

ON SOME DECOMPOSITIONS OF MATRICES OVER ALGEBRAICALLY CLOSED AND FINITE FIELDS

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ABSTRACT. We prove that every square matrix over an algebraically closed field or over a finite field is decomposable into a sum of a potent matrix and a nilpotent matrix of order 2. This somewhat continues the study from our recent paper, published in *Linear & Multilinear Algebra* (2021).

We also completely address the question when each square matrix over an infinite field can be decomposed into a periodic matrix and a nilpotent matrix of order 2.

Key words: Nilpotent matrix, Potent matrix, Jordan normal form, Rational form, Field
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1. INTRODUCTION AND CONVENTIONS

As in the present article we will be mainly concerned with nilpotent and potent elements in matrix rings, let us recall that an element q of an arbitrary ring R is said to be a *nilpotent* if there is an integer $n \geq 1$ depending on q such that $q^n = 0$ (the minimal n with this property is called an *exponent* for q ; in particular, if $n = 2$, the non-zero nilpotent will be shortly named just *square-zero*). As an opposite element to this one, we recall that $p \in R$ is said to be a *potent* if there is a natural number $m \geq 2$, depending on p , with $p^m = p$ (we will concretely call p just m -potent). If $m = 2$, this element is just termed an *idempotent*. As a common generalization of potent elements, we recall the definition of a *periodic* element as being such an element t for which there are two different natural numbers m, n , both depending on t , with $t^m = t^n$.

As a pioneer work in the representation of an arbitrary matrix over a field as a sum of a nilpotent matrix and an idempotent matrix, we can mention [3], where it was proved that this presentation is possible precisely when the field contains only two elements. This was further extended in [12] and [11] by showing in some cases the exact exponent of the nilpotent matrix. An important work was done in [7], where a valuable discussion on the decomposition of a matrix as a sum of an idempotent and a square-zero matrix is given. On the other hand, as a valuable generalization of the aforementioned main fact from [3], it was proved in [1] that every matrix over any finite field of cardinality d is representable as a sum of a nilpotent and a d -potent. Furthermore, this presentation was refined in [2] by proving that, if d is odd, the nilpotent matrix is of exponent at most 3. Moreover, it was constructed in [2, Example 6] a 3×3 matrix over the field of three elements

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which matrix is *not* presentable as the sum of a 3-potent matrix and a square-zero matrix.

We thus come rather naturally to the following intriguing problem.

Question 1: When every square matrix over a field K can be expressed as

$$P + Q,$$

where P is a potent matrix and Q is a square-zero matrix?

We shall examine below two situations, namely of algebraically closed fields (see Corollary 2.4) and of finite fields (see Corollary 3.2). Our results can be viewed and treated as a further development of method and ideas from our investigation in [8] and [6], respectively. Some closely related things of this branch can also be found in [5].

It is well known that in finite rings any element is periodic, so any matrix over a finite ring is periodic itself, too. That is why, a question which immediately arises is what is the situation of matrices over infinite rings. We will concentrate our attention on infinite fields only, so that we, thereby, will also examine the following interesting problem.

Question 2: When each square matrix over an infinite field F can be expressed as

$$T + Q,$$

where T is a periodic matrix and Q is a square-zero matrix?

2. A DECOMPOSITION INTO POTENT AND ZERO-SQUARE OVER ALGEBRAICALLY CLOSED FIELDS

As a first approach to the problem, we will show that all square matrices over an algebraically closed field admit a decomposition into a diagonalizable matrix and a nilpotent matrix of order two. This decomposition will be significantly improved in the next section, where the same result will be proved for (non necessarily algebraically closed) fields by using the rational canonical form. Nevertheless, in this section we provide an easy argument in terms of Jordan blocks and we include it here for the sake of completeness. Moreover, it provides a decomposition for nilpotent matrices over (non necessarily algebraically closed) fields.

Our construction is based on Jordan blocks and roots of unity, so we begin by giving an explicit decomposition for a Jordan block (see also Remarks 2.8 and 2.9 from [8]).

Lemma 2.1. *Let K be a field and let J be a Jordan block in $\mathbb{M}_n(K)$, $n \geq 3$, associated to $a \in K$*

$$J = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 1 & a & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & a \end{pmatrix}.$$

- (i) Suppose that $\text{char}(K)$ does not divide n . If K contains the n (different) roots of the polynomial $x^n - 1 \in K[x]$, then J decomposes as

$$J = \underbrace{(J + e_{1n})}_D + \underbrace{(-e_{1n})}_Q$$

where e_{1n} denotes the nilpotent matrix with 1 in the $(1n)$ -entry and zero in the rest of entries, and the matrix D is diagonalizable. Moreover, if $a = 0$, $D^n = I$.

