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ON SOLVABILITY OF SOME CLASSES OF TRANSMISSION PROBLEMS IN A CYLINDRICAL SPACE DOMAIN

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Abstract. In the article we examine the questions of regular solvability in the Sobolev spaces of the transmission problems with transmission conditions of imperfect contact type for parabolic second order systems in cylindrical space domains. A solution has all generalized derivatives occurring in the system summable to some power $p \in (1, \infty)$. At the interface the limit values of the conormal derivatives are expressed through the limit values of a solution. The problem does not belong to the class of classical diffraction problems and arises when describing heat-and-mass transfer processes in layered media. The proof relies on a priori bounds and the method of continuation in a parameter.

Keywords: transmission problem, discontinuous coefficients, parabolic system, heat-and-mass transfer, cylindrical space domain.

1. Introduction

We consider a second order parabolic system of the form

$$Mu = u_t - Lu = f(x,t), \quad (x,t) \in Q = G \times (0,T), \tag{1}$$

where u is a vector of length h, $Lu = \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} a_i(x,t)u_{x_i} + a_0(x,t)u$, $G \in \mathbb{R}^n$ is a bounded domain with boundary Γ, a_{ij}, a_i, a_0 are $h \times h$-matrix-functions, $h \in \mathbb{N}$. We assume that G is divided into two open subsets G^+ and G^- not necessarily connected such that $\overline{G^+} \cup \overline{G^-} = \overline{G}, G^+ \cap G^- = \emptyset$. Denote

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$\Gamma_0 = \partial G^+ \cap \partial G^-$, $S_0 = \Gamma_0 \times (0,T)$. The equation (1) is supplemented with the initial-boundary conditions

$$Ru|_S = \varphi \quad (S = \Gamma \times (0,T)), \quad u|_{t=0} = u_0(x), \tag{2}$$

where either $Ru = u$ or $Ru = \sum_{i,j=1}^n a_{ij}\nu_i \frac{\partial u}{\partial x_j} + \sigma(x,t)u$, with $\vec{\nu}$ is the unit outward normal, and the transmission conditions

$$\frac{\partial u^+}{\partial N}(x,t) - \alpha_1(x,t)u^+(x,t) - \alpha_2(x,t)u^-(x,t) = g^+(x,t), \quad (x,t) \in S_0, \tag{3}$$

$$\frac{\partial u^-}{\partial N}(x,t) - \beta_1(x,t)u^+(x,t) - \beta_2(x,t)u^-(x,t) = g^-(x,t), \quad (x,t) \in S_0, \tag{4}$$

where

$$\frac{\partial u^\pm}{\partial N}(x_0,t) = \lim_{x \in G^\pm,\ x \to x_0 \in \Gamma_0} \sum_{i,j=1}^n a_{ij} u_{x_i} \nu_j, \quad u^\pm = \lim_{x \in G^\pm,\ x \to x_0 \in \Gamma_0} u(x,t),$$

and ν is the unit outward normal to ∂G^-. The problem is to find a solution to the equation (1) satisfying (2)-(4). The transmission conditions (3), (4) on the boundary of two media generalize those known in the heat and mass transfer theory in the case of an imperfect contact (see settings in [1]). They can be written as follows:

$$\frac{\partial u^+}{\partial N}\Big|_{S_0} = \frac{\partial u^-}{\partial N}\Big|_{S_0}, \quad \frac{\partial u^+}{\partial N}\Big|_{S_0} = \alpha(u^+ - u^-). \tag{5}$$

If $\alpha \to \infty$ then we arrive at the conventional diffraction problem (see [2, §16, Ch. 3]), when the conditions are of the form $u^+ = u^-$, $\frac{\partial u^+}{\partial N}|_{S_0} = \frac{\partial u^-}{\partial N}|_{S_0}$. First results on generalized solvability and the simplest results about differential properties of solutions to the diffraction problem in the case of $\overline{G^-} \subset G$ for second order parabolic and elliptic equations (the L_2 theory) were obtained by Ladyzhenskaya O.A. and Oleinik O.A. (see [3, 4, 6, 7]) and some other authors in 50-60 years. The general theory of diffraction problems for higher order elliptic equations in the case of $\overline{G^-} \subset G$ can be found in Sheftel's articles [8, 9], where solvability conditions are stated for the cases of the Sobolev and Hölder spaces. We can refer also to the articles [10] and [11] devoted to generalized and classical solvability of the diffraction type problems for higher order elliptic equations and higher order elliptic systems, respectively, in the Sobolev and Hölder spaces. The diffraction type problems in the general setting for higher order parabolic systems were examined in [12] in the Sobolev and Hölder spaces as well. Note the monograph [13, §6.5] which contains some results on solvability of the diffraction problems in the Sobolev-Besov spaces for second order parabolic operators with coefficients independent on time whose solutions take values in a Hilbert space (in particular, the usual systems of second order parabolic equations are included into this class). The problem (1)-(4) is not a diffraction problem in the classical sense and it does not belong the class of problems studied in the above-mentioned articles. Under the condition $\Gamma_0 \cap \partial G = \emptyset$, the generalized solvability of the problem (1)-(4) in the case of the quasilinear operator M written in the divergent form is proven in [14], where actually the theory of monotone operators is employed and the regular solvability of this problem is studied in [15]. The results presented in the latter article are quite similar to those of the present article but the case $\Gamma_0 \cap \partial G \neq \emptyset$ which is studied here is much more difficult. This case is not studied even in case of the classical diffraction problems due to difficulties connected with regular solvability of parabolic problems in Lipschitz domains. Some results connected with the generalized solvability of the problem (1)-(4) (i. e., a solution u belongs to the class $L_2(0,T;W_2^1(G))$) with the conditions of the form (5) in the linear case are obtained in [16] (see also [17]).

In the present article we consider a particular case of the cylinder $G=\Omega\times(0,l)$ ($\Omega\subset\mathbb{R}^{n-1}$), $G^+=\cup_i G^{2i}$, $G^-=\cup_i G^{2i-1}$, $G^i=\Omega\times(l_{i-1},l_i)$, $l_0=0<l_1<\ldots,<l_m=l$ as a space domain. Our problem in this or close statement arises in applications. In particular, we want to use the results obtained in the study of solvability questions for inverse problems of recovering the heat transfer coefficients (the functions α_i,β_i in (3), (4)). General statements of these problems can be found in [18] for $n=2$ and in [19, §3-7,3-8] for $n=1$. Some model statements are exposed in [20]-[27] and some other articles. Almost all articles deal with numerical solving the problems, in very simple situations a solution is constructed explicitly.

In contrast to [16, 14], we provide some results on regular solvability of the problem (1)-(4) in the Sobolev space $W_p^{1,2}(Q)$. We rely on the theorem on regular solvability of boundary value problems in cylindrical domains. Unfortunately, we do not find sharp results (there are solvability theorems in weighted classes) and, hence, its proof is also presented. The foundations of the solvability theory for elliptic and parabolic equations and systems in nonsmooth domains were laid in Kondrat'ev's articles and some other authors. We can refer to the survey [37], where a relevant bibliography can be found.

2. Preliminaries

Let E be a Banach space. By $L_p(G;E)$ (G is a domain $\mathbb{R}^n$) we mean the space of strongly measurable functions defined on G with values in E and the finite norm $\|\|u(x)\|_E\|_{L_p(G)}$ [29]. We also employ the spaces $C^k(\overline{G})$, comprising functions having in G all derivatives up to and including the order k continuous in G and admitting continuous extension to $\overline{G}$. The notation of the Sobolev spaces $W_p^s(G;E)$ and $W_p^s(Q;E)$ are conventional (see [30, 29]). If $E=\mathbb{C}$ or $E=\mathbb{C}^n$ then the latter space is denoted simply by $W_p^s(Q)$. Similarly, we use the notations $W_p^s(G)$ or $C^k(\overline{G})$ rather than $W_p^s(G;E)$ or $C^k(\overline{G};E)$, respectively. Thus, the inclusion $u\in W_p^s(G)$ (or $u\in C^k(\overline{G})$) for a given vector-function $u=(u_1,u_2,\ldots,u_k)$ means that every of the component u_i belongs to $W_p^s(G)$ (or $C^k(\overline{G})$). In this case, the norm of a vector is the sum of the norms of its coordinates. Given an interval $J=(0,T)$, assign $W_p^{s,r}(Q)=W_p^s(J;L_p(G))\cap L_p(J;W_p^r(G))$. Assume that Γ is a smooth surface of dimension $n-1$ and $S=(0,T)\times\Gamma$. In this case, $W_p^{s,r}(S)=W_p^s(J;L_p(\Gamma))\cap L_p(J;W_p^r(\Gamma))$. The definitions of the Hölder spaces $C^{\alpha,\beta}(\overline{Q})$ and $C^{\alpha,\beta}(\overline{S})$ can be found, for instance, in [2].

In what follows, we assume that $\partial\Omega\in C^2$ (see the definition, for instance, in [2, p. 9]). Denote $(u,v)=\int_G u(x)\overline{v(x)}dx$ if u and v are scalar functions and $(u,v)=\int_G\langle u(x,t),v(x)\rangle\,dx$ if u and v are vectors of length h. Here the symbol $\langle\cdot,\cdot\rangle$ stands for the inner product in $\mathbb{C}^h$. Introduce the following notations: $\Gamma=\partial G$, $S_{\alpha,\beta}=(\alpha,\beta)\times\Gamma$, $Q_\phi=(0,\phi)\times G$, $Q_{\alpha,\beta}=(\alpha,\beta)\times G$, $Q^0=(0,T)\times\Omega$, $\Gamma_0=\partial\Omega\times(0,l)$, $S_0=(0,T)\times\Gamma_0$, $Q_\phi^0=(0,\phi)\times\Omega$, $S_0^\phi=(0,\phi)\times\Gamma_0$, $G^i=\Omega\times(l_{i-1},l_i)$, $Q^i=(0,T)\times G^i$, $Q_\phi^i=(0,\phi)\times G^i$, $\Gamma_i=\partial\Omega\times(l_{i-1},l_i)$, $S_i^\phi=(0,\phi)\times\Gamma_i$, $S_i=(0,T)\times\Gamma_i$, where $i=1,2,...,m$.

We endow the space $W_p^s(\alpha,\beta;E)$ ($s\in(0,1)$, E is a Banach space) with the norm $\|q(t)\|_{W_p^s(\alpha,\beta;E)}=(\|q\|^p_{L_p(\alpha,\beta;E)}+<q>^p_{s,p})^{1/p}$, $<q>^p_{s,p}=\int_\alpha^\beta\int_\alpha^\beta\frac{\|q(t_1)-q(t_2)\|^p_E}{|t_1-t_2|^{1+sp}}dt_1dt_2$. If $E=\mathbb{C}$, we obtain the usual space $W_p^s(\alpha,\beta)$. Given $s\in(0,1)$, assign $\tilde W_p^s(\alpha,\beta;E)=\{q\in W_p^s(\alpha,\beta;E):\ (t-\alpha)^{-s}q(t)\in L_p(\alpha,\beta;E)\}$. This is a Banach space with the norm $\|q(t)\|^p_{\tilde W_p^s(\alpha,\beta;E)}=\|\frac{q}{(t-\alpha)^s}\|^p_{L_p(\alpha,\beta;E)}+<q>^p_{s,p}$. If $s>1/p$ then all functions q in this space possess the property $q(\alpha)=0$ and, for $s\neq 1/p$, this norm and the conventional norm $\|\cdot\|_{W_p^s(\alpha,\beta;E)}$ are equivalent for functions $q(t)$ such that $q(\alpha)=0$ for $s>1/p$. Equivalence results from Lemma 1 of Subsect. 3.2.6 [29], for example. The spaces $\tilde W_p^s(\alpha,\beta;L_p(G))$, $\tilde W_p^{s,2s}(Q_{\alpha,\beta})=\tilde W_p^s(\alpha,\beta;L_p(G))\cap L_p(\alpha,\beta;W_p^{2s}(G))$ for $s\neq 1/p$ consist of functions $v(t,x)$ in $W_p^s(\alpha,\beta;L_p(G))$ and $W_p^{s,2s}(Q_{\alpha,\beta})$, respectively, such that $v(\alpha,x)=0$ whenever $s>1/p$. The norms $\|\cdot\|_{\tilde W_p^{s,2s}(Q_{\alpha,\beta})}$, $\|\cdot\|_{\tilde W_p^s(\alpha,\beta;L_p(G))}$ are defined conventionally with the use of the above norm in $\tilde W_p^s(\alpha,\beta;E)$. We have

$$(6)\quad \|u\|_{\tilde W_p^{s,2s}(Q_{\alpha,\beta})}=\Big(\int_G\int_\alpha^\beta\frac{|u(x,t)|^p}{(t-\alpha)^{sp}}\,dtdx+\int_G\int_\alpha^\beta\int_\alpha^\beta\frac{|u(x,t)-u(x,\tau)|^p}{|t-\tau|^{1+sp}}\,dtd\tau dx+\|u\|^p_{L_p(\alpha,\beta;W_p^{2s}(G))}\Big)^{1/p}.$$

The spaces $\tilde W_p^s(\alpha,\beta;L_p(\Gamma))$, $\tilde W_p^{s,2s}(S_{\alpha,\beta})$ are defined similarly. In the latter case we need some explanations. We say that a function $v\in L_p(S_{\alpha,\beta})$ belongs to the class $\tilde W_p^{s,2s}(S_{\alpha,\beta})$ if $v|_{S_0}\in\tilde W_p^{s,2s}(S_0)$ and $v(t,x',0),v(t,x',l)\in\tilde W_p^{s,2s}(Q^0)$. By the norm in the space $\tilde W_p^{s,2s}(S_{\alpha,\beta})$ we mean the corresponding sum of the norms.

The next lemma is actually known (see, for instance, Lemma 1 in [31], where $s\in(1/p,1)$, or Prop. 7.1, where $p=2$, in [28]). Hence, we omit its proof which relies on the definitions of the norms in the corresponding spaces and some simple inequalities. In this lemma, we assume that either $S_\phi=S_0^\phi$ and $S_{-1,\phi}=(-1,\phi)\times\Gamma_0$ or $S_\phi=Q_\phi^0$ and $S_{-1,\phi}=(-1,\phi)\times\Omega$.

Lemma 1. *The exist constants C_1 and C_2 independent of $\phi\in(0,T]$ such that*

$$(7)\qquad C_1\|v\|_{\tilde W_p^{s,2s}(S_\phi)}\le\|v\|_{W_p^{s,2s}(S_{-1,\phi})}\le C_2\|v\|_{\tilde W_p^{s,2s}(S_\phi)},\ s\in(0,1),$$

for all $v\in W_p^{s,2s}(S_{-1,\phi})$ such $v=0$ for $t<0$.

Put $s_0=1/2-1/2p$ and $s_1=1-1/2p$.

Lemma 2. *There exists a constant C independent of $\phi\in(0,T]$ such that*

$$(8)\quad \|v\|_{\tilde W_p^{s_1,2s_1}(S_{0,\phi})}\le C\|v\|_{W_p^{1,2}(Q_\phi)},\quad p\neq 3/2,$$
$$\|\frac{\partial v}{\partial\nu}\|_{\tilde W_p^{s_0,2s_0}(S_{0,\phi})}\le C\|v\|_{W_p^{1,2}(Q_\phi)},\ p\neq 3,$$

for all $v\in W_p^{1,2}(Q_\phi)$ such that $v(0,x)=0$. Here the symbol $\frac{\partial v}{\partial\nu}$ designates the derivative with respect to the outward normal to $S_{0,\phi}$.

Proof. This lemma is a simple consequence of Lemma 1 and the trace theorems (see, for instance, Theorem 7.3 in [32], or Lemma 3.4 §2, Ch. 2 in [2]). □

Next, we present an auxiliary result. Consider the problem

$$Mu = u_t - Lu = f, \quad Lu = a_{nn}u_{x_nx_n} + \sum_{i,j=1}^{n-1} a_{ij}u_{x_ix_j} - \sum_{i,j=1}^{n} a_iu_{x_i} - a_0u, \tag{9}$$

where $x \in G = \Omega \times (0,l) \subset \mathbb{R}^n$, Ω is a bounded domain with the boundary of class C^2, a_{ij}, a_i, a_0 are $h \times h$-matrix-functions, $a_{ij}(t,x) = a_{ji}(t,x)$ for $(t,x) \in Q = (0,T) \times G$ and all i,j,

$$Ru|_{S_0} = \varphi, R_iu(x', r_i, t) = \varphi_i(x',t),\ i = 0,1, r_0 = 0, r_1 = l,\ u(x,0) = u_0(x), \tag{10}$$

where $Ru = u$ or $Ru = \sum_{i,j=1}^{n-1} a_{ij}(x,t)\nu_i \frac{\partial u}{\partial x_j} + \sigma(x,t)u$; $R_0u = u$ or $R_0u = -u_{x_n}$, $R_1u = u$ or $R_1u = u_{x_n}$. Fix $p \in (1,\infty)$ and assume that

$$a_i \in L_q(Q), \quad a_0 \in L_r(Q),\ \sigma \in C^{s_0+\varepsilon_0, 2s_0+2\varepsilon_0}(\overline{S_0}),\ i = 1,\dots,n, \tag{11}$$

$$a_{ij} \in C^{s_0+\varepsilon_0, 2s_0+2\varepsilon_0}(\overline{Q}), \quad i,j = 1,\dots,n, \tag{12}$$

where $q > n+2$ for $p \le n+2$ and $q = p$ for $p > n+2$, $r > (n+2)/2$ for $p \le (n+2)/2$ and $r = p$ for $p > (n+2)/2$, $\varepsilon_0 \in (0,1/2)$ is an arbitrarily small positive parameter.

