A FRICTIONAL CONTACT PROBLEM WITH DAMAGE IN VISCOPLASTICITY

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ABSTRACT. In this paper, we study a quasistatic contact problem with damage between a viscoplastic body and an obstacle the so-called foundation. The contact is modelled with a general normal compliance condition and the associated version of Coulomb's law of dry friction. We provide a variational formulation of the mechanical problem for which we establish an existence theorem of a weak solution including a regularity result.

Keywords: viscoplastic material, damage, Coulomb's law of dry friction, normal compliance, quasistatic, Rothe method, variational inequalities.

1. INTRODUCTION

Mechanical damage may appear in many applications of material science and solid mechanics. In an isotropic and a homogeneous linear elastic material, the damage function is given by

$$\zeta = \frac{\mathcal{E}_{eff}}{\mathcal{E}_Y},$$

where $\mathcal{E}_Y$ is the Young modulus which measures the stiffness of the original material and $\mathcal{E}_{eff}$ is the current one. It follows from this definition that the damage function ζ is restricted to values between zero and one. General models with damage were derived from thermodynamical considerations in [8, 9]. Related contact problems involving the material damage can be found in [3, 7, 10, 14, 16, 22, 23] and the references therein.

The aim of this paper is to present a new result in the study of a quasistatic contact problem for a rate-type elastic-viscoplastic body with a general constitutive

law of the form

$$(1) \qquad\qquad\qquad\qquad \dot{\sigma} \;=\; \mathcal{A}\varepsilon(\dot{u}) + \mathcal{B}(\sigma, \varepsilon(u), \zeta),$$

$$(2) \qquad\qquad \dot{\zeta} - \kappa\Delta\zeta + \partial I_{[0,1]}(\zeta) \;\ni\; \mathcal{G}(\sigma, \varepsilon(u), \zeta),$$

where u denotes the displacement field, σ represents the stress tensor, $\varepsilon(u)$ is the linearized strain tensor, ζ is the damage function, $\mathcal{A}$ is a fourth order tensor which describes the elastic behaviour of the material and $\mathcal{B}$ is a constitutive function which describes the viscoplastic properties of the body. Here in (1)-(2) and everywhere in this paper the dot above a variable represents its derivative with respect to the time variable. In (2), the evolution of the damage field is described by a differential parabolic inclusion, where Δ is the Laplace operator, $\kappa > 0$ denotes the microcrack diffusion constant and $\mathcal{G}$ represents the damage source function, see [22, Section 3.4]. The indicator function $I_{[0,1]} : \mathbb{R} \to\;]-\infty, \infty]$ is defined by

$$I_{[0,1]}(s) = \begin{cases} 0 \text{ if } s \in [0,1], \\[2mm] \infty \text{ otherwise.} \end{cases}$$

The subdifferential of the function $I_{[0,1]}$ is given by

$$\partial I_{[0,1]}(s) = \begin{cases}]-\infty, 0] \text{ if } s = 0, \\[2mm] 0 \text{ if } s \in\;]0,1[, \\[2mm] [0, \infty[\text{ if } s = 1, \\[2mm] \emptyset \qquad \text{otherwise.} \end{cases}$$

Therefore, the term $\partial I_{[0,1]}(\zeta)$ in (2) guarantees that the function ζ has values between zero and one; when $\zeta = 1$ there is no damage in the material; when $\zeta = 0$ the material is completely damaged; when $0 < \zeta < 1$ there is partial additional damage.

Analysis of various contact problems for viscoplastic materials with a constitutive law of the form (1)-(2) can be found for instance in [3, 7, 22, 23]. The variational and numerical analysis of two quasistatic frictional contact problems arising in viscoplasticity including the mechanical damage of the material was performed in [3]. In [7], the Signorini frictionless contact problem for viscoplastic materials with damage was modelled and analyzed, and, moreover, fully discrete scheme based on the finite element method was introduced and numerical examples were presented to show the performance of the method. The viscoplastic problem with dissipative friction potential and damage was studied in [23], a weak formulation for the model was obtained and an existence and uniqueness result was proved.

The novelty, of this paper, consists in dealing with a frictional contact condition for which we use the normal compliance model and the associated version of Coulomb's law of dry friction, which leads to a new and nonstardard mathematical problem. The main difficulties are generated by the additional dependence of the nondifferentiable functional employed on the solution of the problem.

We focus on the weak solvability of the problem within the framework of variational inequalities. Our analysis is based on the Rothe time-discretization method, we transform the quasistatic contact problem into a sequence of elliptic variational inequalities for which at each time step, under a smallness assumption, we prove

the existence of a unique solution. Then, after obtaining the necessary estimates, we use arguments of compactness, lower semicontinuity and the Banach fixed point theorem to prove that the limit of a subsequence of approximate solutions is a solution of the continuous problem. We recall that the Rothe method was first introduced in [20] and since then used to investigate various types of boundary value problems by many authors, see for instance [12, 13, 21, 22] and the references therein.

The rest of this paper is organized as follows. Section 2 is dedicated to present the notation and some preliminary materials. In Section 3 we describe the mechanical problem and after state the assumptions on the data we derive its variational formulation. In Section 4 we establish an existence result of a weak solution to the model.

2. NOTATION AND PRELIMINARIES

Here we introduce the notation we shall use and some preliminary materials. For further details we refer the reader to [5, 11, 19]. We use the notation $\mathbb{N}^*$ for the set of positive integers. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$, (d=2, 3), and we define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$w \cdot v = \sum_{i=1}^{d} w_i v_i, \ |w| = \sqrt[2]{w \cdot w}, \forall w, \ v \ \in \mathbb{R}^d;$$

$$\sigma \cdot \tau = \sum_{1 \leq i,j \leq d} \sigma_{ij} \tau_{ij}, \ |\sigma| = \sqrt[2]{\sigma \cdot \sigma}, \forall \sigma, \ \tau \in \mathbb{S}^d.$$

Let $\Omega \subset \mathbb{R}^d$, ($d$=2,3), be a bounded domain with a Lipschitz boundary Γ and let ν denote the unit outer normal on Γ. Let $[0, T]$, $T > 0$ be the time interval of interest and let $x \in \bar{\Omega}$ and $t \in [0, T]$ be the spatial variable and the temporal variable, respectively. We introduce the spaces

$$H = L^2(\Omega; \mathbb{R}^d), \ \mathcal{Q} = L^2(\Omega; \mathbb{S}^d),$$

$$H_1 = \{u \in H; \ \varepsilon(u) \in \mathcal{Q}\}, \ \mathcal{Q}_1 = \{\sigma \in \mathcal{Q}; \ Div\sigma \in H\},$$

where $\varepsilon : H_1 \to \mathcal{Q}$ is the deformation operator defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \ \varepsilon_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \ 1 \leq i,j \leq d, \forall u \in H_1,$$

$Div : \mathcal{Q}_1 \to H$ is the divergence operator for tensor functions defined by

$$Div\sigma = ((Div\sigma)_i)_{1 \leq i \leq d} = \left(\sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j} \right)_{1 \leq i \leq d}, \forall \sigma \in \mathcal{Q}_1.$$

Note that H, $\mathcal{Q}$, H_1 and $\mathcal{Q}_1$ are Hilbert spaces equipped with the respective canonical inner products

$$(u, v)_H = \int_{\Omega} u \cdot v dx, \ (\sigma, \tau)_{\mathcal{Q}} = \int_{\Omega} \sigma \cdot \tau dx,$$

$$(u, v)_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{Q}}, \ (\sigma, \tau)_{\mathcal{Q}_1} = (Div\sigma, Div\tau)_H + (\sigma, \tau)_{\mathcal{Q}}.$$

The associated norms on the spaces H, $\mathcal{Q}$, H_1 and $\mathcal{Q}_1$ are denoted by $\|.\|_H$, $\|.\|_{\mathcal{Q}}$, $\|.\|_{H_1}$ and $\|.\|_{\mathcal{Q}_1}$, respectively.

Let $\tilde{\gamma}\,:\,H_1\,\rightarrow\,L^2\left(\Gamma;\mathbb{R}^d\right)$ be the trace map. We recall that $\tilde{\gamma}$ is a compact operator, i.e. for any bounded sequence $\{v_n\}$ in H_1 there is a subsequence of $\{v_n\}$ which is convergent in $L^2\left(\Gamma;\mathbb{R}^d\right)$. For every element $v \in H_1$ we use the notation v to denote the trace $\tilde{\gamma}(v)$ of v on Γ and for all $v \in H_1$ we denote by v_ν and v_τ the normal and the tangential components of v on the boundary Γ

$$v_\nu = v \cdot \nu, \ v_\tau = v - v_\nu\nu \text{ on } \Gamma.$$

In a similar manner, the normal and the tangential components of a regular (say C^1) tensor field σ are defined by

$$\sigma_\nu = \sigma\nu \cdot \nu, \ \sigma_\tau = \sigma\nu - \sigma_\nu\nu \text{ on } \Gamma.$$

Moreover, the following Green formula holds

$$(3) \qquad (Div\sigma, v)_H + (\sigma, \varepsilon(v))_Q = \int_\Gamma \sigma\nu \cdot v da, \ \forall \ v \in H_1,$$

where da is the surface measure element.

Let Z and E be real Hilbert spaces such that Z is dense in E and the injection map is continuous; the space E is identified with its own dual and with a subspace of the dual Z^* of Z, i.e. $Z \subset E \subset Z^*$ is a Gelfand triplet. Denote by $(.,.)_E$, $\|.\|_Z$, $\|.\|_E$, $\|.\|_{Z^*}$ and $\langle.,.\rangle_{Z^*\times Z}$ the inner product on the space E, the norms on the spaces Z, E, Z^* and the duality pairing between Z^* and Z, respectively. We note that if $w \in E$ then

$$\langle w, v \rangle_{Z^*\times Z} = (w, v)_E, \ \forall v \in Z.$$

For every real Banach space $(Z, \|.\|_Z)$, we denote by $C([0, T]; Z)$ the space of continuous functions from $[0, T]$ to Z, which is a real Banach space with the norm

$$\|v\|_{C([0,T];Z)} = \max_{t\in[0,T]} \|v(t)\|_Z.$$

Also, we use the standard notation for the spaces $L^p(0, T; Z)$ and $W^{k,p}(0, T; Z)$, $p \in [1, \infty]$ and $k \geq 1$. We have the following result which may be found in [1, p. 140].

Lemma 1. *Let $Z \subset E \subset Z^*$ be a Gelfand triplet and let K be a nonempty, closed and convex set of Z. Assume that $\tilde{a}(.,.) : Z \times Z \rightarrow \mathbb{R}$ is a continuous symmetric bilinear form and there are two real constants $c_2 > 0$ and c_1 such that*

$$(4) \qquad \tilde{a}(v, v) + c_1\|v\|_E^2 \geq c_2\|v\|_Z^2, \ \forall v \in Z.$$

Then, for each $w_0 \in K$ and each $l \in L^2(0, T; E)$, there exists a unique fuction $w \in W^{1,2}(0, T; E) \cap L^2(0, T; Z)$ such that

$$
\begin{aligned}
w(t) &\in K, \ \forall t \in [0, T], \\
\langle \dot{w}(t), v - w(t) \rangle_{Z^*\times Z} + \tilde{a}(w(t), v - w(t)) &\geq (l(t), v - w(t))_E, \\
\forall v &\in K, \ a.e. \ t \in (0, T), \\
w(0) &= w_0.
\end{aligned}
$$

Finally, we conclude this section with two Gronwall type inequalities. Other versions of Gronwall inequalities can be found for instance in [6] and references therein.

Lemma 2. *Assume that* y *and* z : $[0,T] \rightarrow \mathbb{R}$ *are two functions in* $L^1(0,T)$ *satisfying*

$$y(t) \leq z(t) + \alpha \int_0^t y(s)\,ds, \ \forall t \in [0,T],$$

where α *is a nonnegative constant. Then, it follows that*

$$y(t) \leq z(t) + \alpha \int_0^t e^{\alpha(t-s)} z(s)\,ds, \ \forall t \in [0,T].$$

Proof. Use arguments similar to those in [6, proof of Proposition 2.1]. $\qquad\square$

Lemma 3. *Let* $T > 0$ *be a constant. Let* α_1 *and* α_2 *be two nonnegative constants. Let* $m \in \mathbb{N}^*$, *let* $\{w_i\}_{i=0}^m \subset \mathbb{R}$ *be a nonnegative sequence which satisfies*

$$w_{i+1} \leq \alpha_1 + \alpha_2 h \sum_{j=0}^i w_j, \ 0 \leq i \leq m-1,$$

where $h = \dfrac{T}{m}$. *Then, it holds*

$$w_{i+1} \leq (\alpha_1 + \alpha_2 T w_0)\, e^{\alpha_2 T}, \ 0 \leq i \leq m-1.$$

The proof of Lemma 3 may be found in [12].