- (ii) Suppose that $\text{char}(K)$ divides n . If K contains the $n - 1$ (different) roots of the polynomial $x^{n-1} - 1 \in K[X]$, then J decomposes as

$$J = \underbrace{(J + e_{2n})}_D + \underbrace{(-e_{2n})}_Q$$

where e_{2n} denotes the nilpotent matrix with 1 in the $(2n)$ -entry and zero in the rest of entries, and the matrix D is diagonalizable. Moreover, if $a = 0$, $D^n = D$ and D^{n-1} is similar to the diagonal matrix $\text{diag}(1, \dots, 1, 0)$.

Proof. (i) If $q = \text{char}(K)$ does not divide n , then J can be written as

$$J = \underbrace{\begin{pmatrix} a & 0 & 0 & 0 & 1 \\ 1 & a & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & a \end{pmatrix}}_D + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_Q$$

The minimal polynomial of D is $p(x) = (x - a)^n - 1$ and it has n different roots in K (by hypothesis K contains all roots of $p(x) = (x - a)^n - 1$ and they are all different because $p'(x) = n(x - a)^{n-1} \neq 0$ since $q \nmid n$). In particular, D is diagonalizable. Moreover, one plainly sees that Q satisfies $Q^2 = 0$.

- (ii) If $q = \text{char}(K)$ divides n , then decompose J as

$$J = \underbrace{\begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 1 & a & 0 & 0 & 1 \\ 0 & 1 & a & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & a \end{pmatrix}}_D + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_Q$$

The minimal polynomial of D is $p(x) = (x - a)^n - (x - a)$ and its n roots belong to K by hypothesis and are all different because $p'(x) = -1 \neq 0$ (recall $q|n$). In particular, one follows that D is diagonalizable. Moreover, $Q^2 = 0$, as required. \square

Remark 2.2. The decomposition of each Jordan block into $D + Q$ given in Lemma 2.1 has the following properties:

- Each D is diagonalizable with no multiple eigenvalues.
- $Q^2 = 0$ and $\text{rank}(Q) \leq 1$.

Proposition 2.3. *Let K be an algebraically closed field. Then any matrix $A \in \mathbb{M}_n(K)$ can be written as $D + Q$, where D is a diagonalizable matrix and Q is a nilpotent matrix for which $Q^2 = 0$.*

Proof. Since K is algebraically closed, A is similar to a direct sum of Jordan blocks and it suffices to decompose each Jordan block. Let J be a Jordan block of size $m \times m$ for some $m \leq n$ and let q be characteristic of K . If $m \leq 2$ the decomposition is straightforward (see Section 1 of [8]). When $m \geq 3$, if q does not divide m , decompose J as in Lemma 2.1(i) and if q does divide m , decompose J as in 2.1(ii). \square

The hypothesis of algebraic closeness of the field can be removed when dealing with nilpotent matrices over a field, for which the decomposition into Jordan blocks always hold. Notice that this can be related to [9, §2] where the authors gave minimal conditions for a nilpotent element in a ring to admit a decomposition into Jordan blocks. As a consequence, we obtain that any nilpotent matrix can be expressed as the sum of a potent matrix and a nilpotent matrix of zero square. This result can be related to [4, Corollary 8] of Breaz and Megiesan, where they decomposed any nilpotent matrix into an idempotent matrix and a nilpotent matrix.

Corollary 2.4. *Every nilpotent matrix over a field can be written as $D + Q$, where D is a potent matrix (i.e., $D^q = D$ for a certain $q \in \mathbb{N}$) and Q is a nilpotent matrix with $Q^2 = 0$.*