Lemma 3. *Assume that $u \in W_p^{1,2}(Q_\tau)$ and $u(0,x) = 0$. If $b \in L_q(Q_\tau)$ c $q > (n+2)/2$ for $p \le (n+2)/2$ and $q = p$ for $p > (n+2)/2$ then there exists a constant $c > 0$ such that*

$$\|bu\|_{L_p(Q_\tau)} \le c\tau^{\beta_1}\|u\|_{W_p^{2,1}(Q_\tau)},$$

where $\beta_1 = 1 - \frac{n+2}{2q}$ for $q > (n+2)/2$ and $\beta_1 < 1 - \frac{n+2}{2p}$ for $q = p > (n+2)/2$. If $b \in L_r(Q^\tau)$ with $r > n+2$ for $p \le n+2$ and $r = p$ for $p > n+2$ then

$$\|b\nabla u\|_{L_p(Q_\tau)} \le c\tau^{\beta_2}\|u\|_{W_p^{2,1}(Q_\tau)},$$

where $\beta_2 = 1/2 - \frac{(n+2)}{2q}$ for $r > n+2$ and $\beta_2 < 1/2 - \frac{(n+2)}{2p}$ for $r = p > n+2$. The constant $c > 0$ is independent of $\tau \le T$ and $u \in W_p^{2,1}(Q_\tau)$.

The proof is based on the Hölder inequality and the inequalities of Lemma 3.3 in Ch. 2 of [2], where we take $\delta = \sqrt{\tau}$ and use the estimate $\|u\|_{L_p(Q_\tau)} \le c\tau\|u_t\|_{L_p(Q_\tau)}$. Note that the inequalities of this type and even more general are proven and used in Ch. 5 of [33], moreover, this lemma is actually obtained and used in the proof of Theorem 9.1, Ch.4 in [2].

Lemma 4. *Assume that the conditions* (11), (12) *are fulfilled and*

$$\sigma_0 \in C^{s_0+\varepsilon_0, 2s_0+2\varepsilon_0}(\overline{Q_\phi^0}) \quad (\varepsilon_0 > 0).$$

Then there exist constants C_0, C_1, C_2 independent of $\phi \in (0,T]$ such that

$$\|\sigma_0 v\|_{\tilde{W}_p^{s_0,2s_0}(Q_\phi^0)} \le C_0\|v\|_{\tilde{W}_p^{s_0,2s_0}(Q_\phi^0)},\ s_0 = 1/2 - 1/2p,$$

for all $v \in \tilde{W}_p^{s_0,2s_0}(Q_\phi^0)$,

$$\|Rv\|_{\tilde{W}_p^{s_0,2s_0}(S_0^\varphi)} \le C_1\|v\|_{W_p^{1,2}(Q_\phi)},\ \|L_0v\|_{L_p(Q_\phi)} \le C_2\|v\|_{W_p^{1,2}(Q_\phi)} \tag{13}$$

for all $v \in W_p^{1,2}(Q_\phi)$ such that $v(x,0) = 0$.

The proof of the first inequality relies on the definition of the norm and theorems on pointwise multipliers (see, for instance, [34, Theorem 3.3.2, p.198]). The proof of the second inequality additionally employs Lemma 1. The last estimate (13) (and even more general estimates) is obtained in the proof of Theorem 5.4 in [33]. It also follows from the Lemma 3.

Next we describe some conditions ensuring parabolicity of the problem and the Lopatinskii conditions. Let $A_0(t,x,\xi)=a_{nn}\xi_n^2+\sum_{i,j=1}^{n-1}a_{ij}(t,x)\xi_i\xi_j$ $(\xi\in\mathbb{R}^n)$.

The strong ellipticity condition (see [2, Definition 7, §8, Ch.7]**)**. There exists a constant $\delta_0>0$ such that

$$Re\,\langle A_0(t,x,\xi)\eta,\eta\rangle\geq\delta_0|\xi|^2|\eta|^2\quad\forall\xi\in\mathbb{R}^n,\ \eta\in\mathbb{C}^h. \tag{14}$$

The normal strong ellipticity condition. The condition (14) holds and there exists a constant $\delta_1>0$ such that

$$Re\sum_{i,j=1}^{n}\langle a_{ij}(t,x)(\xi_iu+\eta_iv),\xi_ju+\eta_jv\rangle\geq\delta_1|Im\langle\ u,v\rangle|\ \forall(t,x)\in Q, \tag{15}$$

for all $u,v\in\mathbb{C}^h,\xi,\eta\in\mathbb{R}^n$ such that $\langle\xi,\eta\rangle=0,|\xi|=|\eta|=1$.

The Legengre condition. There exists a constant $\delta_3>0$ such that

$$Re\sum_{i,j=1}^{n}\sum_{k,r=1}^{h}a_{ij}^{k,r}(t,x)\xi_i^k,\overline{\xi}_j^r\geq\delta_3\sum_{i,k}|\xi_i^k|^2,\quad\forall(t,x)\in Q \tag{16}$$

and all symmetric matrices with entries $\{\xi_i^k\}$. Here $a_{ij}^{k,r}$ are the entries of the matrix a_{ij}.

The strong Legendre condition. There exists a constant $\delta_2>0$ such that

$$Re\sum_{i,j=1}^{n}\langle a_{ij}(t,x)\xi_i,\xi_j\rangle\geq\delta_2\sum_{i=1}^{n}|\xi_i|^2,\quad\forall\xi_i\in\mathbb{C}^h,\ \forall(t,x)\in Q. \tag{17}$$

Lemma 5. *The following implications are valid:* (17) $\Rightarrow$ (16)$\Rightarrow$ (15) $\Rightarrow$ (14)*. The parabolicity condition and the Lopatinskii condition on S_0 in the case of $Ru=u$ follows from* (14) *and in the case of $Ru=\sum_{i,j=1}^{n-1}a_{ij}(t,x)\nu_i\frac{\partial u}{\partial x_j}+\sigma(t,x)u$ from* (15).

All statements of the lemma can be found in Subsec. 2.5, §6.2, Ch. 6 in [13]) and in Subsec. 4.3 in [35]. In the case of the Dirichlet condition the fulfilment of the parabolicity condition and the Lopatiskii condition is stated in [3].

Next, we assume that

$$u_0(x)\in W_p^{2-2/p}(G),\quad\varphi\in W_p^{k_0,2k_0}(S_0),\qquad \varphi_0\in W_p^{k_1,2k_1}(Q^0),\ \varphi_1\in W_p^{k_2,2k_2}(Q^0), \tag{18}$$

where $k_i=s_1$ in the case of the Dirichlet condition on Γ_0 (or on $\Omega_0=\{(t,x',0):(t,x')\in Q^0\}$, or on $\Omega_m=\{(t,x',l):(t,x')\in Q^0\}$, respectively) and $k_i=s_0$ otherwise.

We need the agreement conditions ensuring existence of a function $\Phi\in W_p^{1,2}(Q)$ satisfying (10). The cases $p=3/2$, $p=3$ are critical if we treat the agreement conditions at $t=0$. However, the most natural case $p=2$ is critical as well and we need some additional integral agreement conditions. Assume that $\Phi\in W_2^{1,2}(Q)$ satisfies (10) and, for example, $R_0u=u$ and $Ru\neq u$. By Theorem

7.3 in [32], if $p > 2$ and $\Phi \in W_p^{1,2}(Q)$ then $R\varphi_0|_{\partial\Omega} = \varphi(t, x', 0)$; if $p = 2$ then this condition is replaced with an integral agreement condition. In this case we have $R\Phi|_{S_0} \in L_2(0, T; W_2^{1/2}(\Gamma_0))$, $R_0\Phi(t, x', 0) \in L_2(0, T; W_2^{3/2}(\Omega))$. Expose the complete collection of the agreement conditions. Given $\varphi(t, x', x_n)$, put

$$S_{\alpha,\beta}(\varphi) = \int_\alpha^\beta \int_0^\delta \int_{\partial\Omega} |\varphi(t, x', \tau)|^2 d\Gamma \frac{d\tau}{\tau} dt,$$

where $0 \le \alpha < \beta \le T$. Let $n(x') = (n_1, \dots, n_{n-1})$ be the unit interior normal to $\partial\Omega$ at x'. Assume that $\delta \in (0, \delta_0)$, with $\delta_0 \in (0, l)$ a constant such that $x' + \tau n(x') \in \Omega$ for all $x' \in \partial\Omega$ and $\tau \le \delta_0$. In what follows, we fix δ_0. Assign $J_{\alpha,\beta}(\varphi, \varphi_0) = S_{\alpha,\beta}(\varphi(t, x', \tau) - R(t, x', 0)\varphi_0(t, x' + \tau n(x')))$, $J^{\alpha,\beta}(\varphi, \varphi_1) = S_{\alpha,\beta}(\varphi(t, x', l - \tau) - R(t, x', l)\varphi_1(t, x' + \tau n(x')))$, $I_{\alpha,\beta}(\varphi, \varphi_0) = S_{\alpha,\beta}(\varphi_\tau(t, x', \tau) + \varphi_0(t, x' + \tau n(x')))$, $I^{\alpha,\beta}(\varphi, \varphi_1) = S_{\alpha,\beta}(\varphi_\tau(t, x', l - \tau) - \varphi_1(t, x' + \tau n(x')))$. Here the symbol $R(t, x', 0)$ ($R(t, x', l)$) stands for the operator R whose coefficients are taken at the point $x_n = 0$ ($x_n = l$).

The agreement conditions.

A) if $R_i u = u$ $(i = 0, 1)$ and $Ru = u$ then $\varphi_i(x', t)|_{\partial\Omega} = \varphi(x', r_i, t)$; if $p > 2$ and $R_i u = u$ $(i = 0, 1)$ and $Ru \ne u$ then $R(x', r_i, t)\varphi_i(x', t)|_{\partial\Omega} = \varphi(x', r_i, t)$; if $p > 2$, $R_i u = (-1)^{i+1} u_{x_n}$ $(i = 0, 1)$ and $Ru = u$ then $(-1)^{i+1}\varphi_{x_n}(x', r_i, t) = \varphi_i(x', t)|_{\partial\Omega}$; if $R_i u = u$ $(i = 0, 1)$ then $u_0(x', r_i)|_\Omega = \varphi_i(x', 0)$ for $p > 3/2$; if $R_i u = (-1)^{i+1} u_{x_n}$ $(i = 0, 1)$ then $(-1)^{i+1} u_{0x_n}(x', r_i)|_\Omega = \varphi_i(x', 0)$ for $p > 3$; if $Ru = u$ then $u_0(x)|_{\Gamma_0} = \varphi(0, x)$ for $p > 3/2$; if $Ru \ne u$ then $R(0, x)u_0(x)|_{\Gamma_0} = \varphi(0, x)$ for $p > 3$; if $p = 2$ and if $Ru \ne u$ and $R_0 u = u$ or $Ru \ne u$ and $R_1 u = u$ then there exists a constant $\delta \in (0, \delta_0)$ such that $J_{0,T}(\varphi, \varphi_0) < \infty$ or, respectively, $J^{0,T}(\varphi, \varphi_1) < \infty$; if $p = 2$ and $R_0 u \ne u$ and $Ru = u$ or $R_1 u \ne u$ and $Ru = u$ then there exists a constant $\delta \in (0, \delta_0)$ such that $I_{0,T}(\varphi, \varphi_0) < \infty$ or, respectively, $I^{0,T}(\varphi, \varphi_1) < \infty$. All agreement conditions except for those integral results from the conventional trace theorems (see Lemma 3.4 and Theorem 2.3 of Ch. 1 in [2] and Lemma 2). Note that, for the data satisfying (18), the fulfillment of an integral agreement condition for some parameter $\delta \in (0, \delta_0)$ implies its fulfillment for all such parameters δ.

Lemma 6. *Assume that the conditions* (11), (18), *and* (12) *in the case of* $Ru \ne u$ *hold and there exists a function* $\Phi \in W_2^{1,2}(Q)$ *satisfying* (10). *If* $Ru \ne u$ *and* $R_0 u = u$ *or* $R_1 u = u$ *then there exists a constant* $\delta \in (0, \delta_0)$ *such that* $J_{0,T}(\varphi, \varphi_0) < \infty$ *or, respectively,* $J^{0,T}(\varphi, \varphi_1) < \infty$; *if* $R_0 u \ne u$ *and* $Ru = u$ *or* $R_1 u \ne u$ *and* $Ru = u$ *then there exists* $\delta \in (0, \delta_0)$ *such that* $I_{0,T}(\varphi, \varphi_0) < \infty$ *or, respectively,* $I^{0,T}(\varphi, \varphi_1) < \infty$.

Proof. First, we note that the second part of the claim follows from Theorem 7.3 in [32]. Moreover, its proof is quite similar to that for the first part. So we omit it. Demonstrate the former half of the claim. Consider, for example, the expression

$$(19) \quad S_{0,T}(\varphi(t, x', \tau) - R(t, x', 0)\varphi_0(t, x' + \tau n(x'))) =$$

$$\int_0^T \int_0^\delta \int_{\partial\Omega} |R(t, x', \tau)\Phi(t, x', \tau) - R(t, x', 0)\Phi(t, x' + \tau n(x'), 0)|^2 d\Gamma \frac{d\tau}{\tau} dt \le$$

$$2\int_0^T \int_0^\delta \int_{\partial\Omega} |R(t, x', 0)\Phi(t, x', \tau) - R(t, x', 0)\Phi(t, x' + \tau n(x'), 0))|^2 d\Gamma \frac{d\tau}{\tau} dt +$$

$$2\int\limits_0^T\int\limits_0^\delta\int\limits_{\partial\Omega}|(R(t,x',\tau)-R(t,x',0))\Phi(t,x',\tau)|^2 d\Gamma\frac{d\tau}{\tau}dt \le$$

$$\int\limits_0^T\int\limits_0^\delta\int\limits_{\partial\Omega} c_1(\sum_{i=1}^{n-1}|\Phi_{x_i}(t,x',\tau)-\Phi_{x_i}(t,x'+\tau n(x'),0)|^2+|\Phi(t,x',\tau)-$$

$$\Phi(t,x'+\tau n(x'),0)|^2)+c_2\tau^{2\varepsilon_0}(\sum_{i=1}^{n-1}|\Phi_{x_i}(t,x',\tau)|^2+|\Phi(t,x',\tau)|^2)d\Gamma\frac{d\tau}{\tau}dt$$

(we use the condition that all coefficients of R meet the Hölder condition in the variable x with the exponent $2\varepsilon_0$). To estimate the last summand, we use the inequality

$$\int_0^\delta \tau^{2\varepsilon_0}(\sum_{i=1}^{n-1}|\Phi_{x_i}(t,x',\tau)|^2+|\Phi(t,x',\tau)|^2)\,\frac{d\tau}{\tau}\le$$
$$c(\sum_{i=1}^{n-1}\|\Phi_{x_i}\|^2_{W_2^{1/2-\varepsilon_0}(0,\delta)}+\|\Phi\|^2_{W_2^{1/2-\varepsilon_0}(0,\delta)})$$

and Theorem 7.3 in [32]. We obtain

$$(20)\quad \int_0^T\int_0^\delta\int_{\partial\Omega}\tau^{2\varepsilon_0}(\sum_{i=1}^{n-1}|\Phi_{x_i}(t,x',\tau)|^2+|\Phi(t,x',\tau)|^2)d\Gamma\frac{d\tau}{\tau}dt\le$$
$$c_3\|\Phi\|^2_{L_2(0,T;W_2^2(\Omega\times(0,\delta)))}.$$

Estimate the first summand under the integral sign. We have

$$\Phi_{x_i}(t,x',\tau)-\Phi_{x_i}(t,x'+\tau n(x'),0)=\int_0^\tau \partial_\xi\Phi_{x_i}(t,x'+(\tau-\xi)n(x'),\xi)\,d\xi=$$
$$\int_0^\tau -\sum_{j=1}^{n-1}n_j\Phi_{x_ix_j}(t,x'+(\tau-\xi)n(x'),\xi)+\Phi_{x_ix_n}(t,x'+(\tau-\xi)n(x'),\xi)\,d\xi.$$

As a result, we obtain the inequality

$$|\Phi_{x_i}(t,x',\tau)-\Phi_{x_i}(t,x'+\tau n(x'),0)|^2\le$$
$$\tau c\int_0^\tau\sum_{j=1}^n|\Phi_{x_ix_j}(t,x'+(\tau-\xi)n(x'),\xi)|^2\,d\xi.$$