3. Problem statement

The physical setting is as follows. A deformable body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ (with d=2, 3) with a Lipschitz boundary Γ that is partitioned into three disjoint measurable parts Γ_1, Γ_2, Γ_3, such that meas(Γ_1) > 0. The material's behaviour is modelled with a rate-type constitutive law with damage and the process is quasistatic in the time interval of interest $[0,T]$. The body is clamped on Γ_1 and therefore the displacement field vanishes there, while volume forces of density f_0 act in Ω and surface tractions of density f_2 act on Γ_2. On the other hand, the body is supposed to be in contact over Γ_3 with a foundation such that both normal compliance and a version of Coulomb's law of dry friction are included. To simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $x \in \Omega \cup \Gamma$. Under the above assumptions, the classical formulation of our problem is the following.

Problem 1. *Find a displacement field* $u \; : \Omega \times [0,T] \to \mathbb{R}^d$, *a stress field* $\sigma \; :$ $\Omega \times [0,T] \to \mathbb{S}^d$ *and a damage field* $\zeta : \Omega \times [0,T] \to \mathbb{R}$, *such that*

$$(5) \qquad\qquad \dot{\sigma} = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{B}(\sigma, \varepsilon(u), \zeta), \; in \; \Omega \times (0,T),$$

$$(6) \; \dot{\zeta} - \kappa\Delta\zeta + \partial I_{[0,1]}(\zeta) \ni \mathcal{G}(\sigma, \varepsilon(u), \zeta), \; in \; \Omega \times (0,T),$$

$$(7) \qquad\qquad Div\sigma + f_0 = 0, \; in \; \Omega \times (0,T),$$

$$(8) \qquad\qquad \frac{\partial \zeta}{\partial \nu} = 0, \; on \; \Gamma \times (0,T),$$

$$(9) \qquad\qquad u = 0, \; on \; \Gamma_1 \times (0,T),$$

$$(10) \qquad\qquad \sigma\nu = f_2, \; on \; \Gamma_2 \times (0,T),$$

$$(11) \qquad\qquad -\sigma_\nu = p_\nu(u_\nu - g), \; on \; \Gamma_3 \times (0,T),$$

$$(12) \qquad\qquad \begin{cases} |\sigma_\tau| \le p_\tau(u_\nu - g), \\[2mm] |\sigma_\tau| < p_\tau(u_\nu - g) \Rightarrow \dot{u}_\tau = 0, \\[2mm] |\sigma_\tau| = p_\tau(u_\nu - g) \Rightarrow \exists \lambda \ge 0, \\[2mm] such \; that \;\; \sigma_\tau = -\lambda\dot{u}_\tau, \end{cases} \; on \; \Gamma_3 \times (0,T),$$

$$(13) \qquad\qquad u(0) = u_0, \; \sigma(0) = \sigma_0, \; \zeta(0) = \zeta_0 \; in \; \Omega.$$

Equations (5)-(6) represent the rate-type elastic-viscoplastic constitutive law with damage. Rate-type viscoplastic constitutive law which does not depend on the material damage was considered by many authors, see for instance [4, 11, 12, 22] and the references therein. Equation (7) is the equilibrium equation posed on the domain Ω. Condition (8) means that the normal derivative of ζ, denoted by $\frac{\partial\zeta}{\partial\nu}$, vanishes on Γ. Therefore, there is no influx of microcracks across the boundary. (9)-(10) are the displacement-traction boundary conditions where $\sigma\nu$ represents the Cauchy stress vector. Condition (11) is a general expression of the normal reactive traction on the potential surface contact Γ_3, where the normal compliance function p_ν is a nonnegative prescribed function which vanishes for negative arguments, such that when $u_\nu < g$ there is no contact and the normal pressure vanishes; and when contact takes place $u_\nu - g \ge 0$ is a measure of the penetration of the surface asperities into those of the foundation. We note that an early attempt to study the contact problem with normal compliance was done in [15, 17]. A possible choice of the function p_ν is

$$p_\nu(r) = c_\nu(r)_+,$$

where $(r)_+$ denotes the positive part of r, that is $(r)_+ = \max\{r, 0\}$, c_ν is the surface stiffness coefficient, such that Signorini's nonpenetration condition is obtained in the limit $c_\nu \to \infty$ and thus interpenetration is not allowed. The relations (12) represent a version of Coulomb's law of dry friction where p_τ is a prescribed nonnegative function, the so-called friction bound. A possible choice of the function p_τ is

$$p_\tau(r) = \mu p_\nu(r),$$

where $\mu \ge 0$ is the coefficient of friction (see, e.g., [22]). Finally, (13) are the initial conditions.

In order to prove an existence result concerning the mechanical problem (5)-(13), we need to introduce the convex set $\mathcal{K}$ and the spaces Y, V and W defined, respectively, by

$$\mathcal{K} = \{\zeta \in H^1(\Omega) \colon 0 \leq \zeta \leq 1, \ a.e. \text{ on } \Omega\},$$

$$Y = L^2(\Omega),$$

$$V = \{v \in H_1, \ v = 0 \ on \ \Gamma_1\},$$

$$W = \{\zeta \in H^2(\Omega), \ \frac{\partial \zeta}{\partial \nu} = 0 \ on \ \Gamma\},$$

where $\dfrac{\partial \zeta}{\partial \nu}$ is the normal derivative of ζ on the boundary Γ in the trace sense. Since $meas(\Gamma_1) > 0$, Korn's inequality holds

$$(14) \qquad\qquad C_K \|v\|_{H_1} \leq \|\varepsilon(v)\|_{\mathcal{Q}}, \ \forall v \in V,$$

where $C_K > 0$ is a positive constant depending only on Ω and Γ_1. A proof of Korn's inequality can be found in [18, page 79]. Over the space V, we consider the inner product given by

$$(w, v)_V = (\varepsilon(w), \varepsilon(v))_{\mathcal{Q}}, \ \forall w, v \in V,$$

and let $\|.\|_V$ be the associated norm. It follows from Korn's inequality (14) that $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent norms on V. Therefore $(V, (.,.)_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a positive constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$(15) \qquad\qquad \|v\|_{L^2(\Gamma_3; \mathbb{R}^d)} \leq c_0 \|v\|_V, \ \forall v \in V.$$

In the study of the problem (5)-(13), we consider the following assumptions. We assume that $\mathcal{A} = (\mathcal{A}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is a bounded symmetric positive definite fourth order tensor, i.e.

$$(16) \quad \begin{cases} \text{(i) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \mathcal{A}\varepsilon \cdot \varepsilon \geq m_{\mathcal{A}} |\varepsilon|^2 \ a.e. \ x \in \Omega, \ \forall \varepsilon \in \mathbb{S}^d; \\[2mm] \text{(ii) } \mathcal{A}_{ijkl} \in L^\infty(\Omega), \forall i, \ j, \ k, \ l \in \{1, ..., d\}; \\[2mm] \text{(iii) } \mathcal{A}_{ijkl} = \mathcal{A}_{jikl} = \mathcal{A}_{klij}, \forall i, \ j, \ k, \ l \in \{1, ..., d\}. \end{cases}$$

We assume that the function $\mathcal{B} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies

$$(17) \quad \begin{cases} \text{(i) There exists } L_{\mathcal{B}} > 0 \text{ such that} \\ |\mathcal{B}(x, \sigma_1, \varepsilon_1, \zeta_1) - \mathcal{B}(x, \sigma_2, \varepsilon_2, \zeta_2)| \\[2mm] \leq L_{\mathcal{B}}(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| + |\zeta_1 - \zeta_2|) \\ a.e. \ x \in \Omega, \ \forall \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \forall \zeta_1, \zeta_2 \in \mathbb{R}; \\[2mm] \text{(ii) The mapping } x \mapsto \mathcal{B}(x, \sigma, \varepsilon, \zeta) \text{ is measurable on } \Omega \\ \text{for any } \sigma, \ \varepsilon \in \mathbb{S}^d, \forall \zeta \in \mathbb{R}; \\[2mm] \text{(iii) The mapping } x \mapsto \mathcal{B}(x, 0_{\mathbb{S}^d}, 0_{\mathbb{S}^d}, 0) \text{ belongs to } \mathcal{Q}. \end{cases}$$

We assume that the function $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies

(18)
$$\begin{cases} \text{(i) There exists } L_{\mathcal{G}} > 0 \text{ such that} \\ |\mathcal{G}\left(x, \sigma_1, \varepsilon_1, \zeta_1\right) - \mathcal{G}\left(x, \sigma_2, \varepsilon_2, \zeta_2\right)| \\[4pt] \leq L_{\mathcal{G}}\left(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| + |\zeta_1 - \zeta_2|\right) \\ a.e. \ x \in \Omega, \ \forall \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \forall \zeta_1, \zeta_2 \in \mathbb{R}; \\[4pt] \text{(ii) The mapping } x \mapsto \mathcal{G}\left(x, \sigma, \varepsilon, \zeta\right) \text{ is measurable on } \Omega \\ \text{for any } \sigma, \ \varepsilon \in \mathbb{S}^d, \forall \zeta \in \mathbb{R}; \\[4pt] \text{(iii) The mapping } x \mapsto \mathcal{G}\left(x, 0_{\mathbb{S}^d}, 0_{\mathbb{S}^d}, 0\right) \text{ belongs to } Y. \end{cases}$$

We assume that the function $p_\alpha : \Gamma_3 \times \mathbb{R} \to \mathbb{R}^+$, $(\alpha = \nu, \tau)$, satisfies

(19)
$$\begin{cases} \text{(i) There exists } L_\alpha > 0 \text{ such that} \\ |p_\alpha\left(x, r_1\right) - p_\alpha\left(x, r_2\right)| \leq L_\alpha\left|r_1 - r_2\right|, \\ \forall r_1, \ r_2 \in \mathbb{R}, \ a.e. \ x \in \Gamma_3 \ ; \\[4pt] \text{(ii) } p_\alpha\left(x, r\right) = 0, \ \forall r \leq 0, \ a.e. \ x \in \Gamma_3; \\[4pt] \text{(iii) The mapping } x \mapsto p_\alpha\left(x, r\right) \text{ is Lebesgue measurable on } \Gamma_3, \\ \forall r \in \mathbb{R}. \end{cases}$$

The gap function satisfies

(20) $(i)\ g \in L^2(\Gamma_3),\ (ii)\ g \geq 0,\ a.e.\ \text{on } \Gamma_3 .$

The body forces and surface tractions have the regularity

(21) $(i)\ f_0 \in W^{1,\infty}(0, T; H),\ (ii)\ f_2 \in W^{1,\infty}(0, T; L^2(\Gamma_2; \mathbb{R}^d)).$

Finally, we assume that the initial data satisfy

(22) $(i)\ u_0 \in V,\ (ii)\ \sigma_0 \in \mathcal{Q},$

(23) $\zeta_0 \in \mathcal{K} \cap W.$

It follows from (21) that the function $f : [0, T] \to V$ defined by

(24) $(f(t), w)_V = \displaystyle\int_\Omega f_0(t) \cdot w\,dx + \int_{\Gamma_2} f_2(t) \cdot w\,da, \forall t \in [0, T], \ \forall w \in V,$

has the regularity

(25) $f \in W^{1,\infty}(0, T; V).$

In the sequel, we use the bilinear form $\tilde{a} : H^1\left(\Omega\right) \times H^1\left(\Omega\right) \to \mathbb{R}$ defined by

(26) $\tilde{a}\left(\varphi, \xi\right) = \kappa\left(\nabla\varphi, \nabla\xi\right)_H,\ \forall \varphi, \ \xi \in H^1\left(\Omega\right).$

Also, we use the functional $\psi : V \times V \longrightarrow \mathbb{R}$ defined by

(27) $\psi(z, w) = \displaystyle\int_{\Gamma_3} p_\nu\left(z_\nu - g\right) w_\nu\,da + \int_{\Gamma_3} p_\tau\left(z_\nu - g\right) |w_\tau|\,da, \forall z, \ w \in V$

Using (15), (19), (20) and (27), we deduce that the functional ψ satisfies

(28) $\psi(\eta, v) - \psi(\eta, w) + \psi(z, w) - \psi(z, v) \leq c_0^2\left(L_\tau + L_\nu\right) \|\eta - z\|_V \|v - w\|_V,$

(29) $\psi(\eta, -z) - \psi(\eta, \eta - z) \leq c_0\left(L_\tau + L_\nu\right)\left(c_0 \|\eta\|_V + \|g\|_{L^2(\Gamma_3)}\right) \|\eta\|_V,$

$$(30) \qquad \psi(\eta, v) - \psi(\eta, w) \leq \psi(\eta, v - w),$$

$$(31) \qquad \psi(\eta, w) - \psi(z, w) \leq c_0^2 \left(L_\tau + L_\nu\right) \|\eta - z\|_V \|w\|_V,$$

$$(32) \qquad |\psi(\eta, v) - \psi(\eta, w)| \leq c_0 \left(L_\tau + L_\nu\right) \left(c_0 \|\eta\|_V + \|g\|_{L^2(\Gamma_3)}\right) \|v - w\|_V,$$

$$(33) \qquad \psi(\eta, w) \leq \left(L_\tau + L_\nu\right) \left(c_0 \|\eta\|_V + \|g\|_{L^2(\Gamma_3)}\right) \|w\|_{L^2(\Gamma_3; \mathbb{R}^d)},$$

for all $\eta, v, w, z \in V$. Now, assume u, σ and ζ are smooth functions satisfying (5)-(13). We use Green formula (3) and integration by parts, integrate (5) on $(0, t)$, use the fact that the operator $\mathcal{A}$ defined in (16) is a bounded linear operator, use the initial conditions (13) and the notation (24), (26), (27), to obtain the following variational formulation.

