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## A NOTE ON HARDY TYPE INEQUALITIES FOR QUATERNION-VALUED FUNCTIONS

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**ABSTRACT.** In this note, we present  $L^p$ -Hardy type inequalities and Hardy-Rellich type inequalities for generalized quaternion-valued functions. As consequences, uncertainty principles and embedding theorems are obtained in the quaternion context.

**Keywords:** Hardy inequality, Quaternion-valued function, Dirac operator, Quaternion.

### 1. INTRODUCTION

In the 90s, a new class of quaternion-valued functions introduced and studied intensively by several authors, for example, our motivation to write this paper is based on [2] (see also [1]). This recent theory has been growing intensively and was developed in a series of papers.

In what follows we will work in  $\mathbb{H}$ , the generalized skew-field of quaternions. This means each element  $x' \in \mathbb{H}$  has the following representation

$$x' = x_0 + \sum_{i=1}^n e_i x_i,$$

where  $1, e_1, \dots, e_n$  are the basis elements of  $\mathbb{H}$ . For these elements we have the multiplication rules

- $e_1^2 = \dots = e_n^2 = -1,$

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- $e_i e_j + e_j e_i = -2\delta_{ij}$ , for all  $i, j = 1, \dots, n$ . Here  $\delta_{ij}$  is the Kronecker symbol. Obviously, in the case of  $n = 4$  we get the standard skew-field of the quaternion.

The conjugate element  $\bar{x}'$  is given by  $\bar{x}' = x_0 - \sum_{i=1}^n e_i x_i$  and we have the property

$$|x'|^2 = x' \bar{x}' = \bar{x}' x' = x_0^2 + \sum_{i=1}^n x_i^2,$$

and let us also introduce the following quasi-norm

$$|x|_q = \sqrt{\sum_{i=1}^n x_i^2}$$

on  $\mathbb{H}$ , where  $x = \sum_{i=1}^n e_i x_i$ .

For this setting one can introduce the usual Dirac operator which factorises the  $n$ -dimensional Laplace operator:

$$Du = \sum_{i=1}^n e_i \frac{\partial u}{\partial x_i},$$

and it's conjugate operator

$$\bar{D}u = - \sum_{i=1}^n e_i \frac{\partial u}{\partial x_i}.$$

By multiplying these we have

$$D\bar{D} = \bar{D}D = \Delta_n,$$

where  $\Delta_n$  is the Laplacian for functions defined over domains in  $\mathbb{R}^n$ .

Note that for all  $\gamma \in \mathbb{R}$ , and  $|x|_q \neq 0$ , we have the identities

$$(1.1) \quad D \left( \frac{\bar{x}}{|x|_q^\gamma} \right) = \frac{|x|_q^\gamma D\bar{x} - \bar{x}D|x|_q^\gamma}{|x|_q^{2\gamma}} = \frac{n - \gamma}{|x|_q^\gamma},$$

and

$$(1.2) \quad \bar{D} \left( \frac{x}{|x|_q^\gamma} \right) = \frac{|x|_q^\gamma \bar{D}x - x\bar{D}|x|_q^\gamma}{|x|_q^{2\gamma}} = \frac{n - \gamma}{|x|_q^\gamma},$$

since

$$D\bar{x} = \sum_{i=1}^n e_i \frac{\partial \bar{x}}{\partial x_i} = -e_1^2 - \dots - e_n^2 = n,$$

and

$$\bar{D}x = - \sum_{i=1}^n e_i \frac{\partial x}{\partial x_i} = -e_1^2 - \dots - e_n^2 = n.$$

The main goal of this short note is to prove a version of the Hardy inequality for the Dirac operator  $D$ . As a consequence, we get the uncertainty principle for generalized quaternion-valued functions. We refer a recent open access book [3] for detailed discussions on the Hardy type inequalities and closely related topics. In the present note, in the proofs we use techniques from the papers [4] and [5] by M. Ruzhansky and the second author.

In Section 2,  $L^p$ -Hardy type inequalities are obtained. In turn, these imply Hardy type inequalities for generalized quaternion-valued functions. In Section 3, we present Hardy-Rellich type inequalities and introduce some embedding results.