Similarly, we derive that

$$|\Phi(t,x',\tau)-\Phi(t,x'+\tau n(x'),0)|^2\le\tau c\int_0^\tau\sum\nolimits_{j=1}^n|\Phi_{x_j}(t,x'+(\tau-\xi)n(x'),\xi)|^2\,d\xi.$$

The last two inequalities imply that

$$(21)\quad (\sum_{i=1}^{n-1}|\Phi_{x_i}(t,x',\tau)-\Phi_{x_i}(t,x'+\tau n(x'),0)|^2+|\Phi(t,x',\tau)- \Phi(t,x'+\tau n(x'),0)|^2)\leq c\tau\int_0^{\tau}\sum_{j=1}^{n}|\Phi_{x_j}(t,x'+(\tau-\xi)n(x'),\xi)|^2+ \sum_{i,j=1}^{n}|\Phi_{x_ix_j}(t,x'+(\tau-\xi)n(x'),\xi)|^2\,d\xi.$$

Denote the integrand by $J(t,x'+(\tau-\xi)n(x'),\xi)$. Let $x_0'\in\partial\Omega$. There exists a neighborhood U_0 and the coordinate system y obtained with the help of a rotation and a translation of the origin such that $U_0\cap\Omega=\{y:\ \omega(y'')<y_{n-1}<\omega(y'')+\delta,\ |y''|\leq r_0\}$, $U_0\cap\partial\Omega=\{y:\ \ y_{n-1}=\omega(y''),\ |y''|\leq r_0\}$ $(\delta,r_0>0,\ y''=(y_1,y_2,\dots,y_{n-2}))$; without loss of generality, we can assume that the axis y_{n-1} is directed as the normal to $\partial\Omega$ at x_0'. In this case the normal to $\partial\Omega$ in the coordinate system y has the coordinates $n_{y'}=\frac{1}{1+|\nabla\omega|^2}(-\omega_{y_1},\dots,-\omega_{y_{n-2}},1)$ and $n(x'(y'))=x_0'+An_{y'}$, where A is an orthogonal matrix. The boundary $\partial\Omega$ can be covered by finitely many sets of the form $U_0\cap\Omega$. In order to estimate the integral (19), in view of (20) it suffices to establish the estimate for integrals of the form (we employ (21) and the parametrization $x'=x_0'+Ay'$, $y'=(y'',\omega(y''))$ of the surface $U_0\cap\partial\Omega$)

$$(22)\quad \int_0^T\int_0^{\delta}\int_{U_0\cap\partial\Omega}\sum_{i=1}^{n-1}|\Phi_{x_i}(t,x',\tau)-\Phi_{x_i}(t,x'+\tau n(x'),0)|^2+|\Phi(t,x',\tau)- \Phi(t,x'+\tau n(x'),0)|^2)d\Gamma\frac{d\tau}{\tau}dt=\int_0^T\int_0^{\delta}\int_{|y''|<r_0}(\sum_{i=1}^{n-1}|\Phi_{x_i}(t,x'(y'),\tau)- \Phi_{x_i}(t,x'(y')+\tau n(x'(y')),0)|^2+|\Phi(t,x'(y'),\tau)-\Phi(t,x'(y')+ \tau n(x'(y')),0)|^2)\sqrt{1+|\nabla_{y''}\omega(y'')|^2}dy''\frac{d\tau}{\tau}dt\leq c_1\int_0^T\int_0^{\delta}\int_{|y''|<r_0}\int_0^{\tau}J(t,x'(y')+(\tau-\eta)n(x'(y')),\eta)dy''d\eta d\tau dt= c_1\int_0^T\int_0^{\delta}\int_{|y''|<r_0}\int_{\eta}^{\delta}J(t,x'(y')+(\tau-\eta)n(x'(y')),\eta)dy''d\tau d\eta dt.$$

In the first two integrals we make the change of variables $(y'',\tau)\to z'=x'(y')+(\tau-\eta)n(x'(y'))=x_0'+A\tilde{y'}$, $\tilde{y'}=y'+(\tau-\eta)n_{y'}$. The image of the domain $\{y'':\ |y''|<r_0\}\times(0,\delta)$ under this map $z'=z'(y'',\tau)$ is contained in $\Omega_\delta=\{x'\in\Omega:\ \ \rho(x',\partial\Omega\cap U_0)<\delta\}$ ($\rho(x,M)$ is the distance from a point x to the set M) for every $\eta\in(0,\delta)$. Moreover, the Jacobian of the transformation $\frac{\partial(z_1,\dots,z_{n-1})}{\partial(y'',\tau)}=\pm\frac{\partial(\tilde{y}_1,\dots,\tilde{y}_{n-1})}{\partial(y'',\tau)}$ is separated from zero and this mapping is one-to-one if the parameter δ is sufficiently small (as is easily seen). Without loss of generality we assume this condition to be fulfilled. In this case the right-hand side in (22) is estimated from above by the quantity

$$c_2 \int_0^T \int_0^\delta \int_{\Omega_\delta} J(t,z',\eta)dz'd\eta dt \le c_3 \|\Phi\|^2_{L_2(0,T;W_2^2(\Omega\times(0,\delta)))}.$$

This estimate and the estimate (20) validate the inequality

$$S_{0,T}(\varphi(t,x',\tau) - R(t,x',0)\varphi_0(t,x'+\tau n(x'))) \le c_4\|\Phi\|^2_{L_2(0,T;W_2^2(\Omega\times(0,\delta)))} < \infty,$$

if δ is sufficiently small, where the constant c_4 is independent of the functions φ, φ_0, Φ. The same arguments are used in estimating the integral $J^{0,T}(\varphi,\varphi_1)$. □

Lemma 7. *Assume that the conditions* (11), (18), A), *and* (12) *in the case of $Ru \neq u$ hold. Then there exists a function $\Phi \in W_p^{1,2}(Q)$ satisfying the initial condition, the second and third boundary condition in* (10), *and such that, for $p = 2$, if $R_0u = u$ and $Ru \neq u$ or $R_1u = u$ and $Ru \neq u$ then for some $\delta \in (0,\delta_0)$ we have*

$$J_{0,T}(\varphi - R\Phi, 0) < \infty, \quad J^{0,T}(\varphi - R\Phi, 0) < \infty; \tag{23}$$

if $R_0u = -u_{x_n}$ and $Ru = u$, or $R_1u = u_{x_n}$ and $Ru = u$ then, for some $\delta \in (0,\delta_0)$, respectively,

$$I_{0,T}(\varphi - \Phi, 0) < \infty, \quad \text{or} \quad I^{0,T}(\varphi - \Phi, 0) < \infty. \tag{24}$$

Proof. First, we construct a function $v_0 \in W_p^{1,2}(Q)$ such that $v_0(x,0) = u_0(x)$. It suffices to extend the function u_0 to $\mathbb{R}^n$ preserving the class as a compactly supported function (the existence of this extension follows from Theorem 4.2.3 in [29]) and to apply Theorem 5.5 in [33] taking a solution to the Cauchy problem $v_{0t} - \Delta v_0 = 0,\ v_0(x,0) = u_0(x)$ as v_0. By Lemmas 2, 3, the inclusions (18) remains valid for the new functions $\varphi^1 = \varphi - Rv_0|_{S_0}$, $\varphi_0^1 = \varphi_0 - R_0v_0|_{\Omega_0^T}$, $\varphi_1^1 = \varphi_1 - R_1v_0|_{\Omega_m^T}$. Next, it is easy to find a function $v^0 \in W_p^{1,2}(Q)$ such that $R_iv^0(x',r_i,t) = \varphi_i^1(x',t)$ $(i = 0,1)$, $v^0(x,0) = 0$. Indeed, construct extensions of the functions $\varphi_i^1(x',t)$ to some domain Ω^0 such that $\overline{\Omega} \subset \Omega^0$ preserving the class. The extensions are denoted by the same symbols. Since $\partial\Omega \in C^2$, we can apply the Hestenes method (see, for example, Lemma 2.9.1 in [29], where this method is described for the half-space, in the general case a partition of unity and local straitening of the boundary is employed (see Subsec. 4.2.2 in [29])). Construct a function $\psi_0(x') \in C_0^\infty(\Omega^0)$ such that $\psi_0 = 1$ for $x' \in \Omega$ and a domain G_0 such that $\partial G_0 \in C^2$, $\{(x',0) : x' \in \Omega^0\} \cup \{(x',l) : x' \in \Omega^0\} \subset \partial G_0$, $G \subset G_0$. Define also the function $\psi_1(x_n) \in C_0^\infty(\mathbb{R})$ such that $\psi_1(x_n) = 1$ on $(0,l/3)$, $\psi_1(x_n) = 0$ for $x_n > l/2$. Take, for example, $R_0u = u$ and $R_1u = u_{x_n}$. Define a function $\tilde{\varphi}_0 = \varphi_0^1\psi_0$ for $x' \in \Omega^0$, $x_n = 0$ and $\tilde{\varphi}_0 = 0$ for $x \in \partial G_0 \setminus \{(x',0) : x' \in \Omega^0\}$ and $\tilde{\varphi}_1 = \varphi_1^1\psi_0$ for $x' \in \Omega^0$, $x_n = l$ и $\tilde{\varphi}_1 = 0$ for $x \in \partial G_0 \setminus \{(x',l) : x' \in \Omega^0\}$. The first function $\tilde{\varphi}_0$ belongs to $W_p^{k_1,2k_1}(\partial G_0 \times (0,T))$ $(k_1 = 1 - 1/2p)$ and the second $\tilde{\varphi}_1$ to the class $W_p^{k_2,2k_2}(\partial G_0 \times (0,T))$ $(k_2 = 1/2 - 1/2p)$. Hence, there exist functions $v_i \in W_p^{1,2}(G_0 \times (0,T))$ such that $v_1|_{\partial G_0\times(0,T)} = \tilde{\varphi}_0$, $\frac{\partial}{\partial n}v_2|_{\partial G_0\times(0,T)} = \tilde{\varphi}_1$, where n is the unit exterior normal to ∂G_0 and $v_i(x,0) = 0$. As these functions, we can take solutions to the parabolic problems $v_{it} - \Delta v_i = 0$, $v_i(x,0) = 0$, $v_1|_{\partial G_0\times(0,T)} = \tilde{\varphi}_0$ and, respectively, $\frac{\partial}{\partial n}v_2|_{\partial G_0\times(0,T)} = \tilde{\varphi}_1$ (see Theorem 10.4 in [2]). The needed function v^0 such that $v^0 \in W_p^{1,2}(Q)$ and $R_iv^0(x',r_i,t) = \varphi_i^1(x',t)$ $(i = 0,1)$ is written as follows: $v^0 = v_1\psi_1 + v_2(1-\psi_1)$. We even have $v^0 \in W_p^{1,2}((0,T)\times G_0)$. Similar arguments are used in the case of all other operators R_0, R_1. The function $\Phi = v_0 + v^0$ is a required function. Consider the case of $p = 2$ and show that the condition (23)-(24) are fulfilled. Assume, for example, that $R_0u = -u_{x_n}$, $Ru = u$, $I_{0,T}(\varphi,\varphi_0) < \infty$, and Φ is the above-constructed function satisfying the initial condition as well as the second and third boundary conditions in (10). Since $\Phi_{x_n}(t,x',0) = -\varphi_0(t,x')$, we infer

$$\int_0^T \int_0^\delta \int_{\partial\Omega} |\varphi_\tau(t,x',\tau) - \Phi_{x_n}(t,x',\tau)|^2 d\Gamma \frac{d\tau}{\tau} dt \leq$$
$$2\int_0^T \int_0^\delta \int_{\partial\Omega} |\varphi_\tau(t,x',\tau) + \varphi_0(t,x'+\tau n(x'))|^2 d\Gamma \frac{d\tau}{\tau} dt +$$
$$2\int_0^T \int_0^\delta \int_{\partial\Omega} |\Phi_{x_n}(t,x',\tau) - \Phi_{x_n}(t,x'+\tau n(x'),0)|^2 d\Gamma \frac{d\tau}{\tau} dt.$$

The last integral is estimated by $c\|\Phi\|^2_{L_2(0,T;W_2^2(\Omega\times(0,\delta)))}$ and the proof coincides with that of the previous lemma. Hence, $I_{0,T}(\varphi - \Phi, 0) < \infty$. The remaining inequalities in the claim are validated by the same scheme. $\square$

Corollary 1. *Assume that the conditions of Lemma 7 holds and Φ is a function constructed in this lemma. Then, without loss of generality, we can state that there exists a constant $c > 0$ such that*

$$\|\Phi\|_{W_p^{1,2}(Q)} \leq c\big[\|\varphi_0\|_{W_p^{k_1,2k_1}(Q^0)} + \|\varphi_1\|_{W_p^{k_2,2k_2}(Q^0)} + \|u_0\|_{W_p^{2-2/p}(G)}\big]. \tag{25}$$

Moreover, there exists a constant $c_1 > 0$ such that

$$J_{0,T}(\varphi - R\Phi, 0) \leq c_1(\|\Phi\|_{L_2(0,T;W_2^2(G))} + J_{0,T}(\varphi,\varphi_0)),$$
$$J^{0,T}(\varphi - R\Phi, 0) \leq c_1(\|\Phi\|_{L_2(0,T;W_2^2(G))} + J^{0,T}(\varphi,\varphi_1)),$$
$$I_{0,T}(\varphi - \Phi, 0) \leq c_1(\|\Phi\|_{L_2(0,T;W_2^2(G))} + I_{0,T}(\varphi,\varphi_0)),$$
$$I^{0,T}(\varphi - \Phi, 0) \leq c_1(\|\Phi\|_{L_2(0,T;W_2^2(G))} + I^{0,T}(\varphi,\varphi_1)).$$

The function Φ is not determined uniquely and the constant c in (25) changes in dependence on the method of its construction. However, as it is easy to see from the proof of Lemma 7, the statement is valid for some fixed method of its construction.

Theorem 1. *Assume that the conditions* (11), A), (18) *hold and $f \in L_p(Q)$ ($p \neq 3/2, 3$). In the cases $Ru = u$ and $Ru \neq u$, we also assume that the condition* (14) *and* (15), (12), *respectively, take place. Then there exists a unique solution $u \in W_p^{1,2}(Q)$ to the problem* (9), (10). *There is the estimate*

$$\|u\|_{W_p^{1,2}(Q)} \leq \tag{26}$$
$$c\big[\|f\|_{L_p(Q)} + \|\varphi\|_{W_p^{k_0,2k_0}(S_0)} + \|\varphi_0\|_{W_p^{k_1,2k_1}(Q^0)} + \|\varphi_1\|_{W_p^{k_2,2k_2}(Q^0)} + \|u_0\|_{W_p^{2-2/p}(G)}\big],$$

where, in the case of $p = 2$, in dependence on the form of the boundary conditions one or two summands of the form

$$c_1(J_{0,T}(\varphi,\varphi_0))^{1/2}, \quad c_1(J^{0,T}(\varphi,\varphi_1))^{1/2}, \quad c_1(I_{0,T}(\varphi,\varphi_0))^{1/2}, \quad c_1(I^{0,T}(\varphi,\varphi_1))^{1/2},$$

where c_1 is a constant, are added to the right-hand side.