Problem 2. *Find a displacement field $u : [0, T] \rightarrow V$, a stress field $\sigma : [0, T] \rightarrow \mathcal{Q}$ and a damage field $\zeta : [0, T] \rightarrow Y$ such that*

$$(34) \quad \sigma(t) = \mathcal{A}\varepsilon(u(t)) + \int_0^t \mathcal{B}\left(\sigma(s), \varepsilon(u(s)), \zeta(s)\right) ds + \sigma_0 - \mathcal{A}\left(\varepsilon(u_0)\right), \ \forall t \in [0, T],$$

$$(35) \qquad \begin{cases} \left(\sigma(t), \varepsilon\left(w - \dot{u}(t)\right)\right)_{\mathcal{Q}} + \psi(u(t), w) - \psi(u(t), \dot{u}(t)) \\ \\ \geq \left(f(t), w - \dot{u}(t)\right)_V, \ \forall \ w \in V, \ a.e. \ t \in (0, T), \end{cases}$$

$$(36) \qquad \begin{cases} \zeta(t) \in \mathcal{K}, \\ \left(\dot{\zeta}(t), \vartheta - \zeta(t)\right)_Y + \tilde{a}\left(\zeta(t), \vartheta - \zeta(t)\right) \\ \\ \geq \left(\mathcal{G}\left(\sigma(t), \varepsilon\left(u(t)\right), \zeta(t)\right), \vartheta - \zeta(t)\right)_Y, \ \forall \ \vartheta \in \mathcal{K}, \ a.e. \ t \in (0, T), \end{cases}$$

$$(37) \qquad \qquad \zeta(0) \ = \ \zeta_0,$$

$$(38) \qquad \qquad u(0) \ = \ u_0.$$

To study Problem (34)-(38), we need the following additional assumption on the initial data

$$(39) \qquad \left(\sigma_0, \varepsilon(w)\right)_{\mathcal{Q}} + \psi(u_0, w) \geq \left(f(0), w\right)_V, \ \forall \ w \in V,$$

and we make the following smallness assumption

$$(40) \qquad L_\tau + L_\nu < \frac{m_{\mathcal{A}}}{c_0^2},$$

where c_0, $m_{\mathcal{A}}$ and L_α $(\alpha = \nu, \tau)$ are given in (15), (16) and (19), respectively.

4. Existence of a weak solution

The following theorem is the main result of this paper.

Theorem 1. *Assume that (16)-(66) and (39)-(40) are fulfilled. Then Problem (34)-(38) has at least one solution $\{u, \sigma, \zeta\}$ which satisfies*

$$(41) \qquad \qquad u \ \in \ W^{1,\infty}\left(0, T; V\right).$$

$$(42) \qquad \qquad \sigma \ \in \ W^{1,\infty}\left(0, T; \mathcal{Q}_1\right).$$

$$(43) \qquad \qquad \zeta \ \in \ W^{1,2}\left(0, T; Y\right) \cap L^2\left(0, T; H^1\left(\Omega\right)\right).$$

We will divide the proof into several steps. Let $m \in \mathbb{N}^*$ with $m > TL_{\mathcal{G}}$, where $L_{\mathcal{G}}$ is given in (18). We introduce a uniform partition of the time interval $[0,T]$, denoted by $t_i^m = ih_m$, $h_m = \dfrac{T}{m}$, $i = 0,...,m$. For a sequence $\left\{ w_m^i \right\}_{i=0}^m$, we denote $\delta w_m^{i+1} = \dfrac{w_m^{i+1} - w_m^i}{h_m}$ and for a continuous function $z \in C\left([0,T];X\right)$ with values in a normed space X, we use the notation $z_i^m = z\left(t_i^m\right)$, $i = 0,...,m$.

First step. We consider the following problem.

Problem 3. *Let $\sigma_1, \sigma_2 \in \mathcal{Q}$, let $\xi \in Y$ and let $m \in \mathbb{N}^*$ with $m > TL_{\mathcal{G}}$. Find an element $\varphi \in \mathcal{K}$, such that*

$$(44) \quad \left(\frac{\varphi - \xi}{h_m}, \vartheta - \varphi \right)_Y + \tilde{a}\left(\varphi, \vartheta - \varphi\right) \geq \left(\mathcal{G}\left(\sigma_1, \sigma_2, \varphi\right), \vartheta - \varphi\right)_Y, \text{ for all } \vartheta \in \mathcal{K}.$$

Lemma 4. *Problem (44) has a unique solution.*

Proof. It follows from (26), that the bilinear form $\tilde{b}\left(.,.\right): H^1\left(\Omega\right) \times H^1\left(\Omega\right) \to \mathbb{R}$, defined by

$$\tilde{b}\left(\varphi, \vartheta\right) = \frac{1}{h_m}\left(\varphi, \vartheta\right)_Y + \tilde{a}\left(\varphi, \vartheta\right), \, \forall \varphi, \, \vartheta \in H^1\left(\Omega\right),$$

is continuous and $H^1\left(\Omega\right)$-elliptic. Also, for each $\theta \in Y$, the function

$$\vartheta \longmapsto \left(\theta + \frac{1}{h_m}\xi, \vartheta \right)_Y$$

is a continuous linear functional on $H^1\left(\Omega\right)$. Moreover, $\mathcal{K}$ is a closed convex, non-empty subset of $H^1\left(\Omega\right)$. Therefore, using a standard result on elliptic variational inequalities of the first kind see [11], we deduce that for each $\theta \in Y$ there exists a unique element $\varphi_\theta \in \mathcal{K}$ which satisfies

$$(45) \quad \frac{1}{h_m}\left(\varphi_\theta, \vartheta - \varphi_\theta\right)_Y + \tilde{a}\left(\varphi_\theta, \vartheta - \varphi_\theta\right) \geq \left(\theta + \frac{1}{h_m}\xi, \vartheta - \varphi_\theta \right)_Y, \, \forall \, \vartheta \in \mathcal{K}.$$

Now, let σ_1, $\sigma_2 \in \mathcal{Q}$. We define the operator $\Theta : Y \to Y$ by

$$(46) \quad \Theta\left(\theta\right) = \mathcal{G}\left(\sigma_1, \sigma_2, \varphi_\theta\right), \, \forall \theta \in Y.$$

Let θ_1, $\theta_2 \in Y$, using the notation $\varphi_1 = \varphi_{\theta_1}$ and $\varphi_2 = \varphi_{\theta_2}$. Then, by taking $\left(\theta, \varphi_\theta, \vartheta\right) = \left(\theta_1, \varphi_1, \varphi_2\right)$, $\left(\theta, \varphi_\theta, \vartheta\right) = \left(\theta_2, \varphi_2, \varphi_1\right)$ in (45), adding the two inequalities and using (26), we get

$$\left\|\varphi_1 - \varphi_2\right\|_Y^2 + \kappa h_m \left\|\nabla\varphi_1 - \nabla\varphi_2\right\|_H^2 \leq h_m \left(\theta_1 - \theta_2, \varphi_1 - \varphi_2\right)_Y.$$

So, we have

$$\left\|\varphi_1 - \varphi_2\right\|_Y \leq h_m \left\|\theta_1 - \theta_2\right\|_Y,$$

which, with (46) and (18), gives

$$\left\|\Theta\left(\theta_1\right) - \Theta\left(\theta_2\right)\right\|_Y \leq h_m L_{\mathcal{G}} \left\|\theta_1 - \theta_2\right\|_Y.$$

Thus, if $m > TL_{\mathcal{G}}$, then Θ is a contraction in the Banach space Y. Therefore, Θ has a unique fixed point $\theta^* \in Y$. Now, let θ^* be the unique fixed point of Θ and let $\varphi = \varphi_{\theta^*}$ be the unique solution of the problem (45) for $\theta = \theta^*$, then we deduce that φ is a solution to the problem (44). Finally, the uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Θ and of the uniqueness of the solution of the problem (45). $\qquad \square$

Using the Riesz representation theorem, we can introduce the operator $\mathcal{F} : V \rightarrow V$ defined by

$$(47) \qquad (\mathcal{F}v, w)_V = (\mathcal{A}\varepsilon(v), \varepsilon(w))_{\mathcal{Q}} + (\sigma_0 - \mathcal{A}\varepsilon(u_0), \varepsilon(w))_{\mathcal{Q}}, \ \forall v, w \in V.$$

It follows from (47) and (16), that the operator $\mathcal{F}$ satisfies

$$(48) \qquad m_{\mathcal{A}} \|w_1 - w_2\|_V^2 \leq (\mathcal{F}w_1 - \mathcal{F}w_2, w_1 - w_2)_V, \ \forall w_1, w_2 \in V.$$

Moreover, there exists $L_{\mathcal{A}} > 0$ such that

$$(49) \qquad \|\mathcal{F}w_1 - \mathcal{F}w_2\|_V \leq L_{\mathcal{A}} \|w_1 - w_2\|_V, \ \forall w_1, w_2 \in V.$$

We consider the following incremental problems $\mathcal{P}_m^{i+1}$, $i \in \{0, ..., \ m-1\}$.

Problem 4 ($\mathcal{P}_m^{i+1}$). *Find a function $u_m^{i+1} \in V$, such that*

$$(50) \quad \begin{cases} \left(\mathcal{F}u_m^{i+1}, w - \delta u_m^{i+1}\right)_V + \left(h_m \sum_{j=0}^{i} \mathcal{B}\left(\sigma_m^j, \varepsilon\left(u_m^j\right), \zeta_m^j\right), \varepsilon\left(w - \delta u_m^{i+1}\right)\right)_{\mathcal{Q}} \\[2ex] + \psi(u_m^{i+1}, w) - \psi(u_m^{i+1}, \delta u_m^{i+1}) \\[2ex] \geq \left(f_{i+1}^m, w - \delta u_m^{i+1}\right)_V, \ \textit{for all } w \in V, \end{cases}$$

where u_m^j is the unique solution of the problem $\mathcal{P}_m^j$, $j = 1, ..., i$,

$$(51) \quad \sigma_m^{j+1} = \mathcal{A}\varepsilon\left(u_m^{j+1}\right) + h_m \sum_{l=0}^{j} \mathcal{B}\left(\sigma_m^l, \varepsilon\left(u_m^l\right), \zeta_m^l\right) + \sigma_m^0 - \mathcal{A}\varepsilon\left(u_m^0\right), \ 0 \leq j \leq i,$$

ζ_m^{j+1} is the unique solution of the following variational inequality

$$(52) \quad \begin{cases} \zeta_m^{j+1} \in \mathcal{K}, \\[1ex] \left(\dfrac{\zeta_m^{j+1} - \zeta_m^j}{h_m}, \xi - \zeta_m^{j+1}\right)_Y + \tilde{a}\left(\zeta_m^{j+1}, \xi - \zeta_m^{j+1}\right) \\[2ex] \geq \left(\mathcal{G}\left(\sigma_m^{j+1}, \varepsilon\left(u_m^{j+1}\right), \zeta_m^{j+1}\right), \xi - \zeta_m^{j+1}\right)_Y, \ \forall \, \xi \in \mathcal{K}, \ 0 \leq j \leq i. \end{cases}$$

$$(53) \qquad\qquad (i) \ u_m^0 = u_0, \ (ii) \ \sigma_m^0 = \sigma_0, \ (iii) \ \zeta_m^0 = \zeta_0,$$

$$(54) \qquad\qquad\qquad f_{i+1}^m = f(t_{i+1}^m), \ i = 0, ..., m - 1.$$

We notice that the unique solvability of the variational inequality (52) follows from Lemma 4. Now, by setting $w = \dfrac{v - u_m^i}{h_m}$ in (50), it follows that $\mathcal{P}_m^{i+1}$ is formally equivalent to the following problem.

Problem 5 ($\mathcal{Q}_m^{i+1}$). *Find a function $u_m^{i+1} \in V$, such that*

$$(55) \quad \begin{cases} \left(\mathcal{F}u_m^{i+1}, v - u_m^{i+1}\right)_V + \left(h_m \sum_{j=0}^{i} \mathcal{B}\left(\sigma_m^j, \varepsilon\left(u_m^j\right), \zeta_m^j\right), \varepsilon\left(v - u_m^{i+1}\right)\right)_{\mathcal{Q}} \\[2ex] + \psi(u_m^{i+1}, v - u_m^i) - \psi(u_m^{i+1}, u_m^{i+1} - u_m^i) \\[2ex] \geq \left(f_{i+1}^m, v - u_m^{i+1}\right)_V, \ \textit{for all } v \in V, \end{cases}$$

where $\left\{\sigma_m^i\right\}_{0\le i\le m}$, $\left\{\zeta_m^i\right\}_{0\le i\le m}$ and u_m^0 are given by (51)-(53) and u_m^j is the unique solution of the problem $\mathcal{P}_m^j$, $j=1,...,i$.