Proof. Let $A \in \mathbb{M}_n(K)$ be a nilpotent matrix over the field K . Then A is similar to a direct sum of Jordan blocks J_1, \dots, J_s , each of them associated to the eigenvalue 0. For any of these Jordan blocks $J_i \in \mathbb{M}_{m_i}(K)$, one decomposes as in Lemma 2.1: $J_i = D_i + Q_i$. Define

$$k_i := \begin{cases} m_i, & \text{if } \text{char}(K) \text{ does not divide } m_i, \\ m_i - 1 & \text{if } \text{char}(K) \text{ divides } m_i, \end{cases}$$

and let $q = \text{lcm}\{k_i \mid i = 1, \dots, s\} + 1$. Then

$$\left(\bigoplus_{i=1}^s D_i\right)^q = \bigoplus_{i=1}^s D_i^q = \bigoplus_{i=1}^s D_i,$$

i.e., $\bigoplus_{i=1}^s D_i$ is q -potent. Finally, A decomposes into $D + Q$ as in Proposition 2.3 and D is similar to $\bigoplus_{i=1}^s D_i$. \square

It is also worth to notice that the last statement can also be proved by using the rational (Frobenius) canonical form – in fact, we refer to both sources [4] and [9] in which the authors utilize the general theory to obtain decompositions for nilpotent matrices.

In what follows, we shall substantially generalize Corollary 2.4 as well as Remark 2.9 from [8] by using another approach.

Proposition 2.5. *Every nilpotent matrix over a von Neumann regular ring is decomposable as a sum of a potent matrix and a nilpotent matrix of order two.*

Proof. Letting R be a von Neumann regular ring, then any nilpotent matrix A over R satisfies the property that A^s is a von Neumann regular matrix for all $s \in \mathbb{N}$, so A decomposes into a direct sum of Jordan blocks (see, e.g., [9]). Each of these Jordan blocks can be written as a potent matrix and a zero-square matrix. In particular,

it is not too hard to verify that A itself can be written as a potent matrix and a zero-square matrix, as asserted. \square

3. A DECOMPOSITION INTO POTENT AND ZERO-SQUARE OVER FINITE FIELDS

The following technicality appeared in [8] and will be freely used in what follows.

Lemma 3.1. *Let K be a field, let $n \geq 3$ and let $A \in \mathbb{M}_n(K)$ be the companion matrix of a polynomial $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$. Then*

- *If $c_{n-1} = 0$ and $|K| \geq n$ then A admits a decomposition into $D + Q$, where D is diagonalizable with no multiple eigenvalues and $Q^2 = 0$ with $\text{rank}(Q) \leq 1$.*
- *If $c_{n-1} \neq 0$ and $|K| \geq n + 1$ then A admits a decomposition into $D + Q$, where D is diagonalizable with no multiple eigenvalues and $Q^2 = 0$ with $\text{rank}(Q) \leq 1$.*

We consider in this section the following assertion which is devoted to a non-trivial property of matrices over finite fields.

Corollary 3.2. *Let K be a finite field and $n \in \mathbb{N}$. Then every matrix in $\mathbb{M}_n(K)$ admits a decomposition into an r -potent matrix, for certain $1 < r \in \mathbb{N}$, and a square-zero matrix.*

Proof. Let \mathbb{F}_q be the finite field of q elements and take $A \in \mathbb{M}_n(\mathbb{F}_q)$. If we consider the decomposition of A with respect to its invariant factors, then A is similar to a direct sum of s companion matrices, each of them of size $m_i \leq n$, $i = 1, \dots, s$. For each of them, take an irreducible polynomial $q_i(x)$ of degree m_i with the same trace, and write this block as the sum of the companion matrix $C(q_i(x))$ and a nilpotent matrix of zero square (see Lemma 3.1 and its proof). The decomposition field F of $q_i(x)$ is an extension of degree m_i of \mathbb{F}_q , i.e., $F = \mathbb{F}_{q^{m_i}}$. Since the matrix $C(q_i(x))$ is diagonalizable with different eigenvalues in F (finite fields are perfect), $C(q_i(x))$ is similar to a diagonal matrix $D_i \in \mathbb{M}_{m_i}(F)$ and, therefore,

$$D_i^{q^{m_i}-1} = \begin{cases} I, & \text{if } q_i(0) \neq 0; \\ \text{diag}(1, \dots, 1, 0), & \text{if } q_i(0) = 0. \end{cases}$$

Define $r = \text{lcm}\{q^{m_i} - 1 \mid i = 1, \dots, s\} + 1$. Since each $D_i^r = D_i$, we can express A as a sum of an r -potent matrix and a square-zero matrix. \square

Actually, the more general intriguing question is of whether or not every square matrix over an arbitrary (possibly infinite) field is presentable as a sum of a nilpotent and a potent – notice that for finite fields this was settled independently in Corollary 3.2 alluded to above and [1]. However, the answer seems to be definitely “not” as the next example illustrates, but such a matrix is rather a sum of a non-singular matrix and a nilpotent matrix (see, e.g., [10]) – in particular, this fact surely implies that the matrix over the field \mathbb{F}_4 is a sum of a potent and a nilpotent (compare also with our Corollary 3.2 quoted above and [1]).