Proof. For example, consider the case of $Ru = u$. Using Lemma 7, we construct a function Φ and make the change of variables $u = v + \Phi$. In this case the function

v is a solution to the equation (9) with the right-hand side $\tilde{f} = f - M\Phi \in L_2(Q)$, i. e.,

$$Mv = v_t - a_{nn}v_{x_nx_n} - \sum_{i,j=1}^{n-1} a_{ij}v_{x_ix_j} + \sum_{i=1}^{n} a_iv_{x_i} + a_0u = \tilde{f},$$

satisfying the initial and boundary conditions

$$v|_{S_0} = \tilde{\varphi} = \varphi - \Phi|_{S_0}, \quad R_iv(t,x',r_i) = 0 \ (i=0,1), \ v|_{t=0} = 0.$$

First, assume that $R_0u = -u_{x_n}$, $R_1u = u_{x_n}$. In this case the condition A) implies that $\tilde{\varphi}_{x_n}|_{x_n=0} = 0$ and $\tilde{\varphi}_{x_n}|_{x_n=l} = 0$ for $p > 2$ and the inequalities (24) hold for $p = 2$. In any case the function $\tilde{\varphi}$ admits even extensions with respect to the planes $x_n = 0$ and $x_n = l$ to the domains $Q_{-l,0}$ and $Q_{l,2l}$, respectively, with preservation of the class $W_p^{k_0,2k_0}$. The extension (denote it by the same symbol) belong to class $W_p^{k_0,2k_0}((0,T)\times\partial\Omega\times(-l,2l))$. To prove this fact we employ the definition of the norm and conventional properties of the Sobolev spaces. In particular, for $p \neq 2$, we have the inequality

$$\text{(27)} \quad \|\tilde{\varphi}\|_{W_p^{k_0,2k_0}((0,T)\times\partial\Omega\times(-l,2l))} \le c(\|\tilde{\varphi}\|_{W_p^{k_0,2k_0}((0,T)\times\partial\Omega\times(-l,0))} + \|\tilde{\varphi}\|_{W_p^{k_0,2k_0}((0,T)\times\partial\Omega\times(0,l))} + \|\tilde{\varphi}\|_{W_p^{k_0,2k_0}((0,T)\times\partial\Omega\times(l,2l))}) \le c_1\|\tilde{\varphi}\|_{W_p^{k_0,2k_0}(S_0)}.$$

The last inequality follows from the definition of the function $\tilde{\varphi}$, and the previous inequality from the additivity property of a Sobolev space with respect to partitioning a domain (see Remark 3, Subsec. 4.4.1 in [29] and the bibliography there). However, the exponent $p = 2$ is critical and the last inequality is not valid. We can write out its analog. We have ($k_0 = 3/4$)

$$\text{(28)} \quad \|\tilde{\varphi}\|^2_{W_2^{k_0,2k_0}((0,T)\times\partial\Omega\times(-l,2l))} = \|\tilde{\varphi}\|^2_{L_2(0,T;W_2^{2k_0}(\partial\Omega\times(-l,2l))} + \|\tilde{\varphi}\|^2_{L_2(\partial\Omega\times(-l,2l);W_2^{k_0}(0,T))}.$$

The last norm is written as

$$\|\tilde{\varphi}\|^2_{L_2(\partial\Omega\times(-l,2l);W_2^{k_0}(0,T))} = \|\tilde{\varphi}\|^2_{L_2(\partial\Omega\times(-l,0);W_2^{k_0}(0,T))} + \|\tilde{\varphi}\|^2_{L_2(\partial\Omega\times(0,l);W_2^{k_0}(0,T))} + \|\tilde{\varphi}\|^2_{L_2(\partial\Omega\times(l,2l);W_2^{k_0}(0,T))} \le 3\|\tilde{\varphi}\|^2_{L_2(\partial\Omega\times(0,l);W_2^{k_0}(0,T))}.$$

We can endow the space $W_2^{2k_0}(\partial\Omega\times(-l,2l))$ with the equivalent norm

$$\|\tilde{\varphi}\|^2_{W_2^{2k_0}(\partial\Omega\times(-l,2l))} = \|\tilde{\varphi}\|^2_{L_2(\partial\Omega;W_2^{2k_0}(-l,2l))} + \|\tilde{\varphi}\|^2_{L_2(-l,2l;W_2^{2k_0}(\partial\Omega))}.$$

The last summand is estimated as

$$\text{(29)} \quad \|\tilde{\varphi}\|^2_{L_2(-l,2l;W_2^{2k_0}(\partial\Omega))} = \|\tilde{\varphi}\|^2_{L_2(-l,0;W_2^{2k_0}(\partial\Omega))} + \|\tilde{\varphi}\|^2_{L_2(0,l;W_2^{2k_0}(\partial\Omega))} + \|\tilde{\varphi}\|^2_{L_2(l,2l;W_2^{2k_0}(\partial\Omega))} \le 3\|\tilde{\varphi}\|^2_{L_2(0,l;W_2^{2k_0}(\partial\Omega))}.$$

Consider the first summand. The definition yields

$$\|\tilde{\varphi}\|^2_{W_2^{2k_0}(-l,2l))} = \|\tilde{\varphi}\|^2_{W_2^1(-l,2l))} + <\tilde{\varphi}_{x_n}>^2_{1/2,2},$$

$$<\tilde{\varphi}_{x_n}>^2_{1/2,2} = \int_{(-l,2l)^2} \frac{|\tilde{\varphi}_{x_n}(\xi) - \tilde{\varphi}_{x_n}(\eta)|^2}{|\xi-\eta|^2}\,d\xi d\eta.$$

The first norm is obviously estimated by $3\|\tilde{\varphi}\|^2_{W_2^1(0,l)}$. The function $\psi = \tilde{\varphi}_{x_n}$ can be written as the sum of three functions $\psi_1 + \psi_2 + \psi_3$, with $\psi_2 = \tilde{\varphi}_{x_n}$ for $x_n \in (0,l)$ and $\psi_2 = 0$ for $x_n \notin (0,l)$, $\psi_1 = \psi_2(-x_n)$, $\psi_3 = \psi_2(2l - x_n)$. Then the seminorm $< \tilde{\varphi}_{x_n} >_{1/2,2}$ is estimated by

$$c\sum_{i=1}^{3} < \psi_i >^2_{1/2,2} \leq c_1\|\tilde{\varphi}_{x_n}\|^2_{W_2^{1/2}(0,l)} + c_2\Big(\int_0^l |\tilde{\varphi}_{x_n}|^2 \frac{dx_n}{x_n} + \int_0^l |\tilde{\varphi}_{x_n}|^2 \frac{dx_n}{l - x_n}\Big).$$

The last inequality follows from the definition of the seminorm. Similar estimates are proven in many articles (see, for instance, Proposition 7.1 in [28]). The statement that the zero extension of φ from the interval $(0,l)$ to the whole axis $\mathbb{R}$ preserves the class $W_2^{1/2}$ if and only if $\int_0^l |\varphi|^2 \frac{dx_n}{x_n} < \infty$ and $\int_0^l |\varphi|^2 \frac{dx_n}{l-x_n} < \infty$, which actually is used, is known (it follows from the properties of the spaces $\tilde{B}^s_{q,p}$ in [29, 4.3.2] or from Proposition 5.11 in [32]). Integrating the inequality obtained and involving (29), we conclude that

$$\|\tilde{\varphi}\|^2_{W_2^{2k_0}(\partial\Omega\times(-l,2l))} \leq c_3\Big(\|\tilde{\varphi}\|^2_{W_2^{2k_0}(\partial\Omega\times(0,l))} + \Big\|\frac{\tilde{\varphi}_{x_n}}{\sqrt{\rho(x_n)}}\Big\|^2_{L_2(\partial\Omega\times(0,l))}\Big), \tag{30}$$

where $\rho(x_n) = min(x_n, l - x_n)$. The relations (28)-(30) yield

$$\|\tilde{\varphi}\|^2_{W_2^{k_0,2k_0}((0,T)\times\partial\Omega\times(-l,2l))} \leq c\Big(\|\tilde{\varphi}\|^2_{W_2^{k_0,2k_0}((0,T)\times\Gamma_0)} + \Big\|\frac{\tilde{\varphi}_{x_n}}{\sqrt{\rho(x_n)}}\Big\|^2_{L_2(S_0)}\Big),$$

This estimate indicates that, for $p = 2$, the inequality (27) is replaced with the inequality

$$\|\tilde{\varphi}\|_{W_2^{k_0,2k_0}((0,T)\times\partial\Omega\times(-l,2l))} \leq c\big(\|\tilde{\varphi}\|_{W_2^{k_0,2k_0}(S_0)} + (I_{0,T}(\tilde{\varphi},0))^{1/2} + (I^{0,T}(\tilde{\varphi},0))^{1/2}\big). \tag{31}$$

Construct an even function $\psi_1(x_n) \in C^\infty(\mathbb{R})$ such that $supp\,\psi_1 \in (-2l/3, 2l/3)$, $\psi_1 = 1$ for $x_n \in [0, l/2]$. Define a function $\psi_2(x_n) = 1 - \psi_1(x_n)$ for $x_n \in (0,l)$ and $\psi_2(x_n) = 0$ for $x_n < 0$ which is even with respect to the plane $x_n = l$. Construct also functions $\alpha_i(x_n) \in C^\infty(\mathbb{R})$ such that $supp\,\alpha_1 \subset (-2l/3, 2l/3)$, $supp\,\alpha_2 \subset (l/3, 5l/3)$, $\alpha_1 = 1$ on $supp\,\psi_1$ and $\alpha_2 = 1$ on $supp\,\psi_2$. Construct domains G_1 and G_2 such that $G_1 \supset \Omega \times [0, 2l/3]$, $G_1 \subset \Omega \times [0, 3l/4)$, $G_2 \supset \Omega \times [l/3, l]$, $G_2 \subset \Omega \times [l/4, l]$, and the parts of the boundaries ∂G_1 and ∂G_2 lying in the domains $x_n > 0$ and $x_n < l$, respectively, belong to the class C^2. Denote by $\tilde{G}_1$ and $\tilde{G}_2$ the even extension of the domains G_1 and G_2 with respect to the planes $x_n = 0$ and $x_n = l$. We have that $\partial\tilde{G}_1 \in C^2$ and $\partial\tilde{G}_2 \in C^2$. Given a function $g \in L_p(Q)$. Construct an even extensions of g with respect to the planes $x_n = 0$ and $x_n = l$. Thus, we have

$$g(x', x_n, t) = \begin{cases} g(x', x_n, t),\ x_n \in (0,l) \\ g(x', -x_n, t),\ x_n \in (-l, 0) \\ g(x', 2l - x_n, t),\ x_n \in (l, 2l) \end{cases}$$

and $g \in L_p(Q_{-l,2l})$. Extend $a_{ij}, a_i (i \neq n)$ as even functions and a_n as an odd function with respect to the planes $x_n = 0$ and $x_n = l$ to the domain $Q_{-l,2l}$.
Consider the problem

$$u_{it} - Lu_i = g\alpha_i, \quad (x,t) \in \tilde{Q}_i = (0,T) \times \tilde{G}_i,\ i = 1,2, \tag{32}$$

$$u_i|_{(0,T)\times\partial\tilde{G}_i} = a_i = \psi_i\tilde{\varphi},\ u_i|_{t=0} = 0,\ i = 1,2. \tag{33}$$

By construction, $\partial\tilde{G}_1, \partial\tilde{G}_2 \in C^2$. Then referring to [2, Theorem 10.4, Ch. 7] or to Theorem 2.1 in [30], we can state that the problems (32)-(33) are solvable and the estimate

$$\|u_i\|_{W_p^{1,2}(\tilde{Q}_i)} \le c(\|g\|_{L_p(Q)} + \|a_i\|_{W_p^{s_0,2s_0}((0,T)\times\partial\tilde{G}_i)}),\ i=1,2 \tag{34}$$

holds.

The domain $\tilde{G}_1$ is symmetric with respect to the plane $x_n = 0$. In this case a solution u_1 possesses the property $u_{1x_n}|_{x_n=0} = 0$. Indeed, consider the function $\tilde{u}_1(x,t) = u_1(x', -x_n, t)$ for $x \in \tilde{G}_1$, $\tilde{u}_1|_{\partial\tilde{G}_1\times(0,T)} = a_1(x,t)$. We have $\tilde{u}_{1x_n} = -u_{1x_n}(x', -x_n, t)$, $\tilde{u}_{1x_i} = u_{1x_i}(x', -x_n, t)$, $\tilde{u}_{1x_nx_n} = u_{1x_nx_n}(x', -x_n, t)$, $\tilde{u}_{1x_ix_j} = u_{1x_ix_j}(x', -x_n, t)$. In this case the function $\tilde{u}_1$ is a solution to the problem (32)-(33), since the coefficients are even functions. In view of uniqueness, $\tilde{u}_1 = u_1$. I. e. u_1 is even in the variable x_n and thereby $u_1(x', x_n, t) = u_1(x', -x_n, t)$, $u_{1x_n}(x', x_n, t) = -u_{1x_n}(x', -x_n, t)$, and $u_{1x_n}(x', 0, t) = 0$. Similarly, we have that $u_{2x_n}(x', l, t) = 0$ and u_2 is an even function in the variable x_n with respect to the point $x_n = l$. Construct the function $v = u_1\psi_1 + u_2\psi_2$; we have $v|_{S_0} = \tilde{\varphi}$ $v|_{t=0} = 0$. We look for a solution to the initial problem in the form $v = u_1\psi_1 + u_2\psi_2$. The function v satisfies the Dirichlet boundary condition on $\partial\Omega\times(0,l)$ and the Neumann condition on the planes $x_n = 0$ and $x_n = l$. We have $v_t - Lv = u_{1t}\psi_1 - Lu_1\psi_1 - [L,\psi_1]u_1 + u_{2t}\psi_2 - Lu_2\psi_2 - [L,\psi_2]u_2 = g - [L,\psi_1]u_1 - [L,\psi_2]u_2$, where $[L,\psi_i]b = L(\psi_i b) - \psi_i Lb$ $(v_i = v|_{Q^i})$. We look for a function g being a solution to the equation

$$g = V(g) + f, \quad V(g) = [L,\psi_1]u_1 + [L,\psi_2]u_2. \tag{35}$$

Put $u_1 = V_1(g)$, $u_2 = V_2(g)$, $g \in L_p(Q)$. We infer $[L,\psi_k]u_k = L(\psi_k u_k) - \psi_k L u_k = 2a_{nn}\psi_{kx_n}u_{kx_n} + a_{nn}u_k\psi_{kx_nx_n} - a_n\psi_{kx_n}u_k$ $(k=1,2)$. Let $g_1, g_2 \in L_p(Q)$. Consider $V(g_1) - V(g_2)$. The functions $u_k(g_2) - u_k(g_1) = \tilde{u}_k$ are solutions to the problems

$$\tilde{u}_{kt} - L\tilde{u}_k = \tilde{g} = (g_1 - g_2)\alpha_k,\ \tilde{u}_k|_{(0,T)\times\partial\tilde{G}_k} = 0,\ \tilde{u}_k|_{t=0} = 0. \tag{36}$$

Fix $\tau \in (0,T]$. The above arguments (see (34)) validate the estimate

$$\|\tilde{u}_k\|_{W_p^{1,2}(\tilde{Q}_{k\tau})} \le c\|g_1 - g_2\|_{L_p(Q_\tau)}, \quad \tilde{Q}_{k\tau} = (0,\tau)\times\tilde{G}_k, \tag{37}$$

where the constant c is independent of τ. Indeed, take the function $g_\tau = \tilde{g}$ for $t \in (0,\tau)$ and $g_\tau = 0$ for $t > \tau$ as the right-hand side in (36) and solve the problem (36) whose solution meets the estimate (34), i. e., we have $\|\tilde{u}_k\|_{W_p^{1,2}(\tilde{Q}_k)} \le c\|g_\tau\|_{L_p(\tilde{Q}_k)} = c\|\tilde{g}\|_{L_p(Q_\tau)}$. Moreover, this solution $[0,\tau]$ agrees with a solution to the problem (36) with the old right-hand side. Hence, the constant in the estimate (37) is independent of τ. Consider the expression $[L,\psi_k](u_k(g_1) - u_k(g_2)) = J_k$. We have

$$\|J_k\|_{L_p(Q_\tau)} \le 2\|a_{nn}\psi_{kx_n}\tilde{u}_{kx_n}\|_{L_p(Q_\tau)} + \|\tilde{u}_k a_{nn}\psi_{kx_nx_n}\|_{L_p(Q_\tau)} + \|\tilde{u}_k a_n\psi_{kx_n}\|_{L_p(Q_\tau)}.$$

Lemma 3 and the estimate (37) imply that there exist constants $c_i, \beta > 0$ independent of τ such that

$$\|J_k\|_{L_p(Q_\tau)} \le c_k\tau^\beta\|\tilde{u_k}\|_{W_p^{1,2}(Q_{k\tau})} \le \tau^\beta c_{2k}\|g_1 - g_2\|_{L_p(Q_\tau)}.$$

The estimates obtained yield

$$\|V(g_1) - V(g_2)\|_{L_p(Q_\tau)} \le c_0\tau^\beta\|g_1 - g_2\|_{L_p(Q_\tau)}, \tag{38}$$

where the constant c_0 is independent of $\tau \le T$. Choose a parameter $\tau_0 > 0$ such that $c_0\tau_0^\beta = q < 1$. In this case the operator V is a contraction, the equation (35) has a

unique solution $g \in L_p(Q_\tau)$ and the function v is a solution to the initial problem (9), (10) on the segment $[0,\tau]$. Next, we extend a solution. The arguments are similar to those in [33, §21]. Construct a function $V_1(x,t) = \begin{cases} v(x,t), \ t \in (0,\tau) \\ v(x, 2\tau - t), \ t \in (\tau, 2\tau) \\ 0, \ t > 2\tau \end{cases}$.
Make the change of variables $v = V_1 + V$. The initial problem can be rewritten in the form

$$V_t - LV = \tilde{f} - L_0 V_1, \ V|_{S_0} = \tilde{\varphi} - V_1|_{S_0}, \ V|_{t=0} = 0, \ V_{x_n}(t, x', r_k) = 0 \ (k = 1, 2).$$

Note that $V \equiv 0$ on $[0,\tau]$. Next, we repeat the arguments on the segment $[\tau_0, 2\tau_0]$ rather than $[0,\tau_0]$. Without loss of generality, we can assume that all constants c_i in the above estimates are the same, and thereby the solvability interval does not change. Having solvability on $[\tau_0, 2\tau_0]$, we continue the considerations on $[2\tau_0, 3\tau_0]$, and so on. We prove the claim in a particular case of the boundary operators $Ru = u$, $R_i u = (-1)^{i+1} u_{x_n}$. In the remaining cases the arguments are the same. The only distinction is that in the case of the Dirichlet conditions $R_i u = u$ $(i = 1, 2)$ the extension of the right-hand side g and the functions $\tilde{\varphi}$ through the planes $x_n = 0$ or $x_n = l$ is realized as odd extensions and the corresponding solutions u_k are odd functions of the variable x_n. In the case of mixed boundary conditions, for example, $R_0 u = u$ and $R_1 u = u_{x_n}$ one of the extensions is an odd function and the second an even function of x_n. The corresponding auxiliary problems (32), (33) are of the form

$$u_{it} - Lu_i = g\alpha_i, \quad (x,t) \in \tilde{Q}_i = (0,T) \times \tilde{G}_i, \ i = 1, 2,$$
$$Ru_i|_{(0,T)\times\partial\tilde{G}_i} = a_i = \psi_i \tilde{\varphi}, \ u_i|_{t=0} = 0, \ i = 1, 2.$$

The estimate (26) for solutions follows from the estimates (27), (28), (34), and the estimates for the function g resulting from the arguments of the second part of the proof. In particular, the estimate (38) and the equalities (35) validate the estimate

$$\|g\|_{L_p(0,\tau_0)} \leq \frac{1}{1-q}(\|f\|_{L_p(0,\tau_0)} + \|V(0)\|_{L_p(0,\tau_0)}).$$

The estimates of the norms $\|g\|_{L_p((i-1)\tau_0, i\tau_0)}$ and the corresponding estimates for the norms of v are obtained step by step. To derive (26), we employ (27), (31), and Corollary 1. □

As a consequence of Theorem 1, we have the following statement.