Lemma 5. *Problem $\mathcal{P}_m^{i+1}$, $i \in \{0\ ,...,\ m-1\}$, has a unique solution.*

Proof. From (48)-(49), the operator $\mathcal{F}$ is strongly monotone and Lipschitz continuous on V. On the other hand let $\eta \in V$. Using the following inequality

$$
\begin{aligned}
|\lambda w + (1-\lambda)v - z| &= |\lambda(w-z)+(1-\lambda)(v-z)| \\
&\le \lambda|w-z|+(1-\lambda)|v-z|,\ \forall w,\, v,\, z \in \mathbb{R}^d,\ \forall \lambda \in [0,1],
\end{aligned}
$$

it follows, from (27) and (32), that the functional

$$
v \longmapsto \left(h_m \sum_{j=0}^i \mathcal{B}\left(\sigma_m^j, \varepsilon\left(u_m^j\right), \zeta_m^j\right), \varepsilon(v)\right)_{\mathcal{Q}} + \psi(\eta, v - u_m^i),\ \forall v \in V,
$$

is a proper convex and continuous functional on V. Therefore, using a standard result on elliptic variational inequalities of the second kind, see [11], we deduce that the following problem. Find $u_{m\eta}^{i+1} \in V$, such that

$$
(56) \quad
\begin{cases}
\left(\mathcal{F}u_{m\eta}^{i+1}, v - u_{m\eta}^{i+1}\right)_V + \left(h_m \sum_{j=0}^i \mathcal{B}\left(\sigma_m^j, \varepsilon\left(u_m^j\right), \zeta_m^j\right), \varepsilon\left(v - u_{m\eta}^{i+1}\right)\right)_{\mathcal{Q}} + \\
\\
+\psi(\eta, v - u_m^i) - \psi(\eta, u_{m\eta}^{i+1} - u_m^i) \\
\\
\ge \left(f_{i+1}^m, v - u_{m\eta}^{i+1}\right)_V, \text{ for all } v \in\ V,
\end{cases}
$$

has a unique solution $u_{m\eta}^{i+1} \in V$. To continue, we define the operator $\Psi : V \to V$ by

$$
(57) \qquad\qquad\qquad \Psi(\eta) = u_{m\eta}^{i+1},\ \forall \eta \in V.
$$

Let $\eta_1,\ \eta_2 \in V$, using the notation $u_1 = u_{m\eta_1}^{i+1}$ and $u_2 = u_{m\eta_2}^{i+1}$, then by taking $(\eta,\ u_{m\eta}^{i+1}, v) = (\eta_2, u_2, u_1)$, $(\eta,\ u_{m\eta}^{i+1}, v) = (\eta_1, u_1, u_2)$ in (56) and adding the two inequalities, we get

$$
\begin{aligned}
(\mathcal{F}u_1 - \mathcal{F}u_2, u_1 - u_2)_V &\le \psi(\eta_1, u_2 - u_m^i) - \psi(\eta_1, u_1 - u_m^i) \\
\\
&\quad +\psi(\eta_2, u_1 - u_m^i) - \psi(\eta_2, u_2 - u_m^i),
\end{aligned}
$$

which together with (28) and (48) implies that

$$
m_{\mathcal{A}} \|u_1 - u_2\|_V^2 \le c_0^2 (L_\tau + L_\nu) \|\eta_1 - \eta_2\|_V \|u_1 - u_2\|_V,
$$

and using (57), we have

$$
\|\Psi\eta_2 - \Psi\eta_1\|_V \le \frac{c_0^2(L_\tau + L_\nu)}{m_{\mathcal{A}}} \|\eta_1 - \eta_2\|_V.
$$

This last inequality implies that if $c_0^2(L_\tau + L_\nu) < m_{\mathcal{A}}$, then Ψ is a contraction in the Banach space V. Therefore, Ψ has a unique fixed point $\eta^* \in V$. We have now all the ingredients to prove Lemma 5. Let η^* be the unique fixed point of Ψ and let $u_m^{i+1} = \eta^* = u_{m\eta^*}^{i+1}$ be the unique solution of the problem (56) for $\eta = \eta^*$, then we deduce that u_m^{i+1} is a solution for the problem $\mathcal{Q}_m^{i+1}$ which is equivalent to $\mathcal{P}_m^{i+1}$. Finally, the uniqueness of the solution is a consequence of the uniqueness of the

fixed point of the operator Ψ and of the uniqueness of the solution of the problem (56). □

In the rest of this paper, the same letter c will be used to denote different positive constants which do not depend on $m \in \mathbb{N}^*$ nor on $t \in (0, T)$.

Second step. In this step we have the following result.

Lemma 6. *There exists $c > 0$, such that for all $m \in \mathbb{N}^*$ with $m > TL_{\mathcal{G}}$,*

$$(58) \qquad \left\| \sigma_m^{i+1} \right\|_{\mathcal{Q}} + \left\| u_m^{i+1} \right\|_V + \left\| \zeta_m^{i+1} \right\|_Y \le c, \, i \in \, \{0, ..., \, m-1\} \,,$$

$$(59) \qquad \left\| \delta u_m^{i+1} \right\|_V + \left\| \delta \zeta_m^{i+1} \right\|_Y \le c, \, i \in \, \{0, ..., \, m-1\} \,.$$

Proof. It follows from (53), that there exists $c > 0$ such that

$$(60) \qquad \left\| \sigma_m^0 \right\|_{\mathcal{Q}} + \left\| u_m^0 \right\|_V + \left\| \zeta_m^0 \right\|_Y \le c, \, \forall m \in \mathbb{N}^*.$$

Let $i \in \, \{0, ..., \, m-1\}$. Taking $v = 0_V$ in (55) yields

$$\left(\mathcal{F} u_m^{i+1}, u_m^{i+1} \right)_V \quad \le \quad \psi(u_m^{i+1}, -u_m^i) - \psi(u_m^{i+1}, u_m^{i+1} - u_m^i) +$$

$$- \left(h_m \sum_{j=0}^{i} \mathcal{B} \left(\sigma_m^j, \varepsilon \left(u_m^j \right), \zeta_m^j \right), \varepsilon \left(u_m^{i+1} \right) \right)_{\mathcal{Q}} + \left(f_{i+1}^m, u_m^{i+1} \right)_V,$$

which together with (17), (29) and (48) gives

$$m_{\mathcal{A}} \left\| u_m^{i+1} \right\|_V^2 \quad \le \quad c_0^2 \left(L_\tau + L_\nu \right) \left\| u_m^{i+1} \right\|_V^2 + c_0 \left(L_\tau + L_\nu \right) \left\| g \right\|_{L^2(\Gamma_3)} \left\| u_m^{i+1} \right\|_V$$

$$+ c h_m \sum_{j=0}^{i} \left(\left\| \sigma_m^j \right\|_{\mathcal{Q}} + \left\| u_m^j \right\|_V + \left\| \zeta_m^j \right\|_Y \right) \left\| u_m^{i+1} \right\|_V$$

$$+ c \left\| \mathcal{B} \left(0_{\mathbb{S}^d}, 0_{\mathbb{S}^d}, 0 \right) \right\|_{\mathcal{Q}} \left\| u_m^{i+1} \right\|_V + \left\| f_{i+1}^m \right\|_V \left\| u_m^{i+1} \right\|_V + \left\| \mathcal{F} \left(0_V \right) \right\|_V \left\| u_m^{i+1} \right\|_V.$$

Hence, using (40) and (25) in the last inequality, we obtain

$$(61) \qquad \left\| u_m^{i+1} \right\|_V \leqslant c h_m \sum_{j=0}^{i} \left(\left\| \sigma_m^j \right\|_{\mathcal{Q}} + \left\| u_m^j \right\|_V + \left\| \zeta_m^j \right\|_Y \right) + c.$$

On the other hand, from (16)-(17) and (51), we find that

$$(62) \qquad \begin{cases} \left\| \sigma_m^{i+1} \right\|_{\mathcal{Q}} \le c \left\| u_m^{i+1} \right\|_V + c h_m \sum_{j=0}^{i} \left(\left\| \sigma_m^j \right\|_{\mathcal{Q}} + \left\| u_m^j \right\|_V + \left\| \zeta_m^j \right\|_Y \right) \\ \\ + c \left\| \mathcal{B} \left(0_{\mathbb{S}^d}, 0_{\mathbb{S}^d}, 0 \right) \right\|_{\mathcal{Q}} + c, \, i \in \, \{0, ..., \, m-1\} \,. \end{cases}$$

Now, from (52) we have $\zeta_m^{i+1} \in \mathcal{K}$. Therefore, we deduce that

$$\left\| \zeta_m^{i+1} \right\|_Y \le c, \, i \in \{0 \,, ..., \, m-1\} \,,$$

and employing (61)-(62), we get

$$\left\| \sigma_m^{i+1} \right\|_{\mathcal{Q}} + \left\| u_m^{i+1} \right\|_V + \left\| \zeta_m^{i+1} \right\|_Y \le c h_m \sum_{j=0}^{i} \left(\left\| \sigma_m^j \right\|_{\mathcal{Q}} + \left\| u_m^j \right\|_V + \left\| \zeta_m^j \right\|_Y \right) + c.$$

Applying Lemma 3, in the last inequality, we obtain (58). Setting $v = u_m^0$ in (55) for $i=0$, and $w = u_m^1 - u_m^0$ in (39), adding the two inequalities, we obtain

$$
\begin{cases}
\left(\mathcal{F}u_m^1 - \mathcal{F}u_m^0, u_m^1 - u_m^0\right)_V \leq \psi(u_m^0, u_m^1 - u_m^0) - \psi(u_m^1, u_m^1 - u_m^0) \;_{\mathcal{Q}} \\[2mm]
- \left(h_m \mathcal{B}\left(\sigma_m^0, \varepsilon\left(u_m^0\right), \zeta_m^0\right), \varepsilon\left(u_m^1 - u_m^0\right)\right) + \left(f_1^m - f_0^m, u_m^1 - u_m^0\right)_V.
\end{cases}
$$

We use now (17), (31) and (48), to see that

$$
\begin{cases}
m_{\mathcal{A}} \left\|u_m^1 - u_m^0\right\|_V^2 \leq c_0^2 \left(L_\tau + L_\nu\right) \left\|u_m^1 - u_m^0\right\|_V^2 \\[2mm]
+ ch_m \left(\left\|\sigma_m^0\right\|_{\mathcal{Q}} + \left\|u_m^0\right\|_V + \left\|\zeta_m^0\right\|_Y\right) \left\|u_m^1 - u_m^0\right\|_V \\[2mm]
+ ch_m \left\|\mathcal{B}\left(0_{\mathbb{S}^d}, 0_{\mathbb{S}^d}, 0\right)\right\|_{\mathcal{Q}} \left\|u_m^1 - u_m^0\right\|_V + \left\|f_1^m - f_0^m\right\|_V \left\|u_m^1 - u_m^0\right\|_V,
\end{cases}
$$

and thanks to (25) and (40), we get

$$
\begin{aligned}
\left\|\frac{u_m^1 - u_m^0}{h_m}\right\|_V &\leq c + c \left\|\frac{f_1^m - f_0^m}{h_m}\right\|_V \\[3mm]
&\leq c + c \left\|\dot{f}\right\|_{L^\infty(0,T;V)}.
\end{aligned}
$$

Thus, we have

(63) $$\left\|\delta u_m^1\right\|_V \leq c.$$

Now, for all $i \in \{1, ..., m-1\}$, taking $w = 0_V$ in problem $\mathcal{P}_m^{i+1}$, and $w = \dfrac{u_m^{i+1} - u_m^{i-1}}{h_m}$ in problem $\mathcal{P}_m^i$, adding the two inequalities, we obtain

$$
\begin{cases}
\left(\mathcal{F}u_m^{i+1} - \mathcal{F}u_m^i, \delta u_m^{i+1}\right)_V \leq - \left(h_m \mathcal{B}\left(\sigma_m^i, \varepsilon\left(u_m^i\right), \zeta_m^i\right), \varepsilon\left(\delta u_m^{i+1}\right)\right)_{\mathcal{Q}} \\[2mm]
+ \left(\psi(u_m^i, \dfrac{u_m^{i+1} - u_m^{i-1}}{h_m}) - \psi(u_m^i, \dfrac{u_m^i - u_m^{i-1}}{h_m})\right) - \psi(u_m^{i+1}, \dfrac{u_m^{i+1} - u_m^i}{h_m}) \\[2mm]
+ \left(f_{i+1}^m - f_i^m, \delta u_m^{i+1}\right)_V,
\end{cases}
$$

which combined with (48), (17), (30) and (31) implies that

$$
\begin{cases}
m_{\mathcal{A}} \left\|u_m^{i+1} - u_m^i\right\|_V^2 \leq ch_m \left(\left\|\sigma_m^i\right\|_{\mathcal{Q}} + \left\|u_m^i\right\|_V + \left\|\zeta_m^i\right\|_Y + c\right) \left\|u_m^{i+1} - u_m^i\right\|_V \\[2mm]
+ c_0^2 \left(L_\tau + L_\nu\right) \left\|u_m^{i+1} - u_m^i\right\|_V^2 + \left\|f_{i+1}^m - f_i^m\right\|_V \left\|u_m^{i+1} - u_m^i\right\|_V.
\end{cases}
$$

