d_1(z, B_\rho[x_{1e}, r]) > \frac{r}{4}$ for all $e \in \omega$. As a result, $\big(f_1(k+2), f_1(k+3), \dots\big)$ is a ρ-name for z.

Note that we can effectively decide which of the cases 1–3 occurs. It implies that some ρ-name for z can be computed, using f_0 and f_1.

It is not hard to see that $d_0, d_1 \leq_c d_{01}$. To prove that $\deg_c(d_{01})$ is the least upper bound of $\deg_c(d_0)$ and $\deg_c(d_1)$, it suffices to show that every d_{01}-name f can be translated into a d_0-name or a d_1-name for the same element z. Fix $z \in X$ and a d_{01}-name f for z. As earlier, we can compute the indices i, e of a $v^* \in S$ such that $\rho(w_{f(k+2)}, v) > \frac{3r}{2}$ for all $v \in S - \{v^*\}$. Suppose that $v^* = x_{ie}$. Then Lemma 5 readily implies that

$$B_{d_0}[w_{f(k+2)}, \tfrac{r}{2}] \cap \bigcup_{x \in T - \{v^*\}} B_\rho[x, r] = \varnothing.$$

Similarly, if v^* is a follower y_{ie}, then $B_{d_0}[w_{f(k+2)}, \frac{r}{2}] \cap \bigcup_{x \in T - \{x_{ie}\}} B_\rho[x, r] = \varnothing$. Now, suppose that $i = 0$, i.e., $v^* = x_{0e}$ or $v^* = y_{0e}$. By the above, we have $w_{f(n)} \notin \bigcup_{v \in T - \{x_{0e}\}} B_\rho[v, r]$ for all $n \geqslant k+2$. Since we add the same deformation Γ of the ball $B_\rho(x_{0e}, r)$ to both metrics d_0 and d_{01}, then by induction on s it is not hard to show that $d_{01,s}(w_{f(n)}, w_{f(m)}) = d_{0,s}(w_{f(n)}, w_{f(m)})$ for all $n, m \geqslant k+2$ and all s, thus $d_{01}(w_{f(n)}, w_{f(m)}) = d_0(w_{f(n)}, w_{f(m)})$ for these n, m, and $\big(f(k+2), f(k+3), \ldots\big)$ is a d_0-name for x. We argue in the same manner in case $i = 1$.

To show that $\mathcal{P}_{\mathrm{Comp}}(\omega)$ embeds into $\mathcal{CM}_c(\mathbf{X})$, construct an infinite sequence of metrics $d_{\{i\}}$, $i \in \omega$, similarly to the above, and for any computable set A construct the metric d_A that contains all deformations defined for the metrics d_i for all $i \in A$. Then $\deg_c(d_{A \cap B})$ will be the greatest lower bound of $\deg_c(d_A)$ and $\deg_c(d_B)$, and $\deg_c(d_{A \cup B})$ will be the least upper bound of $\deg_c(d_A)$ and $\deg_c(d_B)$, for arbitrary computable sets A and B.

3. The unit interval

3.1. **Proof of Proposition 1.** We will need the following facts.

Lemma 6 ([3], Corollary 6.2.5)**.** *Let f be a continuous real function defined on a compact set $K \subseteq \mathbf{R}$. Then $\max_{x \in K} f(x)$ and $\min_{x \in K} f(x)$ are computable uniformly in f and K.*

Lemma 7. *Let $\mathbf{X}$ be a Polish space and let $\rho, \rho' \in M(\mathbf{X})$ be computable metrics on $\mathbf{X}$ such that $\rho \leq_c \rho'$. Then ρ' is a (ρ, ρ, ρ_R)-computable function.*

Proof. Immediate. □

Suppose that $\rho \in M(\mathbf{I})$ is a computable metric on $\mathbf{I}$ such that $\rho_R \leq_c \rho$. We need to prove that $\rho \leq_c \rho_R$. By Lemma 7, ρ is (ρ_R, ρ_R, ρ_R)-computable. Then the projection functions $\rho_q(x) = \rho(q, x)$ are (ρ_R, ρ_R)-computable uniformly in $q \in \mathbb{Q}$. Denote $q_n = \nu_{\mathbb{Q}} n$.