Theorem 2. *Assume that the conditions* (11), (18), *and* (14) *if $Ru = u$ and the conditions* (15) *and* (12) *if $Ru \neq u$ ($p \neq 3/2, 3$) are fulfilled, $u_0 \equiv 0$, and $\phi \in (0,T]$. Let also the condition A) hold, where the integral conditions for $p = 2$ are replaced with the following conditions: if $R_0 u = u$ and $Ru \neq u$ or $R_1 u = u$ and $Ru \neq u$ then $J_{0,\phi}(\varphi, \varphi_0) < \infty$ or $J^{0,\phi}(\varphi, \varphi_1) < \infty$, respectively; if $R_0 u = -u_{x_n}$ and $Ru = u$ or $R_1 u = u_{x_n}$ and $Ru = u$ then $I_{0,\phi}(\varphi, \varphi_0) < \infty$ or $I^{0,\phi}(\varphi, \varphi_1) < \infty$, respectively. Then on the segment $(0,\phi)$ there exists a unique solution u to the problem* (9), (10) *such that $u \in W_p^{1,2}(Q_\phi)$. A solution meets the estimate*

$$\|u\|_{W_p^{1,2}(Q_\phi)} \leq c(\|f\|_{L_p(Q_\phi)} + \|\varphi\|_{\tilde{W}_p^{k_0,2k_0}(S_0^\phi)} + \|\varphi_0\|_{\tilde{W}_p^{k_1,2k_1}(Q_\phi^0)} + \|\varphi_1\|_{\tilde{W}_p^{k_2,2k_2}(Q_\phi^0)}),$$

where the constant c is independent of $\phi \in (0,T]$ and f, φ and in the case of $p = 2$ in dependence of the boundary conditions one or two expressions of the form $c_1(J_{0,\phi}(\varphi,\varphi_0))^{1/2}$, $c_1(J^{0,\phi}(\varphi,\varphi_1))^{1/2}$, $c_1(I_{0,\phi}(\varphi,\varphi_0))^{1/2}$, $c_1(I^{0,\phi}(\varphi,\varphi_1))^{1/2}$, with c_1 a constant independent of ϕ, are added to the right-hand side.

Proof. Let $p \neq 2$. Extend the functions $\varphi, \varphi_0, \varphi_1$ by zero for $t < 0$ preserving the notations and let $\tilde{\varphi} = \begin{cases} \varphi(t,x), \ t \in (-\infty, \phi) \\ \varphi(2\phi - t, x), \ t \in [\phi, T] \end{cases}$. The functions $\tilde{\varphi_0}, \tilde{\varphi_1}$ are defined similarly. Obviously, we have

$$\tilde{\varphi} \in W_p^{k_0, 2k_0}(S_0), \quad \tilde{\varphi_0} \in W_p^{k_1, 2k_1}(Q_\phi^0), \quad \tilde{\varphi_1} \in W_p^{k_2, 2k_2}(Q_\phi^0).$$

Construct a function $\tilde{f} = f$ for $t \leq \phi$ and $\tilde{f} = 0$ for $t > \phi$. Using Theorem 1, we find a solution to the problem (9)-(10) with functions $\tilde{\varphi}, \tilde{\varphi_0}, \tilde{\varphi_1}, \tilde{f}$ rather than $\varphi, \varphi_0, \varphi_1$, and f such that $u \in W_p^{1,2}(Q)$. By Theorem 1, we infer

$$\|u\|_{W_p^{1,2}(Q_\varphi)} \leq \|u\|_{W_p^{1,2}(Q)} \leq \\ c(\|\tilde{f}\|_{L_p(Q)} + \|\tilde{\varphi}\|_{W_p^{k_0,2k_0}(S_0)} + \|\tilde{\varphi}_0\|_{W_p^{k_1,2k_1}(Q^0)} + \|\tilde{\varphi}_1\|_{W_p^{k_2,2k_2}(Q^0)}).$$

Estimate the right-hand side. We have the inequality

$$\|\tilde{\varphi}\|_{W_p^{k_0,2k_0}(S_0)} \leq \|\tilde{\varphi}\|_{W_p^{k_0,2k_0}((-1,T)\times\Gamma_0)} \leq \\ c(\|\tilde{\varphi}\|_{W_p^{k_0,2k_0}((-1,\phi)\times\Gamma_0)} + \|\tilde{\varphi}\|_{W_p^{k_0,2k_0}((\phi,T)\times\Gamma_0)})$$

We have used here the additivity of the Sobolev spaces with respect to the partitioning the domain (see Remark 3 of Subsec. 4.4.1 in [29]) and the definition of the norm. In view of Lemma 1, the first summand on the right-hand side is estimated by $c_2\|\varphi\|_{\tilde{W}_p^{k_0,2k_0}(S_0^\varphi)}$. After the change of variables in the definition of the norm, we reduce the estimate of the second summand to Lemma 1 as well and obtain that

$$\|\tilde{\varphi}\|_{W_p^{k_0,2k_0}((\phi,T)\times\Gamma_0)} \leq c_3\|\varphi\|_{\tilde{W}_p^{k_0,2k_0}(S_0^\varphi)},$$

where c_3 is a constant independent of ϕ. Similarly we derive that

$$\|\tilde{\varphi_0}\|_{W_p^{k_1,2k_1}(Q^0)} \leq c_4\|\varphi_0\|_{\tilde{W}_p^{k_1,2k_1}(Q_\varphi^0)}, \|\tilde{\varphi_1}\|_{W_p^{k_1,2k_1}(Q_\phi^0)} \leq c_5\|\varphi_1\|_{\tilde{W}_p^{k_2,2k_2}(Q_\varphi^0)},$$

where the constants c_4 and c_5 are independent of ϕ as well. Moreover, we have that $\|\tilde{f}\|_{L_p(Q)} = \|f\|_{L_p(Q_\phi)}$. The claim is proven. In the case of $p = 2$, the arguments are the same but the form of the corresponding inequalities becomes more complicated. □

3. Basic results

As we have already noted, we consider a model case in which $G = \Omega \times (0, l)$, $(\partial\Omega \in C^2)$. Describe the exact statement of the problem and conditions on the data. The operator L is of the form

$$Lu = a_{nn}(t,x)u_{x_nx_n} + \sum_{i,j=1}^{n-1} a_{ij}(t,x)u_{x_ix_j} - \sum_{i=1}^{n} a_i(t,x)u_{x_i} - a_0(t,x)u.$$

The equation (1) can be written as

$$Mu = u_t - Lu = f. \tag{39}$$

The equation (39) is furnished with the initial and boundary conditions

$$Ru|_{S_0} = \varphi, \tag{40}$$

where either $Ru = u$ or $Ru = \sum_{i,j=1}^{n-1} a_{ij}u_{x_j}\nu_i + \sigma u$,

$$u(0,x) = u_0(x) \ (x \in G), \ R_0u(t,x',0) = \varphi_0, \quad R_1u(t,x',l) = \varphi_1, \tag{41}$$

where either $R_0u = u$ or $R_0u = -u_{x_n} + \sigma_0 u$, respectively, either $R_1u = u$ or $R_1u = u_{x_n} + \sigma_1 u$,
and the transmission conditions

$$R_i^+ u = (u_{x_n} - \alpha_i^1(t,x')u)|_{x_n=l_i+0} - \alpha_i^2(t,x')u|_{x_n=l_i-0} = g_i^+, \tag{42}$$

$$R_i^- u = (u_{x_n} - \beta_i^1(t,x')u)|_{x_n=l_i-0} - \beta_i^2(t,x')u|_{x_n=l_i+0} = g_i^-,\ i = 1,2,...,m-1. \tag{43}$$

Write out the conditions on the coefficients. In what follows, $\varepsilon_0 \in (0,1/2)$ is a fixed parameter which can be arbitrarily small. We assume that

$$a_i \in L_q(Q),\quad a_0 \in L_r(Q),\ a_{ij} = a_{ji},\ a_{ij} \in C(Q^k)\ (i,j = 1,\dots,n), \tag{44}$$

where $q > n+2$ for $p \le n+2$ and $q = p$ for $p > n+2$, $r > (n+2)/2$ for $p \le (n+2)/2$ and $r = p$ for $p > (n+2)/2$, the functions $a_{ij}|_{Q^k}$ admits extensions to continuous functions of the class $C(\overline{Q^k})$ $(k = 1,\dots,m)$. These functions a_{ij} can have discontinuities of the first kind at the points $x_n = l_k$. We further assume that

$$\alpha_i^k(x',t), \beta_i^k(x',t) \in C^{s_0+\varepsilon_0, 2s_0+2\varepsilon_0}(\overline{Q^0})\quad (k = 1,2;\ i = 1,2\dots,m-1). \tag{45}$$

$$\sigma \in C^{s_0+\varepsilon_0,2s_0+2\varepsilon_0}(S_i), \tag{46}$$

and $\sigma|_{S_i}$ admits extensions to functions of the class $C^{s_0+\varepsilon_0,2s_0+2\varepsilon_0}(\overline{S_i})$;

$$\sigma_0, \sigma_1 \in C^{s_0+\varepsilon_0,2s_0+2\varepsilon_0}(\overline{Q^0}). \tag{47}$$

$$u_0(x) \in \cap_{i=1}^m W_p^{2-2/p}(G^i),\quad \varphi \in \cap_{i=1}^m W_p^{k_0,2k_0}(S_i), \tag{48}$$

where $k_0 = 1 - 1/2p$ if $Ru = u$ and $k_0 = 1/2 - 1/2p$ otherwise;

$$\varphi_0 \in W_p^{k_1,2k_1}(Q^0),\ \varphi_1 \in W_p^{k_2,2k_2}(Q^0), \tag{49}$$

where $k_1 = 1 - 1/2p$ if $R_0u = u$ and $k_1 = 1/2 - 1/2p$ otherwise and similarly $k_2 = 1 - 1/2p$ if $R_1u = u$ and $k_2 = 1/2 - 1/2p$ otherwise. Let

$$g_i^\pm \in W_p^{s_0,2s_0}(Q^0);\ i = 1,2,...,m-1. \tag{50}$$

The agreement conditions at $t = 0$ are of the form

$$\begin{aligned} &R(0,x)u_0|_{\partial\Omega} = \varphi(0,x), \\ &\qquad R_0(0,x')u_0(x',0) = \varphi_0(0,x'),\ R_1(0,x')u_0(x',l) = \varphi_1(0,x'), \end{aligned} \tag{51}$$

where each of the equalities is fulfilled for $p > 3/2$ whenever the corresponding operator R, R_0, or R_1 defines the Dirichlet condition and for $p > 3$ otherwise;
for $p > 3$, we assume that

$$g_i^+(0,x') = (u_{0x_n} - \alpha_i^1(0,x')u_0)|_{x_n=l_i+0} - \alpha_i^2(0,x')u_0|_{x_n=l_i-0}, \tag{52}$$

$$g_i^-(0,x') = (u_{0x_n} - \beta_i^1(0,x')u_0)|_{x_n=l_i-0} - \beta_i^2(0,x')u_0|_{x_n=l_i+0}. \tag{53}$$

In the case of $Ru \ne u$ we additionally require that

$$a_{ij}|_{Q^k} \in C^{s_0+\varepsilon_0,2s_0+2\varepsilon_0}(Q^k)\ (i,j = 1,\dots,n,\ k = 1,\dots,m) \tag{54}$$

and the functions $a_{ij}|_{Q^k}$ admit extensions to functions of the class $C^{s_0+\varepsilon_0,2s_0+2\varepsilon_0}(\overline{Q^k})$.

Introduce the following additional notations:

$$J_i^+(\varphi,g_i^+)= \quad S_{0,T}(\varphi_{x_n}(t,x',l_i+\tau)-\alpha_i^1\varphi(t,x',l_i+\tau)-\alpha_i^2\varphi(t,x',l_i-\tau)-g_i^+(t,x'+\tau n(x'))),$$
$$J_i^-(\varphi,g_i^-)= \quad S_{0,T}(\varphi_{x_n}(t,x',l_i-\tau)-\beta_i^1\varphi(t,x',l_i-\tau)-\beta_i^2\varphi(t,x',l_i+\tau)-g_i^-(t,x'+\tau n(x'))),\ i=1,2,\dots,m-1,$$
$$J_m^-(\varphi,\varphi_1)=J^{0,T}(\varphi,\varphi_1),\ J_0^+(\varphi,\varphi_0)=J_{0,T}(\varphi,\varphi_0),$$
$$I_0^+(\varphi,\varphi_0)=S_{0,T}(-\varphi_{x_n}(t,x',\tau)+\sigma_0\varphi(t,x',\tau)-\varphi_0(t,x'+\tau n(x'))),$$
$$I_m^-(\varphi,\varphi_1)=S_{0,T}(\varphi_{x_n}(t,x',l-\tau)+\sigma_1\varphi(t,x',l-\tau)-\varphi_1(t,x'+\tau n(x'))).$$

Further, we assume that

B) if $R_iu=u$ $(i=0,1)$ and $Ru=u$ then $\varphi_i(t,x')|_{\partial\Omega}=\varphi(t,x',r_i)$ $(r_0=0,r_1=l)$; if $p>2$ and $R_iu=u$ $(i=0,1)$ and $Ru\neq u$ then $R(t,x',r_i)\varphi_i(t,x')|_{\partial\Omega}=\varphi(t,x',r_i)$; if $p>2$, $R_iu\neq u$ $(i=0,1)$, and $Ru=u$ then $R_i\varphi(t,x',r_i)=\varphi_i(t,x')|_{\partial\Omega}$; if $p>2$ and $Ru=u$ then $R_i^+\varphi=g_i^+(t,x')|_{\partial\Omega}$ and $R_i^-\varphi=g_i^-(t,x')|_{\partial\Omega}$; if $p=2$, $R_0u=u$, and $Ru\neq u$ or $R_1u=u$ and $Ru\neq u$ then $J_0^+(\varphi,\varphi_0)<\infty$ or $J_m^-(\varphi,\varphi_1)<\infty$, respectively; if $p=2$ and $Ru=u$ then $J_i^\pm(\varphi,g_i^\pm)<\infty$ $(i=1,2,\dots,m-1)$ and if $R_0u\neq u$ or $R_1u\neq u$ then $I_0^+(\varphi,\varphi_0)<\infty$ or respectively, $I_m^-(\varphi,\varphi_1)<\infty$ for some $\delta\in(0,\min_i(l_i-l_{i-1}))$, $\delta<\delta_0$.

The main result can be written as follows.