We use now (25), (40) and (58) to obtain

$$
\left\|\delta u_m^{i+1}\right\|_V \leq c + c \left\|\dot{f}\right\|_{L^\infty(0,T;V)},
$$

and keeping in mind (63), we have

(64) $$\left\|\delta u_m^{i+1}\right\|_V \leq c, \; i \in \{0, ..., m-1\}.$$

It follows from (51) that

$$
\sigma_m^{i+1} - \sigma_m^i = \mathcal{A}\varepsilon\left(u_m^{i+1}\right) - \mathcal{A}\varepsilon\left(u_m^i\right) + h_m \mathcal{B}\left(\sigma_m^i, \varepsilon\left(u_m^i\right), \zeta_m^i\right),
$$

which, with (16), (17), (58) and (60), gives

(65) $$\left\|\delta \sigma_m^{i+1}\right\| \leq c \left\|\delta u_m^{i+1}\right\| + c, \; i \in \{0, ..., m-1\}.$$

Now, for $i=0$, taking $\xi = \zeta_m^0$ in (52), we get

$$\left(\delta\zeta_m^1, \zeta_m^1 - \zeta_m^0\right)_Y + \tilde{a}\left(\zeta_m^1, \zeta_m^1 - \zeta_m^0\right) \leq \left(\mathcal{G}\left(\sigma_m^0, \varepsilon\left(u_m^0\right), \zeta_m^0\right), \zeta_m^1 - \zeta_m^0\right)_Y,$$

which gives

(66) $\left(\delta\zeta_m^1, \delta\zeta_m^1\right)_Y + h_m\tilde{a}\left(\delta\zeta_m^1, \delta\zeta_m^1\right) + \tilde{a}\left(\zeta_m^0, \delta\zeta_m^1\right) \leq \left(\mathcal{G}\left(\sigma_m^0, \varepsilon\left(u_m^0\right), \zeta_m^0\right), \delta\zeta_m^1\right)_Y.$

Keeping in mind (23), (26), (53 (iii)) and applying integration by parts, we get

$$\tilde{a}\left(\zeta_m^0, \delta\zeta_m^1\right) = -\kappa\left(\Delta\zeta_0, \delta\zeta_m^1\right)_Y,$$

which combined with (66) implies that

$$\left(\delta\zeta_m^1, \delta\zeta_m^1\right)_Y \leq \left(\mathcal{G}\left(\sigma_m^0, \varepsilon\left(u_m^0\right), \zeta_m^0\right), \delta\zeta_m^1\right)_Y + \kappa\left(\Delta\zeta_0, \delta\zeta_m^1\right)_Y.$$

Thus, we have

(67) $$\left\|\delta\zeta_m^1\right\|_Y \leq c.$$

To continue, for all $j \in \{1, ..., m-1\}$, using (52), we obtain

$$\begin{cases} \left(\delta\zeta_m^{j+1} - \delta\zeta_m^j, \zeta_m^{j+1} - \zeta_m^j\right)_Y + \tilde{a}\left(\zeta_m^{j+1} - \zeta_m^j, \zeta_m^{j+1} - \zeta_m^j\right) \\ \\ \leq \left(\mathcal{G}\left(\sigma_m^j, \varepsilon\left(u_m^j\right), \zeta_m^j\right) - \mathcal{G}\left(\sigma_m^{j-1}, \varepsilon\left(u_m^{j-1}\right), \zeta_m^{j-1}\right), \zeta_m^{j+1} - \zeta_m^j\right)_Y, \end{cases}$$

from which we deduce that

$$\begin{aligned} \left(\delta\zeta_m^{j+1}, \delta\zeta_m^{j+1}\right)_Y &\leq \left(\mathcal{G}\left(\sigma_m^j, \varepsilon\left(u_m^j\right), \zeta_m^j\right) - \mathcal{G}\left(\sigma_m^{j-1}, \varepsilon\left(u_m^{j-1}\right), \zeta_m^{j-1}\right), \delta\zeta_m^{j+1}\right)_Y \\ &\quad + \left(\delta\zeta_m^j, \delta\zeta_m^{j+1}\right)_Y, \end{aligned}$$

and keeping in mind (18), we infer that

$$\left\|\delta\zeta_m^{j+1}\right\|_Y \leq c\left(\left\|\sigma_m^j - \sigma_m^{j-1}\right\|_{\mathcal{Q}} + \left\|u_m^j - u_m^{j-1}\right\|_V + \left\|\zeta_m^j - \zeta_m^{j-1}\right\|_Y\right) + \left\|\delta\zeta_m^j\right\|_Y,$$

which leads us to

$$\left\|\delta\zeta_m^{i+1}\right\|_Y \leq ch_m\sum_{j=1}^{i}\left(\left\|\delta\sigma_m^j\right\|_{\mathcal{Q}} + \left\|\delta u_m^j\right\|_V + \left\|\delta\zeta_m^j\right\|_Y\right) + \left\|\delta\zeta_m^1\right\|_Y,$$

for all $i \in \{1, ..., m-1\}$, and using (64), (65) and (67), we obtain

$$\left\|\delta\zeta_m^{i+1}\right\|_Y \leq ch_m\sum_{j=1}^{i}\left\|\delta\zeta_m^j\right\|_Y + c.$$

Therefore, using (67) and applying again Lemma 3, in the last inequality, we obtain

(68) $$\left\|\delta\zeta_m^{i+1}\right\|_Y \leq c, \ i \in \{0, ..., m-1\}.$$

Finally, (59) is a consequence of (64) and (68). $\qquad\square$

Third step. For each $m \in \mathbb{N}^*$ with $m > TL_{\mathcal{G}}$, let u_m^j be the unique solution of the Problem $\mathcal{P}_m^j$, $j = 1, ..., m$. We introduce the following functions $u_m : [0, T] \to V$, $\tilde{u}_m : [0, T] \to V$, $\tilde{\sigma}_m : [0, T] \to \mathcal{Q}$, $\zeta_m : [0, T] \to H^1(\Omega)$, $\tilde{\zeta}_m : [0, T] \to H^1(\Omega)$, $\mathcal{B}_m : [0, T] \to \mathcal{Q}$ and $f_m : [0, T] \to V$ defined, respectively, by

(69) $$u_m(0) = u_0, \ u_m(t) = u_m^i + (t - t_i^m)\delta u_m^{i+1}, \ \forall t \in \left(t_i^m, t_{i+1}^m\right],$$

(70) $$\tilde{u}_m(0) = u_0, \ \tilde{u}_m(t) = u_m^{i+1}, \ \forall t \in \left(t_i^m, t_{i+1}^m\right],$$

(71) $$\tilde{\sigma}_m(0) = \sigma_0, \ \tilde{\sigma}_m(t) = \sigma_m^{i+1}, \ \forall t \in \left(t_i^m, t_{i+1}^m\right],$$

(72) $\zeta_m(0) = \zeta_0,\ \zeta_m(t) = \zeta_m^i + (t - t_i^m)\,\delta\zeta_m^{i+1},\ \forall t \in \left(t_i^m, t_{i+1}^m\right],$

(73) $\tilde{\zeta}_m(0) = \zeta_0,\ \tilde{\zeta}_m(t) = \zeta_m^{i+1},\ \forall t \in \left(t_i^m, t_{i+1}^m\right],$

(74) $\mathcal{B}_m(t) = \begin{cases} h_m \sum\limits_{j=0}^{i} \mathcal{B}\left(\sigma_m^j, \varepsilon\left(u_m^j\right), \zeta_m^j\right),\ \forall t \in \left(t_i^m, t_{i+1}^m\right], \\ \qquad\qquad \mathcal{B}_m(0) = 0_{\mathcal{Q}}, \end{cases}$

(75) $f_m(0) = f(0),\ f_m(t) = f_{i+1}^m = f(t_{i+1}^m),\ \forall t \in \left(t_i^m, t_{i+1}^m\right],$

for all $i \in \{0, ..., m-1\}$. Here $\{\sigma_m^i\}_{0 \le i \le m}$, $\{\zeta_m^i\}_{0 \le i \le m}$ and u_m^0 are given by (51)-(53). From (69), it follows that the function u_m has a derivative function given by

(76) $\dot{u}_m(t) = \delta u_m^{i+1},\ \forall t \in \left(t_i^m, t_{i+1}^m\right),\ i = 0, ..., m-1.$

Also, from (72), we deduce that the function ζ_m has a derivative function defined by

(77) $\dot{\zeta}_m(t) = \delta\zeta_m^{i+1},\ \forall t \in \left(t_i^m, t_{i+1}^m\right),\ i = 0, ..., m-1.$

We have the following estimate results.

Lemma 7. *There exists $c > 0$, such that for all $m \in \mathbb{N}^*$ with $m > TL_{\mathcal{G}}$,*

(78) $\|\tilde{\sigma}_m(t)\|_{\mathcal{Q}} + \|\tilde{u}_m(t)\|_V + \left\|\tilde{\zeta}_m(t)\right\|_Y \quad \le \quad c,\ \forall t \in [0, T],$

(79) $\|\zeta_m(t)\|_Y + \|u_m(t)\|_V \quad \le \quad c,\ \forall t \in [0, T],$

(80) $\left\|\dot{\zeta}_m(t)\right\|_Y + \|\dot{u}_m(t)\|_V \quad \le \quad c,\ a.e.t \in [0, T],$

(81) $\left\|\tilde{\zeta}_m(t) - \zeta_m(t)\right\|_Y + \|\tilde{u}_m(t) - u_m(t)\|_V \quad \le \quad ch_m,\ \forall t \in [0, T],$

(82) $\|f_m(t) - f(t)\|_V \quad \le \quad ch_m,\ \forall t \in [0, T],$

(83) $\|\zeta_m(t) - \zeta_m(s)\|_Y + \|u_m(t) - u_m(s)\|_V \quad \le \quad c\,|t - s|,\ \forall t, s \in [0, T],$

(84) $\|u_m(t) - u_m(s)\|_{L^2(\Gamma_3; \mathbb{R}^d)} \quad \le \quad c\,|t - s|,\ \forall t, s \in [0, T].$

Proof. It is clear that (78)-(80) are consequences of (69)-(73), (76)-(77) and (58)-(59). For the proof of (81)-(82), see [13, Lemma 4.7]. Now, it follows from (72) and (77) that

$$\begin{aligned} \zeta_m(t) &= \zeta_m^i + \left(\int_{t_i^m}^{t} dr\right)\delta\zeta_m^{i+1} = \left(\zeta_0 + \sum_{j=1}^{i} h_m \delta\zeta_m^j\right) + \left(\int_{t_i^m}^{t} dr\right)\delta\zeta_m^{i+1} \\ &= \zeta_0 + \sum_{j=1}^{i}\int_{t_{j-1}^m}^{t_j^m}\dot{\zeta}_m(r)dr + \int_{t_i^m}^{t}\dot{\zeta}_m(r)dr \\ &= \zeta_0 + \int_0^t \dot{\zeta}_m(r)dr, \end{aligned}$$

for all $t \in \left(t_i^m, t_{i+1}^m\right]$, $i \in \{0, ..., m-1\}$. Also, using (69) and (76), we obtain

$$u_m(t) = u_0 + \int_0^t \dot{u}_m(r)dr,\ \text{for all } t \in [0, T].$$

Therefore, using (80), we get

$$\|\zeta_m(t) - \zeta_m(s)\|_Y + \|u_m(t) - u_m(s)\|_V \leq \left|\int_s^t \left\|\dot{\zeta}_m(r)\right\|_Y dr\right| + \left|\int_s^t \|\dot{u}_m(r)\|_V dr\right|$$
$$\leq c\,|t - s|,$$

for all t, $s \in [0, T]$. Finally, (84) is a direct consequence of (83) and (15). $\qquad\square$

In the next we need the following result.