Example 3.3. Take the matrix $A = 2\text{Id} \in \mathbb{M}_n(\mathbb{R})$, and let us show that A cannot be expressed as the sum of a k -potent and an r -nilpotent. Otherwise, $A = Q + N$ with $Q^k = Q$ and $N^r = 0$ for some natural numbers k and r . This surely implies that $Q = A - N$ satisfies on the one hand the polynomial $X^k - X$ (because $Q^k = Q$), whereas on the other hand, since $0 = N^r = (A - Q)^r = (2\text{Id} - Q)^r$, the matrix Q

also satisfies the polynomial $(2 - X)^r$. This means that the minimal polynomial of Q must divide both $X^k - X$ and $(X - 2)^r$, but these two polynomials have no roots in common in \mathbb{R} , so that the minimal polynomial of Q is 1, a contradiction which finishes the proof.

4. A DECOMPOSITION INTO PERIODIC AND ZERO-SQUARE OVER INFINITE FIELDS

To start our consideration here, we just show that Question 2 is *not* true, unfortunately, even over algebraically closed fields. Specifically, the following construction holds:

Example 4.1. Let \mathbb{C} be the field of complex numbers, and consider the matrix $A = 2Id \in \mathbb{M}_n(\mathbb{C})$. Suppose in a way of contradiction that $A = T + N$, where $N^2 = 0$. Then $N = A - T$ and, therefore, $0 = N^2 = (A - T)^2 = 4Id + T^2 - 4T$. This means that the matrix T satisfies the polynomial $x^2 - 4x + 4 = (x - 2)^2$.

Furthermore, one sees and easily checks that the characteristic polynomial of T is of the form $(x - 2)^n$ for some $n \in \mathbb{N}$, so one calculates that the determinant of T is exactly $(-1)^n 2^n$. Consequently, T cannot be periodic since one verifies that either $\det(P) = 0$, $\det(P) = 1$ or $\det(P) = -1$.

We are now ready to give a complete answer to Question 2 as follows:

Proposition 4.2. *Let F be a field. Then the next three points are valid:*

- (1) *If $\text{char}(F) = 0$, the answer is NO.*
- (2) *If $\text{char}(F) = p$ and the extension F over is transcendental, the answer is NO.*
- (3) *If $\text{char}(F) = p$ and the extension F over the simple field F_p consisting of p elements is algebraic, then the answer is YES.*

Proof. (1) The unique prime field of zero characteristic is precisely the field of rationals \mathbb{Q} , and hence the matrix $2Id$ cannot be decomposed into periodic plus zero-square matrices (compare with the stated above example).

(2) There exists an element $a \in F$ that is not algebraic over F_p . Consider then the matrix aId , and with the aid of the same argument as in (1) this matrix cannot be decomposed into periodic plus zero-square matrices.

(3) Indeed, we may decompose even into potent plus zero-square: Let $A \in \mathbb{M}_n(F)$. Consider the finite field L generated by F_p and by the entries of A . If L has more elements than the matrix size n , then applying the main result from [8] the matrix A decomposes into a diagonalizable matrix over L plus a zero-square matrix. If not, then we may extend L (adding elements from F) until you get a finite field L' with more elements than n . Since still $A \in \mathbb{M}_n(L')$, one finds that A decomposes into a diagonalizable matrix over L' plus a zero-square matrix. By taking into account that a diagonalizable matrix over a finite field is always potent, and hence periodic, we are done. \square

In closing our work, we pose the following question which could be of some interest and importance.

Problem. Can we decompose any square matrix over the indecomposable ring \mathbb{Z}_4 as a sum of a square-zero matrix and a potent matrix?

Let us note that in [12] was established that every such matrix can be decomposed as the sum of a nilpotent matrix of order at most 8 and an idempotent matrix. So, it is rather realistic to replace the idempotent by a potent and thereby to expect that the order of the nilpotent could be decreased to eventually order 2 or, in the more bad variant, to eventually 4.

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