Theorem 3. *Assume that $p\neq 3/2, p\neq 3$, the conditions* (44)-(53), B)*, and the condition* (14) *if $Ru=u$ and the conditions* (15), (54) *if $Ru\neq u$ are fulfilled. Then there exists a unique solution to the problem* (39)-(43) *such that $u\in\cap_{i=1}^m W_p^{1,2}(Q^i)$.*

Proof. First, we reduce the problem to a problem with the homogeneous initial condition and the boundary condition on S_0. Construct functions $v_{0i}\in W_p^{2,1}(Q^i)$ such that $v_{0i}(x,0)=u_0(x)$. It suffices to extend the function u_0 to $\mathbb{R}^n$ as a compactly supported function of the same class (the existence of this extension results from Theorem 4.2.3 in [29]) and apply Theorem 5.5 in [33] taking a solution to the Cauchy problem $v_{0it}-\Delta v_{0i}=0,\ v_{0i}(x,0)=u_0(x)$ as the function v_{0i}. Define the function $\varphi^1=\varphi-Rv_{0i}\in W_p^{k_0,2k_0}(S_i)$, we have that $\varphi^1(0,x)=0$ if this trace has a sense. Extend the function φ^1 from S_i through the planes $x_n=l_{i-1},x_n=l_i$ to the set $\tilde S_i=\partial\Omega\times(2l_{i-1}-l_i,2l_i-l_{i-1})$ preserving the class with the help of the Hestenes method. Next, we construct a domain $\tilde G_i\subset\tilde G^i=\Omega\times(2l_{i-1}-l_i,2l_i-l_{i-1})$ such that $\tilde G_i\supset\Omega\times(3l_{i-1}/2-l_i/2,3l_i/2-l_{i-1}/2)$ and $\partial\tilde G_i\in C^2$. Construct also extensions f_i of f to the domain $(0,T)\times\tilde G_i$ taking $f\equiv 0$ in $\tilde G_i\setminus G_i$. Next, take a function $\psi_i(x_n)\in C_0^\infty(3l_{i-1}/2-l_i/2,3l_i/2-l_{i-1}/2)$ such that $\psi_i(x_n)=1$ for $x_n\in(l_{i-1},l_i)$. Extend the coefficients of the operator R as well as the coefficients of L (if $Ru\neq u$) to the domain $\tilde G^i$ as even functions in x_n with respect to the points $x_n=l_j,\ j=i-1,i$. We look for solutions $V_i\in W_p^{1,2}((0,T)\times\tilde G_i)$ to the problems

$$V_{it}-LV_i=f_i-Mv_{0i},\ RV_i=\varphi^1\psi_i|_{(0,T)\times\partial\tilde G_i},\ V_i|_{t=0}=0. \tag{55}$$

There exist unique solutions V_i from the required class (see Theorem 10.4 in [2] and Theorem 2.1 in [30]). Let $\Phi(t,x)=V_i(t,x)+v_{0i}(t,x)$ for $(t,x)\in Q^i$ $(i=1,2,\dots,m)$. By construction, $\Phi(0,x)=u_0(x)$, $R\Phi|_{S_0}=\varphi$, and $M\Phi=Mv_{0i}+MV_i=f$ in Q^i. Since the extension operators and the operator taking the data of the problem into a solution V_i to the problem (55) are continuous in the corresponding classes then there exists a constant $c > 0$ such that

$$\sum_{i=1}^{m} \|\Phi\|_{W_p^{1,2}(Q^i)} \le c \sum_{i=1}^{m} (\|u_0\|_{W_p^{2-2/p}(G^i)} + \|\varphi\|_{W_p^{k_0,2k_0}(S_i)} + \|f\|_{L_p(Q^i)}),$$

where c is independent of u_0, φ. Make the change of variables $u = v + \Phi$. We arrive at the problem

$$v_t - Lv = 0. \tag{56}$$

$$Rv|_{S_0} = 0,\ v(0,x) = 0\ (x \in G), \tag{57}$$

$$R_0 v(t,x',0) = \tilde{\varphi}_0 = \varphi_0 - R_0\Phi(t,x',0); \quad R_1 v(t,x',l) = \tilde{\varphi}_1 = \varphi_1(t,x') - R_1\Phi(t,x',l), \tag{58}$$

$$R_i^+ v = (v_{x_n} - \alpha_i^1(x',t)v)|_{x_n=l_i+0} - \alpha_i^2(x',t)v|_{x_n=l_i-0} = \tilde{g}_i^+ = g_i^+ - R_i^+\Phi; \tag{59}$$

$$R_i^- v = (v_{x_n} - \beta_i^1(x',t)v)|_{x_n=l_i-0} - \beta_i^2(x',t)v|_{x_n=l_i+0} = \tilde{g}_i^- = g_i^- - R_i^-\Phi. \tag{60}$$

Check that the new data meet the agreement conditions B), (51)-(53). The condition (51) is fulfilled, since the new functions u_0, φ vanish and $\tilde{\varphi}_0(0,x') = \varphi_0(0,x') - R_0\Phi(0,x',0) = \varphi_0(0,x') - R_0u_0(x',0) = 0$ in view of (51), similarly $\tilde{\varphi}_1(0,x') = 0$ whenever the trace exists (for the corresponding p). For $p > 3$ in view of (52), (53), we have

$$\tilde{g}_i^+(0,x') = g_i^+(0,x') - R_i^+u_0 = 0,\ \tilde{g}_i^-(0,x') = g_i^-(0,x') - R_i^-u_0 = 0.$$

The condition B) implies that if $R_iv = v$ $(i = 0,1)$ and $Rv = v$ then $\tilde{\varphi}_i(t,x')|_{\partial\Omega} = \varphi_i(t,x')|_{\partial\Omega} - \Phi(t,x',r_i)|_{\partial\Omega} = \varphi_i(t,x')|_{\partial\Omega} - \varphi(t,x',r_i) = 0$ $(r_0 = 0, r_1 = l)$; if $p > 2$, $R_iv = v$ $(i = 0,1)$, and $Rv \neq v$ then $R(t,x',r_i)\tilde{\varphi}_i(t,x')|_{\partial\Omega} = R(t,x',r_i)\varphi_i(t,x')|_{\partial\Omega} - R\Phi(t,x',r_i)|_{\partial\Omega} = R(t,x',r_i)\varphi_i(t,x')|_{\partial\Omega} - \varphi(t,x',r_i) = 0$; if $p > 2$, $R_iv \neq v$ $(i = 0,1)$, and $Rv = v$ then $\tilde{\varphi}_i(t,x')|_{\partial\Omega} = \varphi_i(t,x')|_{\partial\Omega} - R_i\Phi(t,x',r_i) = 0$. Let $p = 2$. Check that if either $R_0v = v$ and $Rv \neq v$ or $R_1v = v$ and $Rv \neq v$ then $J_0^+(0,\tilde{\varphi}_0) < \infty$ or $J_m^-(0,\tilde{\varphi}_1) < \infty$, respectively; if $Rv = v$ then $J_i^{\pm}(0,\tilde{g}_i^{\pm}) < \infty$ $(i = 1,2,\dots,m-1)$; if $R_0v \neq v$ or $R_1v \neq v$ then $I_0^+(0,\tilde{\varphi}_0) < \infty$ or respectively, $I_m^-(0,\tilde{\varphi}_1) < \infty$, for some $\delta \in (0, \min_i(l_i - l_{i-1}))$, $\delta < \delta_0$.

Consider, for example the case of $R_0v = v$ and $Rv \neq v$. Check that $J_0^+(0,\tilde{\varphi}_0) < \infty$. We have

$$\begin{gathered} J_0^+(0,\tilde{\varphi}_0) = \\ \int_0^T\int_0^\delta\int_{\partial\Omega} |R(t,x',0)(\varphi_0(t,x'+\tau n(x')) - \Phi(t,x'+\tau n(x'),0))|^2 d\Gamma\frac{d\tau}{\tau}dt \le \\ 2\int_0^T\int_0^\delta\int_{\partial\Omega} |R(t,x',0)\varphi_0(t,x'+\tau n(x')) - \varphi(t,x',\tau)|^2 d\Gamma\frac{d\tau}{\tau}dt + \\ 2\int_0^T\int_0^\delta\int_{\partial\Omega} |R(t,x',\tau)\Phi(t,x',\tau) - R(t,x',0)\Phi(t,x'+\tau n(x'),0)|^2 d\Gamma\frac{d\tau}{\tau}dt. \end{gathered} \tag{61}$$

The first integral here is finite in view of the condition B). The second integral is estimated in the proof of Lemma 6 by the quantity $c_1\|\Phi\|^2_{L_2(0,T;W_2^2(G^1))}$. Thus, the condition $J_0^+(0,\tilde{\varphi}_0) < \infty$ is valid. The proof of the remaining agreement conditions for $p = 2$ is in line with the same scheme and we omit it. Consider the auxiliary problems

$$v_{it} - Lv_i = 0,\ (t,x) \in Q^i. \tag{62}$$

$$Rv_i|_{S_i} = 0,\ v_i(0,x) = 0\ (x \in G^i), \tag{63}$$

$$\tilde{R}_{0i}v_i = g^+_{i-1},\ \tilde{R}_{1i}v_i = g^-_i,\ i = 1,2,\dots,m, \tag{64}$$

where $\tilde{R}_{0i}v = -v_{x_n}|_{x_n=l_{i-1}}$ for all i except for $i = 1$, in this case if $R_0v = v$ then $\tilde{R}_{01}v = v|_{x_n=0}$ and $\tilde{R}_{1i}v = v_{x_n}|_{x_n=l_i}$ for all l except for $l = m$, in this case if $R_1v = v$ then $\tilde{R}_{1m}v = v|_{x_n=l}$. Using Theorem 2, we can construct a solution $v_i \in W^{1,2}_p(Q^i_\phi)$ to this problem on any segment $[0,\phi] \subset [0,T]$ which satisfies the estimate

$$\|v_i\|_{W^{1,2}_p(Q^i_\phi)} \le c(\|g^+_{i-1}\|_{\tilde{W}^{k_1,2k_1}_p(Q^0_\phi)} + \|g^-_i\|_{\tilde{W}^{k_2,2k_2}_p(Q^0_\phi))} + J_i(\phi)), \tag{65}$$

where the constant c is independent of $\phi \in (0,T]$ and $g^\pm_i$ and $k_1 = 1/2-1/2p$ except for the case of $i = 1$, in this case if $R_0v = v$ then $k_1 = 1-1/2p$ and $k_2 = 1/2-1/2p$ except for the case of $i = m$, in this case if $R_1v = v$ then $k_2 = 1-1/2p$. If $p \ne 2$ then $J_i(\phi) = 0$. In the case of $p = 2$ and $Rv = v$, if $\tilde{R}_{0i}v = -v_{x_n}|_{x_n=l_{i-1}}$ and $\tilde{R}_{1i}v = v_{x_n}|_{x_n=l_i}$ $(i = 1,2,\dots,m)$ then

$$J_i(\phi) = \int_0^\phi \int_0^\delta \int_{\partial\Omega} |g^+_{i-1}(t,x'+\tau n(x'))|^2 d\Gamma \frac{d\tau}{\tau} dt + \int_0^\phi \int_0^\delta \int_{\partial\Omega} |g^-_i(t,x'+\tau n(x'))|^2 d\Gamma \frac{d\tau}{\tau} dt;$$

if $\tilde{R}_{01}v = v|_{x_n=0}$ and $\tilde{R}_{11}v = v_{x_n}|_{x_n=l_1}$ then

$$J_1(\phi) = \int_0^\phi \int_0^\delta \int_{\partial\Omega} |g^-_1(t,x'+\tau n(x'))|^2 d\Gamma \frac{d\tau}{\tau} dt;$$

if $\tilde{R}_{0m}v = -v_{x_n}|_{x_n=l_{m-1}}$ and $\tilde{R}_{1m}v = v|_{x_n=l}$ then

$$J_m(\phi) = \int_0^\phi \int_0^\delta \int_{\partial\Omega} |g^+_{m-1}(t,x'+\tau n(x'))|^2 d\Gamma \frac{d\tau}{\tau} dt.$$

If $p = 2$ and $Rv \ne v$ then if $i = 1$ and $R_0v = v$ then

$$J_1(\phi) = \int_0^\phi \int_0^\delta \int_{\partial\Omega} |R(t,x',0)g^+_0(t,x'+\tau n(x'))|^2 d\Gamma \frac{d\tau}{\tau} dt;$$

if $i = m$ and $R_1v = v$ then

$$J_m(\phi) = \int_0^\phi \int_0^\delta \int_{\partial\Omega} |R(t,x',0)g^-_m(t,x'+\tau n(x'))|^2 d\Gamma \frac{d\tau}{\tau} dt;$$

if none of these conditions are satisfied then $J_i(\phi) = 0$. Define an operator X taking a vector $(g^+_0, g^-_1, ..., g^+_{m-1}, g^-_m) = \vec{g}$ into a vector $\vec{v} = (v_1, v_1, ..., v_m)$ of solutions to the problems (62)-(64). Thus, $\vec{v} = X(\vec{g})$. Consider the following family of problems depending on the parameter $\tau \in [0,1]$:

$$v_t - Lv = 0, \tag{66}$$

$$Rv|_{S_0} = 0,\ v(0,x) = 0\ (x \in G), \tag{67}$$

$$R^+_{0\tau}v=\tilde{\varphi}_0;\ R^-_{m\tau}v=\tilde{\varphi}_1, \tag{68}$$

$$R^+_{i\tau}v=(v_{x_n}-\tau\alpha^1_i(x',t)v)|_{x_n=l_i+0}-\tau\alpha^2_i(x',t)v|_{x_n=l_i-0}=\tilde{g}^+_i; \tag{69}$$

$$R^-_{i\tau}v=(v_{x_n}-\tau\beta^1_i(x',t)v)|_{x_n=l_i-0}-\tau\beta^2_i(x',t)v|_{x_n=l_i+0}=\tilde{g}^-_i,\ i=1,2,\dots,m, \tag{70}$$

where $R^+_{0\tau}v=(-v_{x_n}+\tau\sigma_0 v)|_{x_n=0}$ if $R_0v\neq v$, otherwise $R^+_{0\tau}v=v|_{x_n=0}$ and $R^-_{m\tau}v=(v_{x_n}+\tau\sigma_1 v)|_{x_n=l}$ if $R_1v\neq v$, otherwise $R^-_{m\tau}v=v|_{x_n=l}$. For $\tau=1$, the problem (66)-(70) agrees with the problem (56)-(60). For $\tau=0$, the problem has a unique solution such that $v|_{Q^i}=v_i$, where v_i is a solution to the problem (62)-(64) with $\vec{g}=(\tilde{\varphi}_0,\tilde{g}^-_1,\tilde{g}^+_1,\dots,\tilde{g}^-_{m-1},\tilde{g}^+_{m-1},\tilde{\varphi}_1)$. In this case the equalities (66)-(70) are rewritten as

$$\vec{v}=\tau XT\vec{v}+X\vec{g}, \tag{71}$$

where the vector $T\vec{v}$ has the coordinates $(Tv)_1=-\sigma_0 v|_{x_n=0}$ if $R_0v\neq v$ and $(Tv)_1=0$ otherwise, $(Tv)_{2m}=-\sigma_1 v|_{x_n=0}$ if $R_1v\neq v$ and $(Tv)_{2m}=0$ otherwise, $(Tv)_{2i}=\beta^1_i(x',t)v|_{x_n=l_i-0}+\beta^2_i(x',t)v|_{x_n=l_i+0}$ and $(Tv)_{2i+1}=\alpha^1_i(x',t)v|_{x_n=l_i+0}+\alpha^2_i(x',t)v|_{x_n=l_i-0}$, where $i=1,2,\dots,m-1$. We identify a function v with the vector $\vec{v}$ whose i-th component is the function $v|_{Q^i}$. We expose the proof in the case of $R_0v\neq v$, $R_1v\neq v$, $Rv=v$. In the remaining cases the proof is even simpler. The operator XT is a bounded linear mapping of the space H_ϕ of functions $\vec{v}=(v_1,v_2,\dots,v_m)\in\prod_{i=1}^m W^{1,2}_p(Q^i_\phi)$ satisfying (67), where $v|_{Q^i}=v_i$, into itself. We endow the space H_ϕ with the norm $\|\vec{v}\|_{H_\phi}=\sum_{i=1}^m\|v_i\|_{W^{1,2}_p(Q^i_\phi)}$. We now establish a priori estimates uniform in the parameter $\tau\in[0,1]$ which, in particular, ensure the boundedness of the operator $XT:H_\phi\to H_\phi$. By Lemma 2, if $v_i\in W^{1,2}_p(Q^i_\phi)$ then $v_i(t,x',l_k)\in\tilde{W}^{s_1,2s_1}_p(Q^0_\phi)$ $(k=i-1,i)$ and

$$\|v_i(t,x',l_k)\|_{\tilde{W}^{s_1,2s_1}_p(Q^0_\phi)}\le c_1\ \|v_i(t,x)\|_{W^{1,2}_p(Q^i_\phi)},$$

where the constant c_1 is independent of ϕ. Let $\vec{v}\in H_\phi$ be a solution to the equation (71). Estimate the norm $\|XT\vec{v}\|_{H_\phi}$ In view of the estimates (65), we infer

$$\begin{gathered}\|XT\vec{v}\|_{H_\phi}=\sum_{i=1}^m\|v_i\|_{W^{1,2}_p(Q^i_\phi)}\le\\ c(\|\sigma_0v_1(t,x',0)\|_{\tilde{W}^{s_0,2s_0}_p(Q^0_\phi)}+\|\sigma_1v_m(t,x',l)\|_{\tilde{W}^{s_0,2s_0}_p(Q^0_\phi)}+\\ \sum_{i=1}^{m-1}(\|\beta^1_iv_i(t,x',l_i)+\beta^2_iv_{i+1}(t,x',l_i)\|_{\tilde{W}^{s_0,2s_0}_p(Q^0_\phi)}+\\ \|\alpha^1_iv_{i+1}(t,x',l_i)+\alpha^2_iv_i(t,x',l_i)\|_{\tilde{W}^{s_0,2s_0}_p(Q^0_\phi)})+I(\phi)),\end{gathered} \tag{72}$$

where $I(\phi) = 0$ for $p \neq 2$ and, for $p = 2$,

$$I^2(\phi) = \int_0^\phi \int_0^\delta \int_{\partial\Omega} |\sigma_0 v_1(t, x' + \tau n(x'), 0)|^2 d\Gamma \frac{d\tau}{\tau} dt + \int_0^\phi \int_0^\delta \int_{\partial\Omega} |\sigma_1 v_m(t, x' + \tau n(x'), l))|^2 d\Gamma \frac{d\tau}{\tau} dt + \sum_{i=1}^{m-1} (\int_0^\phi \int_0^\delta \int_{\partial\Omega} |\beta_i^1 v_i(t, x' + \tau n(x'), l_i) + \beta_i^2 v_{i+1}(t, x' + \tau n(x'), l_i)|^2 d\Gamma \frac{d\tau}{\tau} dt + \int_0^\phi \int_0^\delta \int_{\partial\Omega} |\alpha_i^1 v_{i+1}(t, x' + \tau n(x'), l_i) + \alpha_i^2 v_i(t, x' + \tau n(x'), l_i)|^2 d\Gamma \frac{d\tau}{\tau} dt).$$