Lemma 8. *There exists $c > 0$, such that for all m, $n \in \mathbb{N}^*$ with $m > n > \max(T, TL_{\mathcal{G}})$,*

$$(85) \qquad \|\mathcal{B}_m(t) - \mathcal{B}_n(t)\|_{\mathcal{Q}}^2 \leq c \int_0^t \|u_m(s) - u_n(s)\|_V^2\,ds + ch_n,\ \forall t \in [0, T].$$

Proof. Let m, $n \in \mathbb{N}^*$ with $m > n > \max(T, TL_{\mathcal{G}})$. It is obvious that (85) holds for $t = 0$. Now, for each $t \in (0, T]$, there exist two integers $q \in \{0, ..., m-1\}$ and $p \in \{0, ..., n-1\}$, such that

$$(86) \qquad\qquad t \in \left(t_q^m, t_{q+1}^m\right] \cap \left(t_p^n, t_{p+1}^n\right].$$

It follows from (51), (70), (71) and (73) that

$$(87) \qquad \tilde{\sigma}_m(t) = \mathcal{A}\varepsilon(\tilde{u}_m(t)) + h_m \mathcal{B}\left(\sigma_m^0, \varepsilon\left(u_m^0\right), \zeta_m^0\right) + \sigma_0 - \mathcal{A}\varepsilon(u_0),$$

for all $t \in (t_0^m, t_1^m]$. Also, for all $t \in \left(t_q^m, t_{q+1}^m\right]$, $q \in \{1, ..., m-1\}$, we use (51), (70), (71) and (73), to obtain

$$\tilde{\sigma}_m(t) = \mathcal{A}\varepsilon(\tilde{u}_m(t)) + \sum_{l=1}^q \int_{t_{l-1}^m}^{t_l^m} \mathcal{B}\left(\tilde{\sigma}_m(s), \varepsilon(\tilde{u}_m(s)), \tilde{\zeta}_m(s)\right) ds$$

$$+ h_m \mathcal{B}\left(\sigma_m^0, \varepsilon\left(u_m^0\right), \zeta_m^0\right) + \sigma_0 - \mathcal{A}\varepsilon(u_0),$$

which with (87) gives

$$(88) \quad \begin{cases} \tilde{\sigma}_m(t) = \mathcal{A}\varepsilon(\tilde{u}_m(t)) + \displaystyle\int_0^t \mathcal{B}\left(\tilde{\sigma}_m(s), \varepsilon(\tilde{u}_m(s)), \tilde{\zeta}_m(s)\right) ds \\[2mm] \qquad + \displaystyle\int_t^{t_q^m} \mathcal{B}\left(\tilde{\sigma}_m(s), \varepsilon(\tilde{u}_m(s)), \tilde{\zeta}_m(s)\right) ds \\[2mm] \qquad + h_m \mathcal{B}\left(\sigma_m^0, \varepsilon\left(u_m^0\right), \zeta_m^0\right) + \sigma_0 - \mathcal{A}\varepsilon(u_0), \end{cases}$$

for all $t \in \left(t_q^m, t_{q+1}^m\right]$, $q \in \{0, ..., m-1\}$. Therefore, using (88), (16), (17), (60) and (78), we infer that

$$(89) \quad \begin{cases} \|\tilde{\sigma}_m(t) - \tilde{\sigma}_n(t)\|_{\mathcal{Q}} \leq c\|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V + c\displaystyle\int_0^t \|\tilde{\sigma}_m(s) - \tilde{\sigma}_n(s)\|_{\mathcal{Q}}\,ds \\[2mm] + c\displaystyle\int_0^t \|\tilde{u}_m(s) - \tilde{u}_n(s)\|_V\,ds + c\displaystyle\int_0^t \left\|\tilde{\zeta}_m(s) - \tilde{\zeta}_n(s)\right\|_Y\,ds + ch_m + ch_n, \end{cases}$$

for all $t \in [0, T]$. Now, let $s \in \left(t_q^m, t_{q+1}^m\right) \cap \left(t_p^n, t_{p+1}^n\right)$, $q \in \{0, ..., m-1\}$ and $p \in \{0, ..., n-1\}$. Using (52), (70), (71), (73) and (77), we get

$$
\begin{cases}
\left(\dot{\zeta}_m(s) - \dot{\zeta}_n(s), \tilde{\zeta}_m(s) - \tilde{\zeta}_n(s)\right)_Y \\[2mm]
\quad + \tilde{a}\left(\tilde{\zeta}_m(s) - \tilde{\zeta}_n(s), \tilde{\zeta}_m(s) - \tilde{\zeta}_n(s)\right) \\[2mm]
\leq \left(\mathcal{G}\left(\tilde{\sigma}_m(s), \varepsilon(\tilde{u}_m(s)), \tilde{\zeta}_m(s)\right), \tilde{\zeta}_m(s) - \tilde{\zeta}_n(s)\right)_Y \\[2mm]
\quad - \left(\mathcal{G}\left(\tilde{\sigma}_n(s), \varepsilon(\tilde{u}_n(s)), \tilde{\zeta}_n(s)\right), \tilde{\zeta}_m(s) - \tilde{\zeta}_n(s)\right)_Y,
\end{cases}
$$

which gives

$$
\begin{cases}
\left(\dot{\zeta}_m(s) - \dot{\zeta}_n(s), \zeta_m(s) - \zeta_n(s)\right)_Y \\[2mm]
\leq \left(\mathcal{G}\left(\tilde{\sigma}_m(s), \varepsilon(\tilde{u}_m(s)), \tilde{\zeta}_m(s)\right), \tilde{\zeta}_m(s) - \tilde{\zeta}_n(s)\right)_Y \\[2mm]
\quad - \left(\mathcal{G}\left(\tilde{\sigma}_n(s), \varepsilon(\tilde{u}_n(s)), \tilde{\zeta}_n(s)\right), \tilde{\zeta}_m(s) - \tilde{\zeta}_n(s)\right)_Y \\[2mm]
\quad + \left(\dot{\zeta}_m(s) - \dot{\zeta}_n(s), \zeta_m(s) - \tilde{\zeta}_m(s) + \tilde{\zeta}_n(s) - \zeta_n(s)\right)_Y,
\end{cases}
$$

which combined with (18) and using the inequality

$$
ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2, \ \forall a, \ b \in \mathbb{R},
$$

gives

$$
\begin{cases}
\left(\dot{\zeta}_m(s) - \dot{\zeta}_n(s), \zeta_m(s) - \zeta_n(s)\right)_Y \\[2mm]
\leq c\left\|\tilde{\sigma}_m(s) - \tilde{\sigma}_n(s)\right\|_Q^2 + c\left\|\tilde{u}_m(s) - \tilde{u}_n(s)\right\|_V^2 \\[2mm]
\qquad + c\left\|\zeta_m(s) - \tilde{\zeta}_m(s)\right\|_Y^2 \\[2mm]
\quad + \left(\left\|\dot{\zeta}_m(s)\right\|_Y + \left\|\dot{\zeta}_n(s)\right\|_Y\right)\left\|\zeta_m(s) - \tilde{\zeta}_m(s)\right\|_Y \\[2mm]
\quad + \left(\left\|\dot{\zeta}_m(s)\right\|_Y + \left\|\dot{\zeta}_n(s)\right\|_Y\right)\left\|\zeta_n(s) - \tilde{\zeta}_n(s)\right\|_Y,
\end{cases}
$$

and keeping in mind (80) and (81), we obtain

$$
\begin{cases}
\left(\dot{\zeta}_m(s) - \dot{\zeta}_n(s), \zeta_m(s) - \zeta_n(s)\right)_Y \\[2mm]
\leq c\left\|\tilde{\sigma}_m(s) - \tilde{\sigma}_n(s)\right\|_Q^2 + c\left\|\tilde{u}_m(s) - \tilde{u}_n(s)\right\|_V^2 \\[2mm]
\qquad + c\left\|\zeta_m(s) - \tilde{\zeta}_m(s)\right\|_Y^2 + ch_m + ch_n.
\end{cases}
$$

Integrating both sides of the last inequality on $(0, t)$, we get

$$\|\zeta_m(t) - \zeta_n(t)\|_Y^2 \quad \leq \quad c\int_0^t \|\tilde{\sigma}_m(s) - \tilde{\sigma}_n(s)\|_Q^2\, ds + c\int_0^t \|\tilde{u}_m(s) - \tilde{u}_n(s)\|_V^2\, ds$$

$$+ c\int_0^t \left\|\tilde{\zeta}_m(s) - \tilde{\zeta}_n(s)\right\|_Y^2\, ds + ch_m + ch_n,$$

which together with (89) and using the fact that

$$\left\|\tilde{\zeta}_m(t) - \tilde{\zeta}_n(t)\right\|_Y^2 \quad \leq \quad c\left\|\tilde{\zeta}_m(t) - \zeta_m(t)\right\|_Y^2 + c\|\zeta_m(t) - \zeta_n(t)\|_Y^2 + c\left\|\zeta_n(t) - \tilde{\zeta}_n(t)\right\|_Y^2$$

$$\leq \quad c\|\zeta_m(t) - \zeta_n(t)\|_Y^2 + ch_m^2 + ch_n^2,$$

yields

$$\|\tilde{\sigma}_m(t) - \tilde{\sigma}_n(t)\|_Q^2 + \left\|\tilde{\zeta}_m(t) - \tilde{\zeta}_n(t)\right\|_Y^2 \quad \leq \quad c\int_0^t \left\|\tilde{\zeta}_m(s) - \tilde{\zeta}_n(s)\right\|_Y^2\, ds + c\int_0^t \|\tilde{\sigma}_m(s) - \tilde{\sigma}_n(s)\|_Q^2\, ds$$

$$+ c\|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V^2 + c\int_0^t \|\tilde{u}_m(s) - \tilde{u}_n(s)\|_V^2\, ds$$

$$+ ch_m + ch_n.$$

Using Lemma 2, in the last inequality, one has

$$(90) \qquad \begin{cases} \|\tilde{\sigma}_m(t) - \tilde{\sigma}_n(t)\|_Q^2 + \left\|\tilde{\zeta}_m(t) - \tilde{\zeta}_n(t)\right\|_Y^2 \leq c\|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V^2 \\ \\ \qquad + c\int_0^t \|\tilde{u}_m(s) - \tilde{u}_n(s)\|_V^2\, ds + ch_n. \end{cases}$$

We use now (81) to show that

$$(91) \qquad \|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V^2 \leq \|u_m(t) - u_n(t)\|_V^2 + ch_m^2 + ch_n^2,\ \forall t \in [0, T].$$

To continue, using (70), (71), (73) and (74), we have

$$\mathcal{B}_m(t) \quad = \quad \int_0^t \mathcal{B}\left(\tilde{\sigma}_m(s), \varepsilon(\tilde{u}_m(s)), \tilde{\zeta}_m(s)\right) ds$$

$$(92) \qquad + \int_t^{t_q^m} \mathcal{B}\left(\tilde{\sigma}_m(s), \varepsilon(\tilde{u}_m(s)), \tilde{\zeta}_m(s)\right) ds + h_m\mathcal{B}\left(\sigma_m^0, \varepsilon(u_m^0), \zeta_m^0\right),$$

for all $t \in \left(t_q^m, t_{q+1}^m\right]$, $q \in \{0, ..., m-1\}$. Using (92), (17), (60) and (78), we get

$$\|\mathcal{B}_m(t) - \mathcal{B}_n(t)\|_Q^2 \quad \leq \quad c\int_0^t \|\tilde{\sigma}_m(s) - \tilde{\sigma}_n(s)\|_Q^2\, ds + c\int_0^t \|\tilde{u}_m(s) - \tilde{u}_n(s)\|_V^2\, ds$$

$$+ c\int_0^t \left\|\tilde{\zeta}_m(s) - \tilde{\zeta}_n(s)\right\|_Y^2\, ds$$

$$+ ch_m^2 + ch_n^2,$$

for all $t \in \left(t_q^m, t_{q+1}^m\right] \cap \left(t_p^n, t_{p+1}^n\right]$, with $q \in \{0, ..., m-1\}$ and $p \in \{0, ..., n-1\}$, and employing (90) and (91), we obtain (85). $\qquad\square$

Lemma 9. *There exists a function* $u \in W^{1,2}(0,T;V)$ *and two subsequences of* $\{u_m\}$ *and* $\{\tilde{u}_m\}$ *again denoted by* $\{u_m\}$ *and* $\{\tilde{u}_m\}$, *respectively, such that*

(93) $$u_m \;\rightharpoonup\; u \text{ weakly in } L^2(0,T;V),$$

(94) $$\dot{u}_m \;\rightharpoonup\; \dot{u} \text{ weakly in } L^2(0,T;V),$$

(95) $$\varepsilon(\dot{u}_m) \;\rightharpoonup\; \varepsilon(\dot{u}) \text{ weakly in } L^2(0,T;\mathcal{Q}),$$

(96) $$u_m \;\rightarrow\; u \text{ strongly in } C\left([0,T];L^2(\Gamma_3;\mathbb{R}^d)\right),$$

(97) $$u_m \;\rightarrow\; u \text{ strongly in } C\left([0,T];V\right),$$

(98) $$\tilde{u}_m \;\rightarrow\; u \text{ strongly in } L^2(0,T;V).$$

Proof. To proof (93)-(96), we use Lemma 7 and compactness arguments similar to those in [12, Lemma 7]. To continue, using (55), (70), (71), (73), (74), (75) and (30), we conclude that $\{\mathcal{B}_m\}$, $\{\tilde{u}_m\}$ and $\{f_m\}$ satisfy the following inequality

(99)
$$\begin{cases} (\mathcal{F}\tilde{u}_m(t), v - \tilde{u}_m(t))_V + (\mathcal{B}_m(t), \varepsilon(v - \tilde{u}_m(t)))_{\mathcal{Q}} + \\[2mm] \psi(\tilde{u}_m(t), v - \tilde{u}_m(t)) \geq (f_m(t), v - \tilde{u}_m(t))_V, \ \forall v \in V, \ \forall t \in [0,T]. \end{cases}$$