## 2. $L^p$ -HARDY TYPE INEQUALITIES

In this section, we present  $L^p$ -Hardy type inequalities for the generalized Dirac operator. As a consequence, we obtain the uncertainty principle in terms of the generalised Dirac operator. Note that here and after  $\{0\}$  denotes the set of quaternions  $\{x'\}$  such that  $x = 0$  for  $x' \in \mathbb{H}$ .

**Theorem 2.1.** *Let  $\alpha, \beta \in \mathbb{R}$ . For any  $u \in C_0^\infty(\mathbb{H} \setminus \{0\})$ , and for all  $1 < p < \infty$  we have*

$$(2.1) \quad \frac{|n - \gamma|}{p} \left\| \frac{u}{|x|_q^{\frac{\gamma}{p}}} \right\|_{L^p(\mathbb{H})}^p \leq \left\| \frac{Du}{|x|_q^\alpha} \right\|_{L^p(\mathbb{H})} \left\| \frac{|u|^{p-1}}{|x|_q^\beta} \right\|_{L^{p'}(\mathbb{H})}, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

where  $\gamma = \alpha + \beta + 1$  and  $|\cdot|_q$  is the quasi-norm on  $\mathbb{H}$ . If  $\gamma \neq n$ , then the constant  $\frac{|n-\gamma|}{p}$  is sharp.

Note that for the quaternion case, that is, when

$$x' = x_0 + e_1x_1 + e_2x_2 + e_3x_3,$$

we have the following inequality

$$(2.2) \quad \frac{|4 - \gamma|}{p} \left\| \frac{u}{|x|_q^{\frac{\gamma}{p}}} \right\|_{L^p(\mathbb{H})}^p \leq \left\| \frac{Du}{|x|_q^\alpha} \right\|_{L^p(\mathbb{H})} \left\| \frac{|u|^{p-1}}{|x|_q^\beta} \right\|_{L^{p'}(\mathbb{H})}.$$

*Proof of Theorem 2.1.* The inequality is trivial when  $\gamma = n$ , so we assume that  $\gamma \neq n$ . By using identity (1.2), the divergence theorem and Schwarz's inequality we have

$$\begin{aligned} \int_{\mathbb{H}} \frac{|u|^p}{|x|_q^\gamma} dx' &= \frac{1}{n - \gamma} \int_{\mathbb{H}} |u|^p \bar{D} \left( \frac{x}{|x|_q^\gamma} \right) dx' \\ &= -\frac{1}{n - \gamma} \int_{\mathbb{H}} p |u|^{p-1} \frac{x \cdot Du}{|x|_q^\gamma} dx' \\ &\leq \left| \frac{p}{n - \gamma} \right| \int_{\mathbb{H}} \frac{|u|^{p-1}}{|x|_q^\gamma} |x \cdot Du| dx' \\ &\leq \left| \frac{p}{n - \gamma} \right| \int_{\mathbb{H}} \frac{|u|^{p-1}}{|x|_q^{\alpha+\beta}} |Du| dx' \\ &\leq \left| \frac{p}{n - \gamma} \right| \left( \int_{\mathbb{H}} \frac{|Du|^p}{|x|_q^{\alpha p}} dx' \right)^{\frac{1}{p}} \left( \int_{\mathbb{H}} \frac{|u|^{(p-1)p'}}{|x|_q^{\beta p'}} dx' \right)^{\frac{1}{p'}}. \end{aligned}$$

In the last line, we have used the Hölder inequality. Thus, we obtain the desired inequality

$$(2.3) \quad \left| \frac{n - \gamma}{p} \right| \int_{\mathbb{H}} \frac{|u|^p}{|x|_q^\gamma} dx' \leq \left( \int_{\mathbb{H}} \frac{|Du|^p}{|x|_q^{\alpha p}} dx' \right)^{\frac{1}{p}} \left( \int_{\mathbb{H}} \frac{|u|^{(p-1)p'}}{|x|_q^{\beta p'}} dx' \right)^{\frac{1}{p'}}.$$