Referring to Lemma 4, we can rewrite (72) in the form

$$\|XT\vec{v}\|_{H_\phi} = \sum_{i=1}^m \|v_i\|_{W_p^{1,2}(Q_\phi^i)} \leq c_1(\|v_1(t, x', 0)\|_{\tilde{W}_p^{s_0, 2s_0}(Q_\phi^0)} + \sum_{i=1}^{m-1} (\|v_i(t, x', l_i)\|_{\tilde{W}_p^{s_0, 2s_0}(Q_\phi^0)} + \|v_{i+1}(t, x', l_i)\|_{\tilde{W}_p^{s_0, 2s_0}(Q_\phi^0)}) + \|v_m(t, x', l)\|_{\tilde{W}_p^{s_0, 2s_0}(Q_\phi^0)} + I(\phi)) = c_1(I_1(\phi) + I(\phi)). \tag{73}$$

Next, we have

$$\|v_i(t, x', l_i)\|^p_{\tilde{W}_p^{s_0}(0,\phi)} = \int_0^\phi \frac{|v_i|^p}{t^{s_0 p}} dt + \int_0^\phi \int_0^\phi \frac{|v_i(t, x', l_i) - v_i(\xi, x', l_i)|^p}{|t - \xi|^{1+s_0 p}} dt d\xi \leq \phi^{(s_1 - s_0)p} \|v_i(t, x', l_i)\|^p_{\tilde{W}_p^{s_1}(0,\phi)}$$

and thereby

$$\|v_i(t, x', l_i)\|_{L_p(\Omega; \tilde{W}_p^{s_0}(0,\phi))} \leq \phi^{1/2} \|v_i(t, x', l_i)\|_{L_p(\Omega; \tilde{W}_p^{s_1}(0,\phi))}.$$

By Lemma 2, the right-hand side is estimated by $C\phi^{1/2}\|v_i(t, x)\|_{W_p^{1,2}(Q_\phi^i)}$. Therefore, we have the inequality

$$\|v_i(t, x', l_i)\|_{L_p(\Omega; \tilde{W}_p^{s_0}(0,\phi))} \leq C\phi^{1/2} \|v_i\|_{W_p^{1,2}(Q_\phi^i)}, \tag{74}$$

with a constant C independent of φ. Consider the norms occurring in (73). In view of (74), we have

$$\|v_i(t, x', l_i)\|^p_{\tilde{W}_p^{s_0, 2s_0}(Q_\phi^0)} = \|v_i(t, x', l_i)\|^p_{L_p(\Omega; \tilde{W}_p^{s_0}(0,\phi))} + \|v_i(t, x', l_i)\|^p_{L_p(0,\phi; W_p^{2s_0}(\Omega))} \leq c_3 \phi^{p/2} \|v_i\|_{W_p^{1,2}(Q_\phi^i)} + c_4 \|v_i(t, x)\|^p_{L_p(0,\phi; W_p^1(G^i))} \leq c_5 \phi^{p/2} \|v_i\|^p_{W_p^{1,2}(Q_\phi^i)}. \tag{75}$$

Here we have used the trace theorem ([29, Theorem 4.7, p. 412]) and the corresponding estimate $\|v_i\|_{W_p^{1-1/p}(\Omega)} \leq c_1 \|v_i\|_{W_p^1(G^i)}$, the interpolation inequality

$$\|v_i\|_{W_p^1(G^i)} \leq C \|v_i\|^{1/2}_{L_p(G^i)} \|v_i\|^{1/2}_{W_p^2(G^i)}$$

and the inequality $\|v_i\|_{L_p(0,\phi;L_p(G^i))} \leq \phi\|v_{it}\|_{L_p(0,\phi;L_p(G^i))}$ $(v_i(0,x) = 0)$ which follows from the Newton-Leibnitz formula. All constants are independent of ϕ. Final estimate of I_1 is of the form

$$I_1(\phi) \leq c_6\phi^{1/2}\|\vec{v}\|_{H_\phi}, \tag{76}$$

where the constant c_6 is independent of ϕ. Consider the case of $p = 2$ and estimate the expression $I(\phi)$. Every summand in I is estimated with use of the same scheme. Consider, for example, the first summand

$$I_0 = \int\limits_0^\phi \int\limits_0^\delta \int\limits_{\partial\Omega} |\sigma_0 v_1(t, x' + \tau n(x'), 0)|^2 d\Gamma \frac{d\tau}{\tau} dt \leq$$

$$c_1 \int\limits_0^T \int\limits_0^\delta \int\limits_{\partial\Omega} |v_1(t, x' + \tau n(x'), 0)|^2 d\Gamma \frac{d\tau}{\tau} dt$$

We employ the arguments used in Lemma 6. We have

$$|v_1(t, x' + \tau n(x'), 0)|^2 =$$

$$|\int\limits_0^\tau \sum_{i=1}^{n-1} v_{1x_i}(t, x' + \xi n(x'), \tau - \xi)n_i - v_{1x_n}(t, x' + \xi n(x'), \tau - \xi)\, d\xi|^2 =$$

$$|\int_0^\tau \sum_{i=1}^{n-1} v_{1x_i}(t, x' + (\tau - \eta)n(x'), \eta)n_i - v_{1x_n}(t, x' + (\tau - \eta)n(x'), \eta)\, d\eta|^2 \leq$$

$$c_2\tau \int_0^\tau \sum_{i=1}^{n} |v_{1x_i}(t, x' + (\tau - \eta)n(x'), \eta|^2\, d\eta.$$

As in the estimate of the right-hand side in the inequality (22), we derive that

$$I_0 \leq c_2\|v_1(t, x)\|^2_{L_2(0,\phi;W_p^1(G^1))}, \tag{77}$$

under the condition, that the parameter $\delta > 0$ is chosen as in Lemma 6 (or smaller). Now we use the estimate $\|v_1\|_{W_p^1(G^1)} \leq C\|v_1\|^{1/2}_{L_p(G^1)}\|v_1\|^{1/2}_{W_p^2(G^1)}$ and the inequality $\|v_1\|_{L_p(0,\phi;L_p(G^1))} \leq \phi\|v_{1t}\|_{L_p(0,\phi;L_p(G^1))}$ $(v_1(0,x) = 0)$ which is a consequence of the Newton-Leibnitz formula. Here all constants are independent of ϕ. In this case we obtain that

$$I_0 \leq c_3\phi\|v_1(t, x)\|^2_{W_2^{1,2}(Q_\phi^1)}. \tag{78}$$

The remaining summands in $I(\phi)$ admits similar estimates. Thus, using the inequality (78) and its analogs for the functions $v_i(t, x)$ and (76), we arrive at the inequality

$$\|XT\vec{v}\|_{H_\phi} \leq c\phi^{1/2}\|\vec{v}\|_{H_\phi},$$

where the constant c is independent of ϕ. If $\phi \leq \gamma_0$, $c\gamma_0^{1/2} = 1/2$, then the operator XT is a contraction for every $\tau \in [0, 1]$ and, therefore, a solution to the equation (71) meets the estimate $\|\vec{v}\|_{H_\phi} \leq 2\|X\vec{g}\|_{H_\phi}$, $\forall\phi \leq \gamma_0$. Next, we demonstrate that there exists a parameter $\gamma_1 \leq \gamma_0$ such that the inequality

$$\|\vec{v}\|_{H_\phi} \leq \alpha_1 \tag{79}$$

for solutions to the equation (71) ensures the estimate $\|\vec{v}\|_{H_{min(\phi+\gamma_1,T)}} \leq \alpha_2$, where the parameters α_2 and γ_1 depend on the norms of the data and are independent of $\tau \in [0,1]$ and $\phi \in (0,T)$. Take $\gamma_1 \leq \gamma_0$ and assume that the estimate (79) holds. Extend the function $\vec{v}$ by zero for $t < 0$. Define the vector-function $\vec{v}_0(t,x)$ and the corresponding function v_0 such that $v_0|_{Q^i} = v_i^0$ as follows: $\vec{v}_0(t,x) = \vec{v}(t,x)$ for $t \leq \phi$ and $\vec{v}_0(x,t) = \vec{v}(2\phi - t, x)$ for $t \in (\phi, T)$. We have that $\vec{v}_0 = (v_1^0, \dots, v_m^0) \in H_T$ and $\|\vec{v}_0\|_{H_T} \leq 2\|\vec{v}\|_{H_\phi}$. Define a vector $\vec{\Psi}$ and the corresponding function Ψ which is a solution to the problem

$$M\Psi = -Mv_0,\ R\Psi|_{S_0} = 0,\ \Psi(0,x) = 0,\ -\Psi_{x_n}\big|_{x_n=0} = \tilde{\varphi}_0 - R_{0\tau}^{+} v_0;$$
$$\Psi_{x_n}\big|_{x_n=l} = \tilde{\varphi}_1 - R_{m\tau}^{-} v_0,\ \Psi_{x_n}\big|_{x_n=l_i-0} = \tilde{g}_i^{-} - R_{i\tau}^{-} v_0,\ \Psi_{x_n}\big|_{x_n=l_i+0} = \tilde{g}_i^{+} - R_{i\tau}^{+} v_0.$$

As is easily seen, all agreement conditions are fulfilled. By Theorem 1, there exists a unique solution to this problem such that $\Psi_i \in W_p^{1,2}(Q^i)$ $(i = 1, 2 \dots, m)$ and we have the estimate $\|\vec{\Psi}\|_{H_T} \leq c_1$, where without loss of generality we can assume that the constant c_1 is independent of τ. Next, we make the change $\vec{v} = \vec{v}_0 + \vec{\Psi} + \vec{\omega}$ and thereby $v = v_0 + \Psi + \omega$. The function ω is a solution to the problem

$$M\omega = 0,\ R\omega|_{S_0} = 0,\ R_{0\tau}^{+}\omega = -\tau\sigma_0\Psi|_{x_n=0},\ R_{m\tau}^{-}\omega_m = -\tau\sigma_m\Psi|_{x_n=l},$$
$$\omega(0,x) = 0,\ \omega_{x_n}|_{x_n=l_i-0} = \tau(\beta_i^1\omega|_{x_n=l_i-0} + \beta_i^2\omega|_{x_n=l_i+0}) + \tau\Psi_i^{-},$$
$$\omega_{x_n}|_{x_n=l_i+0} = \tau(\alpha_i^1\omega|_{x_n=l_i+0} + \alpha_i^2\omega|_{x_n=l_i-0}) + \tau\Psi_i^{+},$$

where $\Psi_i^{+} = \alpha_i^1\Psi|_{x_n=l_i+0} + \alpha_i^2\Psi|_{x_n=l_i-0}$, $\Psi_i^{-} = \beta_i^1\Psi|_{x_n=l_i-0} + \beta_i^2\Psi|_{x_n=l_i+0}$. By construction, $\vec{\omega} = 0$ for $t \in [0,\phi], x \in G$. By Lemma 3, $Mv_0 \in L_p(Q)$ and there exists a constant $c_0 > 0$ such that

$$\|Mv_0\|_{L_p(Q)} \leq c_0\|\vec{v}_0\|_{H_T} \leq 2c_0\|\vec{v}_0\|_{H_\phi} \leq 2c_0c_1,$$

where the constant c_0 depends of the norms of the coefficients of the equation in the corresponding classes. By construction,

$$\vec{\omega} = \tau XT\vec{\omega} + \tau X\vec{f}, \tag{80}$$

where $\vec{f} = (-\sigma_0\Psi|_{x_n=0}, \Psi_1^{-}, \Psi_1^{+}, \dots, \Psi_{m-1}^{+}, -\sigma_m\Psi|_{x_n=l})$. Taking the estimates of the form (77) into account, we can rewrite the estimate (73) in the form

$$\|XT\vec{\omega}\|_{H_{\phi+\gamma_1}} \leq c_1(\|\omega_1(t,x',0)\|_{\tilde{W}_p^{s_0,2s_0}(Q^0_{\phi+\gamma_1})} + \|\omega_m(t,x',l)\|_{\tilde{W}_p^{s_0,2s_0}(Q^0_{\phi+\gamma_1})} + \sum_{i=1}^{m-1}(\|\omega_i(t,x',l_i)\|_{\tilde{W}_p^{s_0,2s_0}(Q^0_{\phi+\gamma_1})} + \|\omega_{i+1}(t,x',l_i)\|_{\tilde{W}_p^{s_0,2s_0}(Q^0_{\phi+\gamma_1})})) + c_2\sum_{i=1}^{m}\|\omega_i(t,x)\|_{L_2(0,\phi+\gamma_1;W_p^1(G^i))}, \tag{81}$$

where the constants c_1, c_2 are independent of ϕ and $\gamma_1 > 0$ is some constant. Estimate the norm

$$\|\omega_i(t,x',l_i)\|^p_{\tilde{W}_p^{s_0,2s_0}(Q^0_{\phi+\gamma_1})} = \|\omega_i(t,x',l_i)\|^p_{L_p(0,\phi+\gamma_1;W_p^{2s_0}(\Omega))} + \|\omega_i(t,x',l_i)\|^p_{L_p(\Omega;W_p^{s_0}(0,\phi+\gamma_1))}.$$

For convenience, we put $v(t) = \omega_i(t, x', l_i)$ omitting unessential arguments. We infer

$$J = \int\limits_{\phi}^{\phi+\gamma_1} \frac{|v|^p}{t^{s_0 p}}\, dt + \int\limits_{0}^{\phi+\gamma_1} \int\limits_{0}^{\phi+\gamma_1} \frac{|v(t) - v(\tau)|^p}{|t-\tau|^{1+s_0 p}}\, dt d\tau =$$

$$\int\limits_{\phi}^{\phi+\gamma_1} \frac{|v(t)|^p}{t^{s_0 p}}\, dt + \int\limits_{\phi}^{\phi+\gamma_1} \int\limits_{0}^{\phi} \frac{|v(t) - v(\tau)|^p}{|t-\tau|^{1+s_0 p}}\, dt d\tau +$$

$$\int\limits_{0}^{\phi} \int_{\phi}^{\phi+\gamma_1} \frac{|v(t) - v(\tau)|^p}{|t-\tau|^{1+s_0 p}}\, dt d\tau + \int\limits_{\phi}^{\phi+\gamma_1} \int\limits_{\phi}^{\phi+\gamma_1} \frac{|v(t) - v(\tau)|^p}{|t-\tau|^{1+s_0 p}}\, dt d\tau.$$

Since $v = 0$ for $t \le \phi$, the second integral can be estimated as follows:

$$\int_{\phi}^{\phi+\gamma_1} \int_{0}^{\phi} \frac{|v(t) - v(\tau)|^p}{|t-\tau|^{1+s_0 p}}\, dt d\tau = \int_{\phi}^{\phi+\gamma_1} \int_{0}^{\phi} \frac{|v(\tau)|^p}{|t-\tau|^{1+s_0 p}}\, dt d\tau =$$

$$\int_{\phi}^{\phi+\gamma_1} |v(\tau)|^p \big(\frac{1}{(\tau-\phi)^{s_0 p}} - \frac{1}{\tau^{s_0 p}}\big) \frac{1}{s_0 p}\, d\tau \le \frac{1}{s_0 p} \int_{\phi}^{\phi+\gamma_1} \frac{v(\tau)|^p}{|\tau-\phi|^{s_0 p}}\, d\tau.$$

Similarly, $\int_0^{\phi} \int_{\phi}^{\phi+\gamma_1} \frac{|v(t)-v(\tau)|^p}{|t-\tau|^{1+s_0 p}}\, dt d\tau \le \frac{1}{s_0 p} \int_{\phi}^{\phi+\gamma_1} \frac{|v(t)|^p}{|t-\phi|^{s_0 p}}\, dt$. In this case the estimate for the quantity J is written as

$$J \le (1 + \frac{2}{s_0 p}) \int_{\phi}^{\phi+\gamma_1} \frac{|v(t)|^p}{|t-\phi|^{s_0 p}} dt + \int_{\phi}^{\phi+\gamma_1} \int_{\phi}^{\phi+\gamma_1} \frac{|v(t) - v(\tau)|^p}{|t-\tau|^{1+s_0 p}}\, dt d\tau \le$$

$$(1 + \frac{2}{s_0 p}) \gamma_1^{p/2} \|v(t)\|^p_{\tilde{W}_p^{s_1}(\phi, \phi+\gamma_1)}.$$