Now, let m, $n \in \mathbb{N}^*$, such that $m > n > \max(T, TL_{\mathcal{G}})$. Taking $(\mathcal{B}_m, \tilde{u}_m, f_m, v) = (\mathcal{B}_m, \tilde{u}_m, f_m, \tilde{u}_n)$, $(\mathcal{B}_m, \tilde{u}_m, f_m, v) = (\mathcal{B}_n, \tilde{u}_n, f_n, \tilde{u}_m)$ in (99) and adding the two inequalities, we get

$$\begin{cases} (\mathcal{F}\tilde{u}_m(t) - \mathcal{F}\tilde{u}_n(t), \tilde{u}_m(t) - \tilde{u}_n(t))_V \leq (\mathcal{B}_n(t) - \mathcal{B}_m(t), \varepsilon(\tilde{u}_m(t) - \tilde{u}_n(t)))_{\mathcal{Q}} \\[2mm] +\psi(\tilde{u}_n(t), \tilde{u}_m(t) - \tilde{u}_n(t)) + \psi(\tilde{u}_m(t), \tilde{u}_n(t) - \tilde{u}_m(t)) \\[2mm] + (f_m(t) - f_n(t), \tilde{u}_m(t) - \tilde{u}_n(t))_V, \ \forall t \in [0,T], \end{cases}$$

which combined with (48), (33), (78) and using the inequality

$$ab \leq \frac{a^2}{m_{\mathcal{A}}} + \frac{m_{\mathcal{A}}}{4}b^2, \ \forall a, \ b \in \mathbb{R},$$

leads us to

$$\|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V^2 \;\leq\; c\|\mathcal{B}_n(t) - \mathcal{B}_m(t)\|_{\mathcal{Q}}^2 + c\|\tilde{u}_m(t) - \tilde{u}_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} +$$

(100) $$c\|f_m(t) - f(t)\|_V^2 + c\|f(t) - f_n(t)\|_V^2.$$

Using (81) and (15), we get

$$\begin{aligned} \|\tilde{u}_m(t) - \tilde{u}_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} &\leq \|\tilde{u}_m(t) - u_m(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} + \|u_m(t) - u_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} \\ &\quad + \|u_n(t) - \tilde{u}_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} \end{aligned}$$

(101) $$\leq \|u_m(t) - u_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} + ch_m + ch_n.$$

Now, using (100), (101), (82) and (85), we obtain

$$\begin{aligned} \|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V^2 &\leq c\|u_m(t) - u_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} + c\int_0^t \|u_n(s) - u_m(s)\|_V^2\, ds \\ &\quad + ch_m + ch_n, \end{aligned}$$

and using the fact that

$$\begin{aligned} \|u_m(t) - u_n(t)\|_V^2 &\leq c\|u_m(t) - \tilde{u}_m(t)\|_V^2 + c\|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V^2 + c\|\tilde{u}_n(t) - u_n(t)\|_V^2 \\ &\leq c\|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V^2 + ch_m + ch_n, \end{aligned}$$

we get

$$\|u_m(t) - u_n(t)\|_V^2 \leq c \|u_m(t) - u_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} + c \int_0^t \|u_n(s) - u_m(s)\|_V^2 \, ds$$
$$+ c h_m + c h_n.$$

Now, we use Lemma 2, in the last inequality, to obtain

$$\|u_m(t) - u_n(t)\|_V^2 \leq c \|u_m(t) - u_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} + c \int_0^t \|u_m(s) - u_n(s)\|_{L^2(\Gamma_3;\mathbb{R}^d)} \, ds +$$
$$+ c h_m + c h_n.$$

Thus, we get

$$\|u_m - u_n\|_{C([0,T];V)}^2 \leq c \|u_m - u_n\|_{C([0,T];L^2(\Gamma_3;\mathbb{R}^d))} + c h_n,$$

which combined with (96) implies that $\{u_m\}$ is a Cauchy sequence in $C([0,T];V)$, and using the convergence (93), we obtain (97). Finally, the convergence (98) is a consequence of (81) and (97). □

In the rest of this paper u is the function obtained in Lemma 9, $\{u_m\}$, $\{\mathcal{B}_m\}$, $\{\tilde{u}_m\}$, $\left\{\tilde{\zeta}_m\right\}$ and $\{f_m\}$ represent appropriate subsequences of $\{u_m\}$, $\{\mathcal{B}_m\}$, $\{\tilde{u}_m\}$, $\left\{\tilde{\zeta}_m\right\}$ and $\{f_m\}$, respectively, such that the convergences (93)-(98) hold.

Consider the following problem.

Problem 6. *Find* $(\sigma, \zeta) \in L^2(0,T;\mathcal{Q}) \times L^2(0,T;Y)$, *such that*

$$(102) \quad \sigma(t) = \mathcal{A}\varepsilon(u(t)) + \int_0^t \mathcal{B}(\sigma(s), \varepsilon(u(s)), \zeta(s)) \, ds + \sigma_0 - \mathcal{A}\varepsilon(u_0), \ \forall t \in [0,T],$$

$$(103) \quad \begin{cases} \zeta(t) \in \mathcal{K}, \ \forall \, t \in [0,T], \\[2mm] \left(\dot{\zeta}(t), \vartheta - \zeta(t)\right)_Y + \tilde{a}(\zeta(t), \vartheta - \zeta(t)) \\[2mm] \geq (\mathcal{G}(\sigma(t), \varepsilon(u(t)), \zeta(t)), \vartheta - \zeta(t))_Y, \ \forall \, \vartheta \in \mathcal{K}, \ a.e. \ t \in (0,T), \end{cases}$$

$$(104) \quad \zeta(0) = \zeta_0.$$

We have the following result.

Lemma 10. *Problem (102)-(104) has a unique solution* $\{\sigma, \zeta\}$. *Moreover* ζ *satisfies (43).*

Proof. We notice that $X = \mathcal{Q} \times Y$ and $L^2(0,T;X)$ are Hilbert spaces equipped with the respective canonical inner products

$$((\sigma_1, \zeta_1), (\sigma_2, \zeta_2))_X = (\sigma_1, \sigma_2)_{\mathcal{Q}} + (\zeta_1, \zeta_2)_Y,$$

$$((\beta_1, \theta_1), (\beta_2, \theta_2))_{L^2(0,T;X)} = \int_0^T (\beta_1(s), \beta_2(s))_{\mathcal{Q}} \, ds + \int_0^T (\theta_1(s), \theta_2(s))_Y \, ds,$$

for all (σ_1, ζ_1), $(\sigma_2, \zeta_2) \in X$ and for all (β_1, θ_1), $(\beta_2, \theta_2) \in L^2(0,T;X)$.

Let $(\beta, \theta) \in L^2(0,T;X)$ and let $\sigma_\beta : [0,T] \to \mathcal{Q}$ be the function defined by

$$(105) \quad \sigma_\beta(t) = \mathcal{A}\varepsilon(u(t)) + \int_0^t \beta(s) ds + \sigma_0 - \mathcal{A}\varepsilon(u_0), \ \forall t \in [0,T].$$

It follows from (26) that $\tilde{a}$ is a continuous and symmetric bilinear form on $H^1(\Omega)$ and, moreover, $\tilde{a}$ satisfies (4). Thus, applying Lemma 1 for $Z = H^1(\Omega)$, $E = Y$, $K = \mathcal{K}$, $w_0 = \zeta_0$ and $l = \theta$, we deduce that the following problem. Find $\zeta_\theta \in W^{1,2}(0,T;Y) \cap L^2(0,T;H^1(\Omega))$, such that

$$(106) \quad \begin{cases} \zeta_\theta(t) \in \mathcal{K}, \ \forall \ t \in [0,T], \\[2mm] \left(\dot{\zeta}_\theta(t), \vartheta - \zeta_\theta(t)\right)_Y \\[2mm] +\tilde{a}\left(\zeta_\theta(t), \vartheta - \zeta_\theta(t)\right) \geq \left(\theta(t), \vartheta - \zeta_\theta(t)\right)_Y, \ \forall \ \vartheta \in \mathcal{K}, \ a.e. \ t \in (0,T), \end{cases}$$

$$(107) \quad \zeta_\theta(0) = \zeta_0,$$

has a unique solution. To continue, let $\Lambda : L^2(0,T;X) \to L^2(0,T;X)$ be the operator defined by

$$(108) \quad \Lambda(\beta,\theta)(t) = \left(\mathcal{B}(\sigma_\beta(t), \varepsilon(u(t)), \zeta_\theta(t)), \mathcal{G}(\sigma_\beta(t), \varepsilon(u(t)), \zeta_\theta(t))\right),$$

for all $(\beta,\theta) \in L^2(0,T;X)$ and for all $t \in [0,T]$.

Let (β_1,θ_1), $(\beta_2,\theta_2) \in L^2(0,T;X)$. Then, using (105), we get

$$(109) \quad \|\sigma_{\beta_1}(t) - \sigma_{\beta_2}(t)\|_{\mathcal{Q}}^2 \leq c \int_0^t \|\beta_1(t) - \beta_2(t)\|_{\mathcal{Q}}^2 \, ds, \ \forall \ t \in [0,T].$$

Taking $(\theta, \zeta_\theta, \vartheta) = (\theta_1, \zeta_{\theta_1}, \zeta_{\theta_2})$, $(\theta, \zeta_\theta, \vartheta) = (\theta_2, \zeta_{\theta_2}, \zeta_{\theta_1})$ in (106), adding the two inequalities, we have

$$\int_0^t \left(\dot{\zeta}_{\theta_1}(s) - \dot{\zeta}_{\theta_2}(s), \zeta_{\theta_1}(s) - \zeta_{\theta_2}(s)\right)_Y ds \leq \int_0^t \left(\theta_1(s) - \theta_2(s), \zeta_{\theta_1}(s) - \zeta_{\theta_2}(s)\right)_Y ds,$$

and using (107) yeilds

$$\|\zeta_{\theta_1}(t) - \zeta_{\theta_2}(t)\|_Y^2 \leq c \int_0^t \|\theta_1(s) - \theta_2(s)\|_Y^2 \, ds + c \int_0^t \|\zeta_{\theta_1}(s) - \zeta_{\theta_2}(s)\|_Y^2 \, ds.$$

We use now Lemma 2, in the last inequality, to obtain

$$(110) \quad \|\zeta_{\theta_1}(t) - \zeta_{\theta_2}(t)\|_Y^2 \leq c \int_0^t \|\theta_1(s) - \theta_2(s)\|_Y^2 \, ds, \ \forall \ t \in [0,T].$$

From (108), (109) and (110), it follows that

$$\begin{aligned} \|\Lambda(\beta_1,\theta_1)(t) - \Lambda(\beta_2,\theta_2)(t)\|_X^2 &\leq c \int_0^t \|\sigma_{\beta_1}(s) - \sigma_{\beta_2}(s)\|_{\mathcal{Q}}^2 \, ds + c \int_0^t \|\zeta_{\theta_1}(s) - \zeta_{\theta_2}(s)\|_Y^2 \, ds \\[2mm] &\leq c \int_0^t \|\beta_1(s) - \beta_2(s)\|_{\mathcal{Q}}^2 \, ds + c \int_0^t \|\theta_1(s) - \theta_2(s)\|_Y^2 \, ds \\[2mm] &\leq c \int_0^t \|(\beta_1(s),\theta_1(s)) - (\beta_2(s),\theta_2(s))\|_X^2 \, ds, \ \forall \ t \in [0,T]. \end{aligned}$$

Reiterating the last inequality n times, we infer that

$$\|\Lambda(\beta_1,\theta_1) - \Lambda(\beta_2,\theta_2)\|_{L^2(0,T;X)}^2 \leq \frac{(cT)^n}{n!} \|(\beta_1,\theta_1) - (\beta_2,\theta_2)\|_{L^2(0,T;X)}^2,$$

which implies that, for n sufficiently large, a power Λ^n of Λ is a contraction in the Banach space $L^2(0,T;X)$. Therefore, we deduce that Λ has a unique fixed point $(\beta^*,\theta^*) \in L^2(0,T;X)$. Now, let $\sigma = \sigma_{\beta^*}$ and let $\zeta = \zeta_{\theta^*}$, then we deduce that (σ,ζ) is the unique solution of Problem (102)-(104). Moreover ζ satisfies (43). $\qquad \square$

We have the following convergence results.