Now it remains to show that the constant is sharp. Set

$$f(x) = \begin{cases} e^{-\frac{1}{s}|x|_q^2}, & s := \alpha - \frac{\beta}{p-1} + 1 \neq 0, \\ \frac{1}{|x|_q}, & \alpha - \frac{\beta}{p-1} + 1 = 0, \end{cases}$$

then we have

$$\frac{|Df(x)|^p}{|x|_q^{\alpha p}} = \frac{|f(x)|^p}{|x|_q^{\beta p'}}$$

which satisfies the equality condition in the Hölder inequality. It implies that the constant is sharp.  $\square$

**Corollary 2.2.** *Let  $\beta = (\alpha + 1)(p - 1)$  and  $\gamma = p(\alpha + 1)$ . For any  $u \in C_0^\infty(\mathbb{H} \setminus \{0\})$  we have*

$$(2.4) \quad \frac{|n - p(\alpha + 1)|}{p} \left\| \frac{u}{|x|_q^{\alpha+1}} \right\|_{L^p(\mathbb{H})} \leq \left\| \frac{Du}{|x|_q^\alpha} \right\|_{L^p(\mathbb{H})},$$

where  $1 < p < \infty$  and  $\alpha \in \mathbb{R}$ .

Note that when  $\alpha = 0$  and  $1 < p < n$ , the inequality (2.4) gives the following  $L^p$ -Hardy inequality for the Dirac operator

$$(2.5) \quad \left\| \frac{u}{|x|_q} \right\|_{L^p(\mathbb{H})} \leq \frac{p}{n - p} \|Du\|_{L^p(\mathbb{H})}.$$

Hölder's inequality and inequality 2.5 (with  $p = 2$ ) imply the following Heisenberg-Pauli-Weyl type uncertainly principle for quaternions.

**Corollary 2.3.** *For each  $u \in C_0^\infty(\mathbb{H} \setminus \{0\})$  we have*

$$(2.6) \quad \|u\|_{L^2(\mathbb{H})}^2 \leq \frac{1}{2} \|Du\|_{L^2(\mathbb{H})} \| |x|_q |u| \|_{L^2(\mathbb{H})}.$$

### 3. HARDY-RELLICH INEQUALITY

In this section, we present the Hardy-Rellich type inequality for the Dirac operator and some of its consequences.

**Theorem 3.1.** *Let  $\alpha, \beta \in \mathbb{R}$  such that*

$$(3.1) \quad \frac{p - n}{p - 1} \leq \alpha + \beta + 1 \leq 0,$$

then for all  $1 < p < \infty$  with  $\frac{1}{p} + \frac{1}{p'} = 1$ , we have

$$(3.2) \quad \frac{n + \gamma(p - 1) - p}{p} \left\| \frac{Du}{|x|_q^{\frac{\gamma}{p}}} \right\|_{L^p(\mathbb{H})}^p \leq \left\| \frac{D(|Du|^{p-2} \bar{D}u)}{|x|_q^\alpha} \right\|_{L^p(\mathbb{H})} \left\| \frac{Du}{|x|_q^\beta} \right\|_{L^{p'}(\mathbb{H})},$$

and for any  $u \in C_0^\infty(\mathbb{H} \setminus \{0\})$ , where  $\gamma = \alpha + \beta + 1$ .

*Proof.* Inequality (3.2) is trivial when  $\gamma = n$ , so we assume that  $\gamma \neq n$ . By using identity (1.1), the divergence theorem and Schwarz's inequality we have

$$\begin{aligned}
\int_{\mathbb{H}} \frac{|Du|^p}{|x|_q^\gamma} dx' &= \frac{1}{n-\gamma} \int_{\mathbb{H}} |Du|^p D\left(\frac{\bar{x}}{|x|_q^\gamma}\right) dx' \\
&= -\frac{1}{n-\gamma} \int_{\mathbb{H}} \frac{p}{2} |Du|^{p-2} \frac{\bar{x} \cdot \bar{D}|Du|^2}{|x|_q^\gamma} dx' \\
(3.3) \quad &= -\frac{p}{2(n-\gamma)} \int_{\mathbb{H}} |Du|^{p-2} \frac{\bar{x} \cdot D|Du|^2}{|x|_q^\gamma} dx'.
\end{aligned}$$