In view of Lemma 2 (to apply this lemma, we should make the change of variables $\xi = t - \phi$, $\eta = \tau - \phi$), we obtain the estimate

$$\int_{\Omega} J\, d\Omega = \|\omega_i(t, x', l_i)\|^p_{L_p(\Omega; W_p^{s_0}(0, \phi+\gamma_1))} \le C_0 \|\omega_i\|^p_{W_p^{1,2}((\phi, \phi+\gamma_1)\times G^i)} \gamma_1^{1/2}, \tag{82}$$

where C_0 is independent of ϕ, γ_1. We conclude that

$$\begin{aligned} &\|\omega_i(t, x', l_i)\|_{L_p(0, \phi+\gamma_1; W_p^{2s_0}(\Omega))} = \\ &\|\omega_i(t, x', l_i)\|_{L_p(\phi, \phi+\gamma_1; W_p^{1-1/p}(\Omega))} \le c_1 \|\omega_i(t, x)\|_{L_p(\phi, \phi+\gamma_1; W_p^1(G^i))} \le \\ &c_2 \|\omega_i(t, x)\|^{1/2}_{L_p(\phi, \phi+\gamma_1; W_p^2(G^i))} \|\omega_i(t, x)\|^{1/2}_{L_p(\phi, \phi+\gamma_1; L_p(G^i))} \le \\ &c_3 \|\omega_i(t, x)\|_{W_p^{1,2}((\phi, \phi+\gamma_1)\times G^i)} \gamma_1^{1/2}, \end{aligned} \tag{83}$$

Here we employ the trace theorem ([29, Theorem 4.7, p.412] (as before in the derivation of (75)), the corresponding estimate $\|\omega_i\|_{W_p^{1-1/p}(\Omega)} \le c_1 \|\omega_i\|_{W_p^1(G^i)}$, the interpolation inequality $\|\omega_i\|_{W_p^1(G^i)} \le C \|\omega_i\|^{1/2}_{L_p(G^i)} \|\omega_i\|^{1/2}_{W_p^2(G^i)}$, and the inequality $\|\omega_i\|_{L_p(\phi, \phi+\gamma_1; L_p(G^i))} \le \gamma_1 \|\omega_{it}\|_{L_p(\phi, \phi+\gamma_1; L_p(G^i))}$ $(\omega_i(\phi, x) = 0)$ which is a consequence of the Newton-Leibnitz formula. All constants are independent of ϕ. The estimates (82) and (83) yield

$$\|\omega_i(t, x', l_i)\|_{\tilde{W}_p^{s_0, 2s_0}(Q^0_{\phi+\gamma_1})} \le c_5 \|\omega_i(t, x)\|_{W_p^{1,2}((\phi, \phi+\gamma_1)\times G^i)} \gamma_1^{1/2}.$$

Similarly, we obtain that

$$\|\omega_i(t,x',l_{i-1})\|_{\tilde{W}_p^{s_0,2s_0}(Q^0_{\phi+\gamma_1})} \le c_6\|\omega_i(t,x)\|_{W_p^{1,2}((\phi,\phi+\gamma_1)\times G^i)}\gamma_1^{1/2}.$$

As in the proof of (83), we arrive at the estimate

$$\sum_{i=1}^{m}\|\omega_i(t,x)\|_{L_2(0,\phi+\gamma_1;W_p^1(G^i))} \le c_7\|\omega_i(t,x)\|_{W_p^{1,2}((\phi,\phi+\gamma_1)\times G^i)}\gamma_1^{1/2}.$$

The last three inequalities and (81) imply that

$$\|XT\vec{\omega}\|_{H_{\phi+\gamma_1}} \le c_8\|\vec{\omega}\|_{H_{\phi+\gamma_1}}\gamma_1^{1/2},$$

where the constant c_8 is independent of γ_1,ϕ. The equality (80) implies that

$$\|\omega\|_{H_{\phi+\gamma_1}} \le \|XT\vec{\omega}\|_{H_{\phi+\gamma_1}} + \|X\vec{f}\|_{H_T} \le c_8\|\vec{\omega}\|_{H_{\phi+\gamma_1}}\gamma_1^{1/2} + \|X\vec{f}\|_{H_T}.$$

Take γ_2 such that $C_8\gamma_2^{1/2}=1/2$, then for $\gamma_1\le\gamma_2$ the previous inequality implies the estimate

$$\|\omega\|_{H_{\phi+\gamma_1}} \le 2\|X\vec{f}\|_{H_T} = c_9. \tag{84}$$

The constant c_9 depends on the norms of the data and α_1. Take $\phi=\gamma_0$. The estimate (84) ensures the corresponding estimate for v on the segment $[0,\gamma_0+\gamma_1]$. Repeating the arguments we can write out the estimate for v on $[0,\gamma_0+2\gamma_1]$, $[0,\gamma_0+3\gamma_1]$, etc. After the finitely many steps we obtain our estimate on the whole segment $[0,T]$ and thereby we have

$$\|v\|_{H_T} \le c\Big[\sum_{i=1}^{m-1}(\|g_i^+\|_{W_p^{s_0,2s_0}(Q_i^0)} + \|g_i^-\|_{W_p^{s_0,2s_0}(Q_i^0)}) + \|f\|_{L_p(Q)} + \tag{85}$$
$$\|\varphi\|_{W_p^{k_0,2k_0}(S_0)} + \|\varphi_0\|_{W_p^{k_1,2k_1}(Q^0)} + \|\varphi_1\|_{W_p^{k_2,2k_2}(Q^0)} + \sum_{i=1}^{m}\|u_0\|_{W_p^{2-2/p}(G^i)}\Big]$$

where a constant c is independent of $\tau\in[0,1]$. Next, the method of continuation in a parameter (see Theorem 3.13 of Ch. 3 in [36]) ensures the solvability of the equation (71) for all $\tau\in[0,1]$. A solution satisfies the estimate (85) which provides uniqueness of solutions. □

4. Possible generalizations and refinements of the results.

The first result is a refinement of Theorem 3 in the case of $n=1$. This result is just a consequence of Theorem 3, but the statements of the solvability and agreement conditions can be simplified.

In our case $G=(0,l)$, $G^i=(l_{i-1},l_i)$, $Lu=a(t,x)u_{xx}-a_1(t,x)u_x-a_0(t,x)u$. The equation is rewritten as

$$Mu = u_t - Lu = f. \tag{86}$$

The initial and boundary conditions are as follows:

$$R_0u|_{x=0}=\varphi_0(t), R_1u|_{x=l}=\varphi_1(t),\ u(0,x)=u_0(x)\ (x\in G), \tag{87}$$

where $R_iu=u$ or $R_iu=u_x+\sigma_i(t)u$. The transmission conditions can be written in the form

$$R_i^+u=(u_x-\alpha_i^1(t)u)|_{x=l_i+0}-\alpha_i^2(t)u|_{x_n=l_i-0}=g_i^+(t), \tag{88}$$

(89) $$R_i^- u = (u_x - \beta_i^1(t)u)|_{x=l_i-0} - \beta_i^2(t)u|_{x=l_i+0} = g_i^-(t),\ i = 1,2,...,m-1.$$

Proceed with the conditions on the coefficients. In the conditions below $\varepsilon_0 > 0$ is a parameter which can be arbitrarily small. We assume that

(90) $$a_i \in L_q(Q),\quad a_0 \in L_r(Q),\ a \in C(Q^i)\ (i = 1,\dots,m)$$

where $q > 3$ for $p \le 3$ and $q = p$ for $p > 3$, $r > 3/2$ for $p \le 3/2$ and $r = p$ for $p > 3/2$, and the function $a|_{Q^i}$ admits an extension to a continuous strictly positive functions of the class $C(\overline{Q^i})$ $(i = 1,\dots,m)$. Next, we assume that

(91) $$\alpha_i^k(t), \beta_i^k(t) \in C^{s_0+\varepsilon_0}([0,T])\ (k = 1,2;\ i = 1,2\dots,m-1),$$
$$\sigma_0, \sigma_1 \in C^{s_0+\varepsilon_0}([0,T]).$$

(92) $$u_0(x) \in \cap_{i=1}^m W_p^{2-2/p}(G^i), \varphi_0 \in W_p^{k_1}(0,T),\ \varphi_1 \in W_p^{k_2}(0,T),$$

where $k_1 = 1 - 1/2p$ if $R_0u = u$ and $k_1 = 1/2 - 1/2p$ otherwise and, similarly, $k_2 = 1 - 1/2p$ if $R_1u = u$ and $k_2 = 1/2 - 1/2p$ otherwise. Moreover,

(93) $$g_i^{\pm} \in W_p^{s_0}(0,T),\ i = 1,2,...,m-1.$$

The agreement conditions at $t = 0$ are of the form

(94) $$R_0u_0(0) = \varphi_0(0),\ R_1u_0(l) = \varphi_1(0),$$

where every of the conditions holds for $p > 3/2$ whenever the corresponding operator R_0 or R_1 defines the Dirichlet condition and $p > 3$ otherwise;
for $p > 3$ we assume that

(95) $$g_i^+(0) = (u_{0x_n} - \alpha_i^1(0)u_0)|_{x=l_i+0} - \alpha_i^2(0)u_0|_{x=l_i-0},$$

(96) $$g_i^-(0) = (u_{0x_n} - \beta_i^1(0)u_0)|_{x=l_i-0} - \beta_i^2(0)u_0|_{x=l_i+0}.$$

Theorem 3 can be stated in the following form.

Theorem 4. *Assume that $p \ne 3/2, p \ne 3$, and the conditions* (90)-(96) *hold. Then there exists a unique solution to the problem* (86)-(89) *such that $u \in \cap_{i=1}^m W_p^{1,2}(Q^i)$.*

Next, we present some corollaries from the main results which can be of interest in their own right. The first generalization refers to the form of the boundary conditions on S_0 of the form $Ru|_{S_0} = \varphi(t,x)$, where $Ru = u$ or

$$Ru = \sum_{i,j=1}^{n-1} a_{ij}u_{x_j}\nu_i + \sigma u.$$

Generally speaking, the results are the same if we replace the boundary conditions of this type by the conditions

(97) $$P_0u|_{S_0} = \varphi^0(t,x),\quad (I - P_0)(\sum_{i,j=1}^{n-1} a_{ij}u_{x_j}\nu_i + \sigma u) = \varphi^1(t,x),$$

where P_0 is a orthoprojection in $\mathbb{C}^h$ and I is the identity, i. e. for different components of the vector u we can impose different conditions on S_0. In this case the conditions on the coefficients are of the same form but the form of the agreement condition differs. We consider the simplest case when

(98) $$[P_0, \sigma_i] = P_0\sigma_i - \sigma_iP_0 \equiv 0\ (i = 0,1),\quad [P_0, \alpha_i^k] \equiv 0,\ [P_0, \beta_i^k] \equiv 0.$$

Introduce the notations

$$J_i^+(\varphi^0, g_i^+) = S_{0,T}(\varphi^0_{x_n}(t,x',l_i+\tau)-$$
$$\alpha_i^1\varphi^0(t,x',l_i+\tau) - \alpha_i^2\varphi^0(t,x',l_i-\tau) - P_0 g_i^+(t,x'+\tau n(x'))),$$
$$J_i^-(\varphi^0, g_i^-) = S_{0,T}(\varphi^0_{x_n}(t,x',l_i-\tau)-$$
$$\beta_i^1\varphi^0(t,x',l_i-\tau) - \beta_i^2\varphi^0(t,x',l_i+\tau) - P_0 g_i^-(t,x'+\tau n(x'))),$$
$$J_m^-(\varphi^1,\varphi_1) = S_{0,T}(\varphi^1(t,x',l-\tau) - (I-P_0)R(t,x',l)\varphi_1(t,x'+\tau n(x'))),$$
$$I_0^+(\varphi^0,\varphi_0) = S_{0,T}(-\varphi^0_{x_n}(t,x',\tau) + \sigma_0\varphi^0(t,x',\tau) - P_0\varphi_0(t,x'+\tau n(x'))),$$
$$J_0^+(\varphi^1,\varphi_0) = S_{0,T}(\varphi^1(t,x',\tau) - (I-P_0)R(t,x',0)\varphi_0(t,x'+\tau n(x')),$$
$$I_m^-(\varphi^0,\varphi_1) = S_{0,T}(\varphi^0_{x_n}(t,x',\tau) + \sigma_1\varphi^0(t,x',\tau) - P_0\varphi_1(t,x'+\tau n(x'))).$$

Below, we assume that

C) if $R_i u = u$ $(i = 0,1)$ then $P_0\varphi_i(t,x')|_{\partial\Omega} = \varphi^0(t,x',r_i)$ $(r_0 = 0, r_1 = l)$; if $p > 2$ and $R_i u = u$ $(i = 0,1)$ then $(I-P_0)R(t,x',r_i)\varphi_i(t,x')|_{\partial\Omega} = \varphi^1(t,x',r_i)$; if $p > 2$, $R_i u \neq u$ $(i = 0,1)$ then $R_i\varphi^0(t,x',r_i) = P_0\varphi_i(t,x')|_{\partial\Omega}$; if $p > 2$ then $R_i^+\varphi^0 = P_0 g_i^+$ and $R_i^-\varphi^0 = P_0 g_i^-$ $(i = 1,2\ldots,m-1)$; for $p = 2$, if $R_0 u = u$ or $R_1 u = u$ then $J_0^+(\varphi^1,\varphi_0) < \infty$ or $J_m^-(\varphi^1,\varphi_1) < \infty$, respectively; $J_i^\pm(\varphi^0, g_i^\pm) < \infty$ $(i = 1,2,\ldots,m-1)$ and if $R_0 u \neq u$ or $R_1 u \neq u$ then $I_0^+(\varphi^0,\varphi_0) < \infty$ or respectively, $I_m^-(\varphi^0,\varphi_1) < \infty$. For $p = 2$ the conditions must be fulfilled for some $\delta \in (0, \min_i(l_i - l_{i-1}))$, $\delta < \delta_0$.

The agreement conditions at $t = 0$ are of the form:

$$(I-P_0)R(0,x)u_0|_{\partial\Omega} = \varphi^1(0,x),\ P_0 u_0|_{\partial\Omega} = \varphi^0(0,x),$$
$$R_0(0,x')u_0(x',0) = \varphi_0(0,x'),\ R_1(0,x')u_0(x',l) = \varphi_1(0,x'), \tag{99}$$

where an equality is fulfilled for $p > 3/2$ if the corresponding operator R, R_0, or R_1 defines the Dirichlet condition and for $p > 3$ otherwise; for $p > 3$, we assume that

$$g_i^+(0,x') = (u_{0x_n} - \alpha_i^1(0,x')u_0)|_{x_n=l_i+0} - \alpha_i^2(0,x')u_0|_{x_n=l_i-0}, \tag{100}$$

$$g_i^-(0,x') = (u_{0x_n} - \beta_i^1(0,x')u_0)|_{x_n=l_i-0} - \beta_i^2(0,x')u_0|_{x_n=l_i+0}. \tag{101}$$

Also suppose that

$$a_{ij}|_{Q^i} \in C^{s_0+\varepsilon_0, 2s_0+2\varepsilon_0}(Q^i)\ (i = 1,\ldots,m,\ s_0 = 1/2 - 1/2p) \tag{102}$$

and $a_{ij}|_{Q^i}$ admit extensions to continuous functions of class $C^{s_0+\varepsilon_0,2s_0+2\varepsilon_0}(\overline{Q^i})$ $(i,j = 1,\ldots,n)$.

The main result is of the form.

Theorem 5. *Assume that* $p \neq 3/2, p \neq 3$, *and the conditions* (44), (45)-(50), (98)-(102), (15), C) *hold. Then there exists a unique solution to the problem* (39),(41)-(43), (97) *such that* $u \in \cap_{i=1}^m W_p^{1,2}(Q^i)$.

The proof follows the same scheme. Note that the Lopatinskii condition on S_0 is fulfilled for the mixed boundary condition (97) (see Proposition 6.2.13 of Ch. 6 in [13]).

Remark 1. The boundary of Ω can consist of several connectedness components. In this case different boundary conditions can be posed on different connectedness components of S_0, for example, the Dirichlet conditions on one component and the

Neumann conditions on the another component. Theorem 3 remains valid. But the agreement conditions differ for different connectedness components.

Remark 2. The question arises whether it is possible to tale the general operator $Lu = \sum_{i,j=1}^{n} a_{ij}u_{x_ix_j} - \sum_{i=1}^{n} a_i u_{x_i} - a_0 u$ in Theorem 3 as L such that the condition (15) hold, for example. The results are valid (the statement of the problem is the same) if the norms of the coefficients $\{a_{in}\}_{i=1}^{n-1}$ in the space $C(\overline{Q})$ are sufficiently small. Also we can prove Theorem 3 under the condition $a_{jn}(t,x',l_i) = a_{nj}(t,x',l_i) = 0$ for $x' \in \partial\Omega$, $j = 1,2,\ldots,n-1$ and $i = 0,1,2\ldots,m$. The proof is realized in accord with the scheme close to that in Theorem 1.

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