Lemma 11. *The following convergences hold.*

(111) $$\mathcal{F}\tilde{u}_m \quad \to \quad \mathcal{F}u \ \text{strongly in } L^2\left(0,T;V\right),$$

(112) $$f_m \quad \to \quad f \ \text{strongly in } L^2\left(0,T;V\right),$$

(113) $$\mathcal{B}_m \quad \to \quad \widetilde{\mathcal{B}} \ \text{strongly in } L^2\left(0,T;\mathcal{Q}\right),$$

where

(114) $$\widetilde{\mathcal{B}}\left(t\right) = \int_0^t \mathcal{B}\left(\sigma(s),\varepsilon(u(s)),\zeta(s)\right)ds, \ \forall \ t \in [0,T].$$

Proof. Obviously, (49) and (98) gives (111). Also, from (82), we get (112). To continue, let $m \in \mathbb{N}^*$ with $m > TL_{\mathcal{G}}$, using (88), (102), (16), (17), (78) and (60), we have

$$\begin{aligned}\|\tilde{\sigma}_m(t) - \sigma\left(t\right)\|_{\mathcal{Q}}^2 \quad &\leq \quad c\left\|\tilde{u}_m(t) - u(t)\right\|_V^2 \ + c\int_0^t \|\tilde{\sigma}_m(s) - \sigma\left(s\right)\|_{\mathcal{Q}}^2\, ds \\ &\quad + c\int_0^t \left\|\tilde{\zeta}_m(s) - \zeta\left(s\right)\right\|_Y^2\, ds \end{aligned}$$

(115) $$+c\int_0^t \|\tilde{u}_m(s) - u(s)\|_V^2\, ds + ch_m^2 \ , \ \forall t \in [0,T].$$

On the other hand, using (52), (103), (70), (71), (73) and (77), we get

$$\left\{\begin{aligned} &\left(\dot{\zeta}_m\left(s\right) - \dot{\zeta}\left(s\right), \zeta_m\left(s\right) - \zeta\left(s\right)\right)_Y \\ &\quad +\tilde{a}\left(\tilde{\zeta}_m\left(s\right) - \zeta\left(s\right), \tilde{\zeta}_m\left(s\right) - \zeta\left(s\right)\right) \\ &\leq \left(\mathcal{G}\left(\tilde{\sigma}_m\left(s\right), \varepsilon\left(\tilde{u}_m\left(s\right)\right), \tilde{\zeta}_m\left(s\right)\right), \tilde{\zeta}_m\left(s\right) - \zeta\left(s\right)\right)_Y \\ &\quad -\left(\mathcal{G}\left(\sigma\left(s\right), \varepsilon\left(u\left(s\right)\right), \zeta\left(s\right)\right), \tilde{\zeta}_m\left(s\right) - \zeta\left(s\right)\right)_Y \\ &\quad +\left(\dot{\zeta}_m\left(s\right) - \dot{\zeta}\left(s\right), \zeta_m\left(s\right) - \tilde{\zeta}_m\left(s\right)\right)_Y, \end{aligned}\right.$$

for all $s \in \left(t_i^m, t_{i+1}^m\right)$, $i = 0,...,m-1$, which combined with (18), and using the inequality

$$ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2, \ \forall a, \ b \in \mathbb{R},$$

gives

$$\left\{\begin{aligned} &\left(\dot{\zeta}_m\left(s\right) - \dot{\zeta}\left(s\right), \zeta_m\left(s\right) - \zeta\left(s\right)\right)_Y \\ &\leq c\left\|\tilde{\sigma}_m(s) - \sigma(s)\right\|_{\mathcal{Q}}^2 + c\left\|\tilde{u}_m(s) - u(s)\right\|_V^2 \\ &\quad +c\left\|\zeta_m\left(s\right) - \zeta\left(s\right)\right\|_Y^2 \\ &\quad +\left(\left\|\dot{\zeta}_m\left(s\right)\right\|_Y + \left\|\dot{\zeta}\left(s\right)\right\|_Y\right)\left\|\zeta_m\left(s\right) - \tilde{\zeta}_m\left(s\right)\right\|_Y. \end{aligned}\right.$$

Integrating both sides of this inequality on $(0, t)$, and using the fact that

$$\int_0^t \left(\left\| \dot{\zeta}_m(s) \right\|_Y + \left\| \dot{\zeta}(s) \right\|_Y \right) \left\| \zeta_m(s) - \tilde{\zeta}_m(s) \right\|_Y ds \leq ch_m, \ \forall \ t \in [0, T],$$

we get

$$\|\zeta_m(t) - \zeta(t)\|_Y^2 \ \leq \ c \int_0^t \|\tilde{\sigma}_m(s) - \sigma(s)\|_Q^2 \, ds + c \int_0^t \|\tilde{u}_m(s) - u(s)\|_V^2 \, ds$$
$$+ c \int_0^t \left\| \tilde{\zeta}_m(s) - \zeta(s) \right\|_Y^2 ds + ch_m,$$

which, together with (115), gives

$$\|\tilde{\sigma}_m(t) - \sigma(t)\|_Q^2 + \|\zeta_m(t) - \zeta(t)\|_Y^2 \ \leq \ c \|\tilde{u}_m(t) - u(t)\|_V^2 \ + c \int_0^t \|\tilde{\sigma}_m(s) - \sigma(s)\|_Q^2 \, ds$$
$$+ c \int_0^t \|\tilde{u}_m(s) - u(s)\|_V^2 \, ds$$
$$+ c \int_0^t \left\| \tilde{\zeta}_m(s) - \zeta(s) \right\|_Y^2 ds + ch_m^2.$$

Using Lemma 2, in the last inequality, one has

$$\|\tilde{\sigma}_m(t) - \sigma(t)\|_Q^2 + \left\| \tilde{\zeta}_m(t) - \zeta(t) \right\|_Y^2 \ \leq \ c \|\tilde{u}_m(t) - u(t)\|_V^2 \ + c \int_0^t \|\tilde{u}_m(s) - u(s)\|_V^2 \, ds$$
$$+ ch_m^2,$$

which combined with (92) and (114), leads us to

$$\left\| \mathcal{B}_m(t) - \widetilde{\mathcal{B}}(t) \right\|_Q^2 \ \leq \ c \int_0^t \|\tilde{\sigma}_m(s) - \sigma(s)\|_Q^2 \, ds + c \int_0^t \left\| \tilde{\zeta}_m(s) - \zeta(s) \right\|_Y^2 ds$$
$$+ c \int_0^t \|\tilde{u}_m(s) - u(s)\|_V^2 \, ds + ch_m^2$$
$$\leq \ c \int_0^t \|\tilde{u}_m(s) - u(s)\|_V^2 \, ds + ch_m^2, \ \forall t \in [0, T],$$

which with (98) gives (113). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 12. *The following properties hold.*

(116) $\displaystyle \lim_{m \to +\infty} \int_0^T \psi(\tilde{u}_m(s), v(s)) ds = \int_0^T \psi(u(s), v(s)) ds,$ *for all* $v \in L^2(0, T; V)$.

(117) $\displaystyle \lim_{m \to +\infty} \int_0^T \left[\psi(\tilde{u}_m(s), \dot{u}_m(s)) - \psi(u(s), \dot{u}_m(s)) \right] ds = 0.$

(118) $\displaystyle \liminf_{m \to +\infty} \int_0^T \psi(\tilde{u}_m(s), \dot{u}_m(s)) ds \geq \int_0^T \psi(u(s), \dot{u}(s)) ds.$

Proof. Let $v \in L^2(0, T; V)$. Using (31), we obtain

(119)
$$
\begin{cases}
\left| \int_0^T [\psi(\tilde{u}_m(s), v(s)) - \psi(u(s), v(s))] \, ds \right| \leq \\
\\
c \, \|\tilde{u}_m - u\|_{L^2(0,T;V)} \, \|v\|_{L^2(0,T;V)} \, .
\end{cases}
$$

Thus, from (119), (80) and (98), we deduce that ψ satisfies the convergences (116)-(117). To continue, let $\Phi : L^2(0, T; V) \to \mathbb{R}$ be the functional defined by

(120)
$$
\Phi(v) = \int_0^T \psi(u(s), v(s)) ds, \ \forall v \in L^2(0, T; V) .
$$

Using (27), (32) and (120), we find that Φ is convex and continuous. Thus, we deduce that Φ is a weakly lower semicontinuous function on $L^2(0, T; V)$, see [2], which with (94) gives

(121)
$$
\liminf_{m \to +\infty} \Phi(\dot{u}_m) \geq \Phi(\dot{u}) .
$$

On the other hand, one has

(122)
$$
\begin{cases}
\int_0^T \varphi(\tilde{u}_m(s), \dot{u}_m(s)) ds = \int_0^T [\varphi(\tilde{u}_m(s), \dot{u}_m(s)) ds - \varphi(u(s), \dot{u}_m(s))] \, ds \\
\qquad\qquad\qquad\qquad\qquad + \Phi(\dot{u}_m) \, .
\end{cases}
$$

Therefore, taking into account (117) and (121) when passing to the lim inf as $m \to +\infty$ in (122), we obtain (118). $\qquad\square$

Fourth step. We have now all the ingredients to prove Theorem 1.

Proof. Let $t \in (0, T)$, let $r > 0$, such that $t + r \in (0, T)$, for each $w \in V$, we define a function $v \in L^2(0, T; V)$ by

(123)
$$
v(s) = \begin{cases}
w & \text{for } s \in (t, t + r) , \\
\\
\dot{u}(s) & \text{elsewhere.}
\end{cases}
$$

We use (50), (70), (74), (75) and (76) to obtain the following inequality

(124)
$$
\begin{cases}
\dfrac{1}{r} \int_0^T (\mathcal{F}\tilde{u}_m(s), v(s) - \dot{u}_m(s))_V \, ds \\
\\
+ \dfrac{1}{r} \int_0^T (\mathcal{B}_m(s), \varepsilon(v(s) - \dot{u}_m(s)))_{\mathcal{Q}} \, ds \\
\\
+ \dfrac{1}{r} \int_0^T [\varphi(\tilde{u}_m(s), v(s)) - \varphi(\tilde{u}_m(s), \dot{u}_m(s))] \, ds \\
\\
\geq \dfrac{1}{r} \int_0^T (f_m(s), v(s) - \dot{u}_m(s))_V \, ds .
\end{cases}
$$

Passing to the lim sup as $m \to +\infty$ in (124), by using Lemma 11, Lemma 12, (94), (95) and (123), we obtain

(125)
$$
\begin{cases}
\dfrac{1}{r}\displaystyle\int_t^{t+r} \left(\mathcal{F}u(s), w - \dot{u}(s)\right)_V ds \; + \dfrac{1}{r}\displaystyle\int_t^{t+r} \left(\widetilde{\mathcal{B}}(s), \varepsilon\left(w - \dot{u}(s)\right)\right)_{\mathcal{Q}} ds \\[4mm]
+ \dfrac{1}{r}\displaystyle\int_t^{t+r} \left[\varphi(u(s), w) - \varphi(u(s), \dot{u}(s))\right] ds \\[4mm]
\geq \dfrac{1}{r}\displaystyle\int_t^{t+r} \left(f(s), w - \dot{u}(s)\right)_V ds, \text{ for all } w \in V.
\end{cases}
$$

Since $u_m(t) \to u(t)$ strongly in V, $\forall t \in [0, T]$, it follows from (69) that $u(0) = u_0$. Let $\{\sigma, \zeta\}$ be the unique solution of Problem (102)-(104). Then, by Lebesgue point Lemma for L^1 functions, letting $r \to 0$ in (125) and keeping in mind (47) and (114), we conclude that $\{u, \sigma, \zeta\}$ is a solution of Problem (34)-(38). On the other hand, from (83), we have

$$
\begin{aligned}
\|u(t) - u(s)\|_V &\leq \|u(t) - u_m(t)\|_V + \|u_m(t) - u_m(s)\|_V + \|u_m(s) - u(s)\|_V \\
&\leq \|u(t) - u_m(t)\|_V + c\,|t - s| + \|u_m(s) - u(s)\|_V, \forall t, s \in [0, T].
\end{aligned}
$$

Passing to the limit as $m \to +\infty$ and using the convergence (97), we get

$$
\|u(t) - u(s)\|_V \leq c\,|t - s|, \; \forall t, s \in [0, T].
$$

Thus, u satisfies the regularity (41). Also, from (102), we have

$$
\begin{aligned}
\|\sigma(t) - \sigma(s)\|_{\mathcal{Q}} &\leq c\,\|u(t) - u(s)\|_V + c\left|\int_s^t \left(\|u(r)\|_V + \|\sigma(r)\|_{\mathcal{Q}} + \|\zeta(r)\|_Y + 1\right) dr\right| \\
&\leq c\,|t - s|, \; \forall t, s \in [0, T].
\end{aligned}
$$

Therefore, σ satisfies

(126)
$$
\sigma \in W^{1,\infty}(0, T; \mathcal{Q}).
$$

To continue, taking $w = \dot{u}(t) \pm z$ with $z \in [D(\Omega)]^d$ in (35), we deduce that

(127)
$$
Div\,\sigma(t) = -f_0(t) \text{ in } \Omega, \; \forall t \in [0, T].
$$

Hence, the regularity (42) follows from (126), (127) and (21 (i)). Finally, the regularity (43) follows from Lemma 10, which concludes the proof. $\qquad\square$

5. Conclusion

In this paper, we have studied a quasistatic contact problem with normal compliance condition associated to a version of Coulomb's law of dry friction for viscoplastic materials with damage. We have shown the existence of a weak solution under a smallness assumption depending only on the normal compliance functions, the elasticity operator and on the geometry of the problem. The important question of uniqueness of the solution remains open.

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