Moreover, we compute

$$\begin{aligned}
&\int_{\mathbb{H}} \frac{D\left(|Du|^{p-2} \bar{D}u\right)}{|x|_q^\gamma} \bar{x} \cdot \bar{D}u dx' \\
&= -\int_{\mathbb{H}} |Du|^{p-2} Du \cdot D\left(\frac{\bar{x} \cdot \bar{D}u}{|x|_q^\gamma}\right) dx' \\
&= -\int_{\mathbb{H}} |Du|^{p-2} \left(\frac{|Du|^2}{|x|_q^\gamma} + \frac{\bar{x} \cdot \bar{D}|Du|^2}{2|x|_q^\gamma} - \frac{\gamma |\bar{x} \cdot \bar{D}u|^2}{|x|_q^{\gamma+2}}\right) dx'.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int \frac{|Du|^{p-2}}{|x|_q^\gamma} \bar{x} \cdot \bar{D}|Du|^2 dx' \\
&= 2\gamma \int_{\mathbb{H}} |Du|^{p-2} \frac{|\bar{x} \cdot \bar{D}u|^2}{|x|_q^{\gamma+2}} dx' - 2 \int_{\mathbb{H}} \frac{|Du|^p}{|x|_q^\gamma} dx' \\
&\quad - 2 \int_{\mathbb{H}} \frac{D\left(|Du|^{p-2} \bar{D}u\right)}{|x|_q^\gamma} \bar{x} \cdot \bar{D}u dx'.
\end{aligned}$$

Combining this with (3.3), we obtain

$$\begin{aligned}
\int_{\mathbb{H}} \frac{|Du|^p}{|x|_q^\gamma} dx' &= \frac{p\gamma}{\gamma-n} \int_{\mathbb{H}} |Du|^{p-2} \frac{|\bar{x} \cdot \bar{D}u|^2}{|x|_q^{\gamma+2}} dx' \\
&\quad - \frac{p}{\gamma-n} \int_{\mathbb{H}} \frac{|Du|^p}{|x|_q^\gamma} dx' - \frac{p}{\gamma-n} \int_{\mathbb{H}} \frac{D\left(|Du|^{p-2} \bar{D}u\right)}{|x|_q^\gamma} \bar{x} \cdot \bar{D}u dx'.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\int_{\mathbb{H}} \frac{D\left(|Du|^{p-2} \bar{D}u\right)}{|x|_q^\gamma} \bar{x} \cdot \bar{D}u dx' \\
&= \frac{n-p-\gamma}{p} \int_{\mathbb{H}} \frac{|Du|^p}{|x|_q^\gamma} dx' + \gamma \int_{\mathbb{H}} |Du|^{p-2} \frac{|\bar{x} \cdot \bar{D}u|^2}{|x|_q^{\gamma+2}} dx'.
\end{aligned}$$

Now we apply the Schwarz inequality to the second term on the right hand side since from the assumption we have  $\gamma = \alpha + \beta + 1 \leq 0$ , so we derive

$$\begin{aligned}
 \int_{\mathbb{H}} \frac{D \left( |Du|^{p-2} \bar{D}u \right)}{|x|_q^\gamma} \cdot Du \, dx' &= \frac{n-p-\gamma}{p} \int_{\mathbb{H}} \frac{|Du|^p}{|x|_q^\gamma} \, dx' + \gamma \int_{\mathbb{H}} |Du|^{p-2} \frac{|\bar{x} \cdot \bar{D}u|^2}{|x|_q^{\gamma+2}} \, dx' \\
 &\geq \frac{n-p-\gamma}{p} \int_{\mathbb{H}} \frac{|Du|^p}{|x|_q^\gamma} \, dx' + \gamma \int_{\mathbb{H}} \frac{|Du|^p}{|x|_q^\gamma} \, dx' \\
 (3.4) \qquad \qquad \qquad &\geq \frac{n+\gamma(p-1)-p}{p} \int_{\mathbb{H}} \frac{|Du|^p}{|x|_q^\gamma} \, dx'
 \end{aligned}$$