Let f be a ρ-name for $x \in I$. We want to compute a ρ_R-name for x. This can be done as follows: for $n \in \omega$, we will progressively compute rational a_n, b_n such that $B_\rho[q_{f(k)}, 2^{-k}] \subseteq [a_n, b_n]$ for some k and the length of intervals $[a_n, b_n]$ tends to zero. Since $\bigcap_k B_\rho[q_{f(k)}, 2^{-k}] = x$, in the end we obtain a ρ_R-name for x.

At stage n, for each $k > 0$ proceed as follows. Let $0 = c_0, c_1, \ldots, c_{kn} = q_{f(k)}$ be rational points forming a partition of interval $[0, q_{f(k)}]$ into kn even subintervals. For $i \leqslant kn$, compute $\min_{y \in [0, c_i]} \rho_{q_{f(k)}}(y)$ with precision 2^{-kn}. If for some i we are able to see that $\min_{y \in [0, c_i]} \rho_{q_{f(k)}}(y) > 2^{-k}$ with said precision, pick the rightmost c_i with this property and denote it by a_n^k, otherwise let $a_n^k = 0$. By construction, a_n^k

bounds $B_\rho[q_{f(k)}, 2^{-k}]$ from below. Similarly, obtain a b_n^k that bounds $B_\rho[q_{f(k)}, 2^{-k}]$ from above. Continue this process until we find such a k that $b_n^k - a_n^k \leqslant 2^{-n}$; this will eventually happen since the balls $B_\rho[q_{f(k)}, 2^{-k}]$ converge to x. This gives us the a_n, b_n satisfying the properties described above.

3.2. **Proof of Proposition 2.** $\mathbf{0}'$-computable metric $\rho^* >_c \rho_R$ will have the form

$$\rho^*(x, y) = \|\gamma(x) - \gamma(y)\|,$$

where $\gamma \colon I \to I^2$ is a polygonal curve defined as follows. Consider the sequence $\mathbf{a}_0 = 0, \mathbf{a}_1 = 2^{-1}, \ldots, \mathbf{a}_n = 1 - 2^{-n}, \ldots$ We diagonalize against the eth functional Φ_e on the interval $[\mathbf{a}_e, \mathbf{a}_{e+1}]$ of length 2^{-e-1}. Suppose that $\mathbf{a}_e = q_a$. Use $\mathbf{0}'$ to figure out whether the computation $\Phi_e(\bar{a})(e+3)$ ever halts. If it does, let $u = \Phi_e(\bar{a})(e+3)$, then it must hold $\rho_R(q_u, q_a) \leqslant 2^{-e-3}$, otherwise we immediately win, keeping the interval $[\mathbf{a}_e, \mathbf{a}_{e+1}]$ undeformed in the metric ρ^*. Let x_e be the midpoint of $[\mathbf{a}_e, \mathbf{a}_{e+1}]$ and let $y_e = x_e - 2^{-e-3}, z_e = x_e + 2^{-e-3}$. In the spirit of the proof of Theorem 1, add to γ a deformation Γ_e of the subinterval $[y_e, z_e] \subseteq [\mathbf{a}_e, \mathbf{a}_{e+1}]$ that makes x_e close to $\mathbf{a}_e$, so we also win against Φ_e. More precisely, define a piecewise linear map $\Gamma_e \colon [y_e, z_e] \to I^2$ by its action on the following points:

$$\Gamma_e(y_e) = (y_e, 0),\ \Gamma_e(x_e) = (\mathbf{a}_e, h_e),\ \Gamma_e(z_e) = (z_e, 0),$$

where $h_e < \min(2^{-e}, 2^{-\varphi_e(\bar{a})(e+3)})$ (see Fig. 3.2).

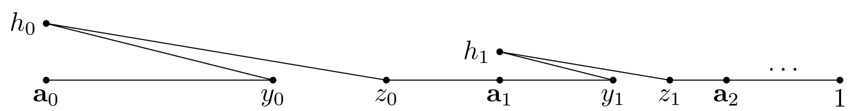

FIGURE 1. First few steps of the construction.

To see why $\rho_R \leq_c \rho^*$, notice that, since the height h_e of the deformation Γ_e is bounded by 2^{-e} and at each step e we define Γ_e in a uniform way, only scaling the length of the interval $[y_e, z_e]$ and distance to the point $\mathbf{a}_e$ down by 2, then all the deformations Γ_e actually share the same computable modulus of continuity m. Thus, m is also a modulus of continiuty of the whole curve γ. Then one can effectively translate any ρ_R-name f into a ρ^*-name g for the same point by letting $g = f \circ m$, which means that $\rho_R \leq_c \rho^*$.

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