We also have

$$\begin{aligned}
 \int_{\mathbb{H}} \frac{D \left( |Du|^{p-2} \bar{D}u \right)}{|x|_q^\gamma} \bar{x} \cdot \bar{D}u \, dx' &\leq \int_{\mathbb{H}} \frac{D \left( |Du|^{p-2} \bar{D}u \right)}{|x|_q^{\gamma-1}} |Du| \, dx' \\
 (3.5) \qquad \qquad \qquad &\leq \left( \int_{\mathbb{H}} \left| \frac{D \left( |Du|^{p-2} \bar{D}u \right)}{|x|_q^\alpha} \right|^p \, dx' \right)^{\frac{1}{p}} \left( \int_{\mathbb{H}} \left| \frac{Du}{|x|_q^\beta} \right|^{p'} \, dx' \right)^{\frac{1}{p'}}
 \end{aligned}$$

Here we have used Schwarz’s inequality and Hölder’s inequality. Combining inequalities (3.4) and (3.5), we arrive at the desired inequality. □

Let us define the following Sobolev type spaces for the quaternion-valued functions when  $p = 2$ :

- Let  $W_\gamma^{1,2}(\mathbb{H})$  be the completion of  $C_0^\infty(\mathbb{H} \setminus \{0\})$  with respect to norm

$$\|u\|_{W_\gamma^{1,2}(\mathbb{H})} = \left( \int_{\mathbb{H}} \frac{|Du|^2}{|x|_q^{2\gamma}} \, dx' \right)^{\frac{1}{2}} .$$

- Let  $W_\gamma^{2,2}(\mathbb{H})$  be the completion of  $C_0^\infty(\mathbb{H} \setminus \{0\})$  with respect to norm

$$\|u\|_{W_\gamma^{2,2}(\mathbb{H})} = \left( \int_{\mathbb{H}} \frac{|\Delta_n u|^2}{|x|_q^{2\gamma}} \, dx' \right)^{\frac{1}{2}} .$$

- Let  $H_{\alpha,\beta}^2(\mathbb{H})$  be the completion of  $C_0^\infty(\mathbb{H} \setminus \{0\})$  with respect to norm

$$\|u\|_{H_{\alpha,\beta}^2(\mathbb{H})} = \left( \int_{\mathbb{H}} \frac{|Du|^2}{|x|_q^{2\beta}} + \frac{|\Delta_n u|^2}{|x|_q^{2\alpha}} \, dx' \right)^{\frac{1}{2}} .$$

**Theorem 3.2.** *For  $\alpha, \beta \in \mathbb{R}$  we have the following continuous embedding*

$$(3.6) \qquad \qquad \qquad H_{\alpha,\beta}^2(\mathbb{H}) \subset W_{\frac{\alpha+\beta-1}{2}}^{2,2}(\mathbb{H}),$$

for  $\alpha + \beta - 1 \neq n$ , and

$$(3.7) \qquad \qquad \qquad W_\alpha^{2,2}(\mathbb{H}) \subset W_{\alpha+1}^{1,2}(\mathbb{H}),$$

for  $\alpha \leq \frac{n}{2} - 2$  with  $\alpha \neq \frac{n}{2}$ .

*Proof.* Since  $\alpha + \beta - 1 \neq n$  from inequality (3.2) with  $p = 2$  we have

$$\begin{aligned} \left\| \frac{Du}{|x|^{\frac{\alpha+\beta-1}{2}}} \right\|_{L^2(\mathbb{H})}^2 &\leq \frac{2}{|n - \alpha - \beta + 1|} \left\| \frac{\Delta_n u}{|x|^\alpha} \right\|_{L^2(\mathbb{H})} \left\| \frac{|Du|}{|x|^\beta} \right\|_{L^2(\mathbb{H})} \\ &\leq \frac{2}{|n - \alpha - \beta + 1|} \left( \int_{\mathbb{H}} \frac{|Du|^2}{|x|^{2\beta}} + \frac{|\Delta_n u|^2}{|x|^{2\alpha}} dx' \right), \end{aligned}$$

for all  $u \in C_0^\infty(\mathbb{H} \setminus \{0\})$ . Here we have used the Cauchy-Bunyakovsky-Schwarz inequality. This proves (3.6). Embedding (3.7) also implies from inequality (3.2) with  $p = 2$ , namely assuming  $\beta = \alpha + 1$  with  $\alpha \leq \frac{n}{2} - 2$  and  $\alpha \neq \frac{n}{2}$ .  $\square$

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